$\frac{\infty}{2}$-Hodge structures: from BCOV theory to Seiberg-Witten geometry

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$\frac{\infty}{2}$-Hodge structure

$\frac{\infty}{2}$-Hodge structure originated from K. Saito's theory of higher residues and primitive form in his study of period maps for isolated singularities. This is generalized and systematically developed in Calabi-Yau geometry by Barannikov-Kontsevich, giving the official name $\frac{\infty}{2}$-HS.

In this talk, we explain the role of $\frac{\infty}{2}$-Hodge structure in

1. B-twisted topological string field theory (BCOV theory)
2. Seiberg-Witten geometry realized via singularities.

Outline

Singularity and $\frac{\infty}{2}$-Hodge Structure

BCOV theory and quantum B-model

Singularity and Seiberg-Witten geometry

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BCOV theory and quantum B-model

Singularity and Seiberg-Witten geometry

Isolated singularity

We motivate $\frac{\infty}{2}$-HS from K.Saito's original version. Let

$$f : X = (\mathbb{C}^{n+1}, 0) \to \Delta = (\mathbb{C}, 0)$$

be the germ of an isolated singularity. We consider the quotient

$$\boxed{\Omega_f := \Omega_X^{n+1} / df \wedge \Omega_X^n}.$$

With a choice of holomorphic volume form $d\mathbf{x} = dx_0 \wedge \cdots \wedge dx_n$

$$\Omega_f = \mathrm{Jac}(f) d\mathbf{x}$$

where $\mathrm{Jac}(f) = \mathbb{C}\{x_0, \cdots, x_n\}/(\partial_i f)$ is the Jacobian algebra of f at 0. There is a non-degenerate pairing on Ω_f given by residue

$$\boxed{\mathrm{Res}_f : \Omega_f \otimes \Omega_f \to \mathbb{C}}.$$

Brieskorn lattice

The space Ω_f is the leading part of the **Brieskorn lattice**

$$\boxed{\mathcal{H}_f^{(0)} := \Omega_X^{n+1}/df \wedge d\Omega_X^{n-1}}.$$

There is a well-defined operator, denoted by a formal variable t,

$$t : \mathcal{H}_f^{(0)} \to \mathcal{H}_f^{(0)}.$$

Given $\alpha \in \mathcal{H}_f^{(0)}$, there exists n-form β such that $\alpha = d\beta$, then

$$t \cdot \alpha := -df \wedge \beta \in \mathcal{H}_f^{(0)}.$$

Symbolically,

$$\boxed{t = -\frac{df}{d}} : \alpha \to -df \wedge d^{-1}\alpha.$$

Descendant forms

Given $\omega \in \mathcal{H}_f^{(0)}$ and $k \geq 0$, we define its k-th descendant form

$$\boxed{\omega^{(-k)} := (-t)^k \omega} \in \mathcal{H}_f^{(0)}.$$

Descendant forms give natural semi-infinite filtrations

$$\cdots \subset \mathcal{H}_f^{(-k)} \subset \mathcal{H}_f^{(-k+1)} \subset \cdots \subset \mathcal{H}_f^{(-1)} \subset \mathcal{H}_f^{(0)}$$

The formal completion of $\mathcal{H}_f^{(0)}$ w.r.t. this t-adic topology identifies

$$\boxed{\hat{\mathcal{H}}_f^{(0)} = \Omega_X^{n+1}[[t]]/(td + df)\Omega_X^n[[t]]}.$$

Higher residue

K. Saito defines a sesqui-linear **higher residue pairing**

$$K_f : \hat{\mathcal{H}}_f^{(0)} \times \hat{\mathcal{H}}_f^{(0)} \to t^n \mathbb{C}[[t]].$$

whose leading term coincides with the residue pairing on

$$\boxed{\Omega_f = \mathcal{H}_f^{(0)} / t\mathcal{H}_f^{(0)}}.$$

We can further extend K_f to

$$\boxed{\hat{\mathcal{H}}_f := \Omega_X^{n+1}((t))/(td + df) = \hat{\mathcal{H}}_f^{(0)}[t^{-1}]}$$

and write

$$K_f : \hat{\mathcal{H}}_f \times \hat{\mathcal{H}}_f \to \mathbb{C}((t)).$$

$\frac{\infty}{2}$-Hodge structure

A $\frac{\infty}{2}$-Hodge structure of weight n consists of $(\mathcal{H}, \mathcal{E}, \nabla, K)$ where

1. $\mathcal{H}$ is a finite dim vector space over $\mathbb{C}((t))$;
2. $\mathcal{E}$ is a $\mathbb{C}[[t]]$-lattice;
3. ∇ is a meromorphic connection on the formal disk such that

$$\nabla_{\frac{\partial}{\partial t}} \mathcal{E} \subset t^{-2}\mathcal{E}$$

4. $K : \mathcal{H} \times \mathcal{H} \to \mathbb{C}((t))$ a ∇-compatible pairing (weight n) s.t.
   - $K(v(t)\alpha, \beta) = K(\alpha, v(-t)\beta) = v(t)K(\alpha, \beta)$;
   - $K(\alpha, \beta) = (-1)^n K(\beta, \alpha)^*$ where $*$-operator takes $t \to -t$;
   - $K : \mathcal{E} \times \mathcal{E} \to t^n\mathbb{C}[[t]]$ with non-degenerate leading pairing

$$t^{-n}K : \mathcal{E}/t\mathcal{E} \times \mathcal{E}/t\mathcal{E} \to \mathbb{C}$$

$(\hat{\mathcal{H}}_f, \hat{\mathcal{H}}_f^{(0)}, \nabla, K_f)$ forms a weight n $\frac{\infty}{2}$-HS, which varies along deformations of f exhibiting good properties.

Oscillatory integral and Period map

Given an element $\omega \in \mathcal{H}_f^{(0)}$, we can consider the oscillatory integral

$$\int_\Gamma e^{f/t}\omega.$$

Under Laplace transformation, this is related to the period map

$$\boxed{\int_\gamma \frac{\omega}{df}}.$$

In Seiberg-Witten curve geometry, we have a 2-form ω with $\omega = d\lambda$. Then the SW period map is related to the period map of the first descendant of ω

$$\int_\gamma \lambda = \int_\gamma \frac{df \wedge d^{-1}\omega}{df} = -\int_\gamma \frac{\omega^{(-1)}}{df}.$$

As we will see, this observation allows us to obtain SW differential arising from higher dimensional geometry (in particular 3-fold fibration) via different choices of the descendants.

Primitive form

Let $\mathcal{M}$ be the miniversal deformation space of $f(x)$, represented by a universal unfolding $F(x^i, \lambda^\alpha), \lambda \in \mathcal{M}$. Using $\frac{\infty}{2}$-HS, K. Saito [1982] constructed a special family of holomorphic volume forms $\xi(x, \lambda) = \varphi(x, \lambda) d\mathbf{x}$, called primitive form. It determines a set of flat coordinates $\{\tau^\alpha\}$ on $\mathcal{M}$ such that

$$\left(t \frac{\partial}{\partial \tau^\alpha} \frac{\partial}{\partial \tau^\beta} - A^\gamma_{\alpha\beta}(\tau) \frac{\partial}{\partial \tau^\gamma} \right) \int e^{F/t} \xi = 0.$$

$A^\gamma_{\alpha\beta}$ is nowadays the Yukawa coupling of 2d LG B-models.

On Calabi-Yau geometries, the analogue of primitive form is called $\frac{\infty}{2}$-period map. In the context of Gromov-Witten theory, this is related to Givental's J-function.

Singularity and $\frac{\infty}{2}$-Hodge Structure

BCOV theory and quantum B-model

Singularity and Seiberg-Witten geometry

B-model string field theory

- Let (X, Ω_X) be a compact Calabi-Yau manifold.
- [Bershadsky-Cecotti-Ooguri-Vafa, 1994]: B-twisted topological string field theory on Calabi-Yau 3-fold

  $\rightarrow$ Kodaira-Spencer gravity

  which describes deformations of Ricci-flat metrics in terms of complex structures (Calabi-Conjecture/Yau-Theorem).
- [Costello-L, 2012] An extension of Kodaira-Spencer gravity in the sense of Zwiebach is formulated on arbitrary Calabi-Yau

  $\rightarrow$ BCOV theory.

  This can be coupled with Witten's open string field theory (HCS) in terms of a cyclic version of Kontsevich's Formality.

Period map for Calabi-Yau 3-fold and mirror symmetry

Let X_0 be a compact Calabi-Yau 3-fold. Consider the pair moduli

$$\mathcal{M} = \{(X, \Omega_X) | X \text{ is a deformation of } X_0, \text{and } \Omega_X \text{ is a CY form on X}\}$$

Let $\mathcal{P}$ be the following period map

$$\mathcal{P} : \mathcal{M} \to H^3(X_0), \quad (X, \Omega_X) \to [\Omega_X] \in H^3(X) \simeq H^3(X_0).$$

- $H^3(X_0)$ is a symplectic space. $(\alpha, \beta) = \int_X \alpha \wedge \beta$.
- $\mathcal{P}$ embeds $\mathcal{M}$ into a Lagrangian submanifold of $H^3(X_0)$.
- A splitting of the Hodge filtration identifies

$$H^3(X_0) \simeq T^* F^2, \quad F^2 = H^{3,0}(X_0) \oplus H^{2,1}(X_0).$$

This allows us to identify $\mathcal{P}(\mathcal{M}) = \text{Graph}(d\mathbf{F}_0)$, where $\mathbf{F}_0$ is a function on F^2 (prepotential) that is mirror to the generating function of Gromov-Witten invariants.

Deformation of Calabi-Yau structure

The pair deformation (X, Ω_X) can be specified by a pair (μ, ρ)

$$\mu \in \mathcal{A}^{0,1}(X, T_X^{1,0}), \quad \rho \in C^\infty(X)$$

where μ specifies the new complex structure, and ρ specifies the new Calabi-Yau volume form. They satisfy the equations

$$\begin{cases} \bar{\partial}\mu + \frac{1}{2}[\mu, \mu] = 0 \\ d(e^\rho e^\mu \lrcorner \Omega_0) = 0 \end{cases}$$

where Ω_0 is the CY form on X_0. This equation is equivalent to

$$\boxed{Q\tilde{\mu} + \frac{1}{2}[\tilde{\mu}, \tilde{\mu}] = 0}$$

where

$$\tilde{\mu} = \mu + t\rho, \quad Q = \bar{\partial} + t\partial.$$

∂ is the divergence operator w.r.t. the CY volume form Ω_0.

BCOV theory: fields

We define the fields of BCOV theory on Calabi-Yau X by

$$\boxed{\mathcal{E} := \mathrm{PV}(X)[[t]]}, \quad \mathrm{PV}(X) = \bigoplus_{i,j} \Omega^{0,j}(X, \wedge^i T_X).$$

$\mathcal{E}$ has a differential $\boxed{Q := \bar{\partial} + t\partial}$ (∂ is the divergence operator) and Schouten-Nijenhuis bracket $[-,-]$. The associated Maurer-Cartan equation describes (extended) deformation of Calabi-Yau structure. This is used by Barannikov-Kontsevich to obtain Frobenius manifold structure for compact Calabi-Yau.

$\mathcal{E}$ carries the analogue of higher residue pairing

$$\boxed{K(f(t)\alpha, g(t)\beta) := f(t)g(-t)\int \alpha \wedge \beta} \in \mathbb{C}[[t]].$$

BCOV theory: equation of motion

- The equation of motion describes deforming CY structure

$$Q\mu + \frac{1}{2}[\mu, \mu] = 0, \quad \mu \in \mathrm{PV}(X)[[t]].$$

  This is a resolution of BCOV's Kodaira-Spencer gravity on divergence free polyvectors $\ker \partial \subset \mathrm{PV}(X)$.

- To generalize Kodaira-Spencer gravity, we need to find an action functional whose variation gives the above equation. Unfortunately such action does not seem exist.
- Solution [Costello-L, 2012]: After we perform a nonlinear

$$L_\infty \text{ transformation}$$

  of the above equation, we can write down a local interaction whose variation gives the transformed equation. We call this BCOV interaction.

BCOV interaction via period map

The nonlinear transformation that allows us to write down local interaction is given by the cochain period map as follows. Let

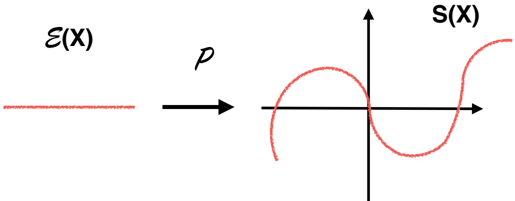
$$S(X) := \mathrm{PV}(X)((t))$$

equipped with a non-degenerate graded skew-symmetric pairing

$$\omega(f(t)\alpha, g(t)\beta) = \mathrm{Res}_{t=0}(f(t)g(-t)dt)\,\mathrm{Tr}(\alpha \wedge \beta).$$

Consider the following period map (strictly speaking, this is a formal map between two functors on Artinian rings)

$$\mathcal{P} : \mathcal{E}(X) = \mathrm{PV}(X)[[t]] \to S(X), \qquad \boxed{\mu \to t\left(e^{\mu/t} - 1\right)}$$



Proposition

$\mathcal{P}$ embeds $\mathcal{E}(X)$ as a formal lagrangain submanifold of $S(X)$ which is tangent to the linear vector field on $S(X)$ generated by the infinitesimal transformation $Q = \bar{\partial} + t\partial$.

The isotropic splitting

$$S(X) = \mathcal{E}(X) \bigoplus t^{-1}\,\mathrm{PV}(X)[t^{-1}]$$

allows us to formally express

$$S(X) \quad \text{“}=\text{”} \quad T^*\mathcal{E}(X).$$

Let $\mathbf{I}_0$ be the generating functional on $\mathcal{E}(X)$ for the image of period map

$$\operatorname{im}\mathcal{P} = \mathrm{Graph}(d\mathbf{I}_0).$$

Theorem (Costello-L)

The functional $\mathbf{I}_0$ *is local and given by*

$$\mathbf{I}_0(\mu) := \int \langle e^{\mu} \rangle_0$$

where

$$\left\langle t^{k_1}\mu_1, \cdots, t^{k_n}\mu_n \right\rangle_0 := \binom{n-3}{k_1, \cdots, k_n} \mu_1 \wedge \cdots \wedge \mu_n.$$

Moreover, $\mathcal{E}(X)$ *has a* (-1)*-shifted degenerate Poisson structure inducing a (degenerate) BV bracket* $\{-,-\}$*. Then* $\mathbf{I}_0$ *satisfies the classical BV master equation*

$$Q\mathbf{I}_0 + \frac{1}{2}\{\mathbf{I}_0, \mathbf{I}_0\} = 0.$$

Remark: Note that the leading cubic part of $\mathbf{I}_0$ is the interaction of Kodaira-Spencer gravity. A.Losev has a similar construction for a finite dimensional model of PV(X).

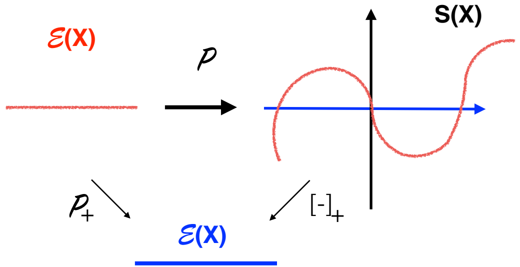
The L_∞ transformation $\mathcal{P}_+$

The cohomological vector field $Q + \{\mathbf{l}_0, -\}$ defines a (local) L_∞-structure on $\mathcal{E}(X)$. The nonlinear transformation

$$\mathcal{P}_+ : \mathcal{E}(X) \to \mathcal{E}(X)$$
$$\mu = \sum_{k \geq 0} t^k \mu_k \to \left[t e^{\mu/t} - t \right]_+$$

identifies

$$\boxed{\text{solutions of } Q\mu + \frac{1}{2}[\mu, \mu] = 0 \overset{\mathcal{P}_+}{\Longrightarrow} \text{zero locus of } Q + \{\mathbf{l}_0, -\}}$$



Quantum master equation

Start with a solution of classical master equation, there is a standard quantization procedure in the BV-formalism. It amounts to find quantum corrected functional

$$I_0 \quad \to \quad I = I_0 + I_1 \hbar + \cdots$$

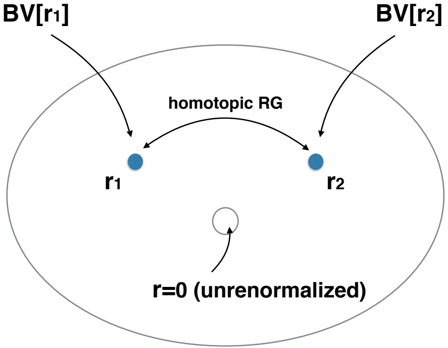
solving the quantum master equation

$$\boxed{QI + \hbar \Delta I + \frac{1}{2}\{I, I\} = 0}.$$

Here Δ is the BV-operator associated to the shifted Poisson structure above. Howover, $\mathrm{PV}(X)$ is infinite dimensional and the above equation is not well-defined suffering from UV divergence.

Costello's homotopic renormalization

One rigorous approach to the above quantum master equation is achieved by Costello's homotopic renormalization method.



Higher genus B-model

After solving quantum master equation, we obtain a generating function $\mathbf{F}_g^B$ on the zero modes

$$H^\bullet(\mathcal{E}(X), Q) \overset{\text{splitting of Hodge}}{\simeq} H^\bullet(X, \wedge^\bullet T_X)[[t]]$$

by collecting the g-loop Feynman diagrams. This gives higher genus B-model invariants in our generalized BCOV theory which are conjectured mirror to higher genus Gromov-Witten invariants (with descendants) . Its dependence on the choice of splitting of the Hodge filtration gives rise to holomorphic anomaly equation.

The geometric interpretation of quantum BV master equation is that it defines a Q-closed element

$$Q|F\rangle = 0$$

in the Fock representation of the Weyl algebra that quantizes the dg symplectic space $(S(X), Q, \omega)$.

Example: elliptic curves

Theorem (Costello-L, 2012, L, 2016)

There exists a canonical solution of homotopic quantum BV-master equation for BCOV theory on elliptic curves.

Theorem (L, 2016)

The BCOV generating function of the elliptic curve with respect to the monodromy splitting around the large complex structure limit coincides with the full descendent Gromov-Witten invariants of the mirror elliptic curve computed by Okounkov-Pandharipande.

This Theorem generalizes the classical result by Dijkgraaf and fully establishes the higher genus mirror symmetry on elliptic curves.

Application: an explaination for integrable hierarchy

Counting curves on CY geometry always leads to (KdV-type) integrable hierarchy. We propose an explanation in the B-model in terms of quantum master equation of BCOV theory. ([L, 2017], also work in progress with He and Yoo):

Let X be a Calabi-Yau geometry, we iterate B-model by considering the product $X \times \mathbb{C}$ (again Calabi-Yau).

B-model (BCOV theory) on $X \times \mathbb{C}$

$\big\downarrow$ integration out over X

Effective 2d chiral theory on $\mathbb{C}$

$\boxed{\text{BV master equation on } \mathbb{C}} \Longrightarrow \boxed{\text{integrable hierarchy associated to } X}$

More precisely, there is an (∞-dim) abelian Lie algebra

$$H^*(X)[[t]] \otimes \text{translation on } \mathbb{C}$$

acting as a symmetry of the effective theory on $\mathbb{C}$, whose Noether currents give commuting currents. It can be shown that at genus 0, such commuting currents lead to Dubrovin's dispersionless integrable hierarchy associated to Frobenius manifolds.

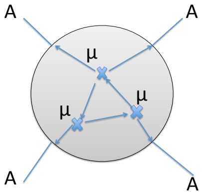
Remark: B-model open-closed string field theory

BCOV theory can be coupled with Witten's holomorphic CS theory to give open-closed string field theory in the B-model. It is required to satisfy Zwiebach's open-closed BV master equation.

The classical open-closed interaction turns out [Cosello-L, 2015]

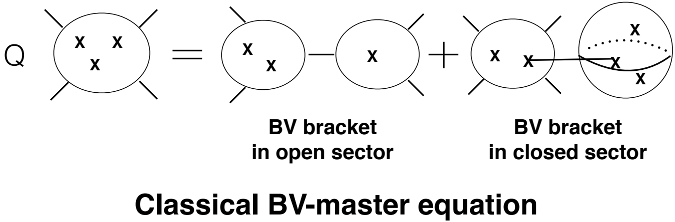
$$I_0(\mu, A) = \sum_{m,n} \int_X \int_{C_{m.n}} \mathcal{L}_{m,n}(\mu, A), \quad \mu \oplus A \in \mathrm{PV}(X)[[t]] \oplus \Omega^{0,*}(X, \mathfrak{g})$$

- $C_{m,n}$: configuration space of the disk
- $\mathcal{L}_{m,n}$ is of the form of Kontsevich's graph formula of deformation quantization (cyclic version)



Open-closed BV master equation

Kontsevich's Formality Theorem is essentially equivalent to Zwiebach's master equation for open-closed string field theory



Here's an example of first order coupling (HKR map)

$$\sum \int_X \mathrm{Tr}\left(\mu^{i_1\cdots i_k}(A \wedge \partial_{i_1} A \wedge \cdots \wedge \partial_{i_k} A)\right) \wedge \Omega_X,$$

for $\mu = \sum \mu^{i_1\cdots i_k}\partial_{i_1} \wedge \cdots \wedge \partial_{i_k} \in \mathrm{PV}(X), \quad A \in \Omega^{0,*}(X, \mathfrak{g}).$

Singularity and $\frac{\infty}{2}$-Hodge Structure

BCOV theory and quantum B-model

Singularity and Seiberg-Witten geometry

Seiberg-Witten geometry

We are interested in 4D N = 2 SCFT. Seiberg-Witten discovered that for many theories the low energy effective theory on the Coulomb branch could be described by a Seiberg-Witten curve fibered over the moduli space:

$$F(x, z; \lambda_\alpha) = 0$$

Here λ_α's are the parameters including coupling constants, mass parameters, and expectation values for Coulomb branch operators. The period integral of an appropriate 1-form over the Riemann surface $F(x, z; \lambda_\alpha) = 0$ determines the low energy photon coupling.

Singularity and 4d $N = 2$ SCFT

We consider type IIB string theory on the following background:

$$\mathbb{R}^{3,1} \times X$$

Here X is a 3-fold weighted homogeneous isolated singularity. It is argued to define 4d $N = 2$ SCFT (Shapere, Vafa, 99; D. Xie, Yau, 15). The SW solution is associated to a three-fold fibration

$$F(x_1, x_2, x_3, x_4; \lambda_\alpha) = 0$$

which may or may not be reduced to the curve geometry. This suggests that the more general SW solution could be three-fold fibrations rather than curves. Our goal is to figure out the corresponding SW differential.

3-fold Singularity

Consider an isolated weighted homogeneous 3-fold singularity:

$$f : \mathbb{C}^4 \to \mathbb{C}, \quad f(\lambda^{q_i} x^i) = \lambda f(x^i), \quad q_i > 0, \quad \lambda \in \mathbb{C}^*.$$

The rational number

$$\hat{c}_f = \sum_i (1 - 2q_i)$$

is the central charge of the 2d (2,2) SCFT defined by LG model with superpotential f. To define a 4d SCFT we require

$$\sum_i q_i > 1 \Longleftrightarrow \hat{c}_f < 2.$$

Coulomb branch and miniversal deformations

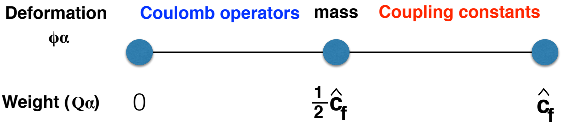
The Coulomb branch of the associated N=2 SCFT is described by the local moduli of miniversal deformations of f:

$$F(x_i, \lambda_\alpha) = f(x_i) + \sum_{\alpha=1}^{\mu} \lambda_\alpha \phi_\alpha$$

where $\{\phi_\alpha\}$ is a basis of the Jacobi algebra $\mathrm{Jac}(f)$. Define

$$\boxed{[\lambda_\alpha] = \frac{1 - Q_\alpha}{\sum_i q_i - 1}}, \quad Q_\alpha = \text{homogeneous weight of } \phi_\alpha.$$

- $[\lambda_\alpha] < 1$: Coupling constants
- $[\lambda_\alpha] = 1$: Mass parameters
- $[\lambda_\alpha] > 1$: Expectation value of Coulomb branch operators



Period map

Given a family of holomorphic volume forms $\xi(x^i, \lambda_\alpha)$, we consider the period map

$$\mathcal{P} : \mathcal{M} \to H^3, \quad \{\lambda_\alpha\} \to \int_\gamma \frac{\xi}{dF}.$$

Here the integration is over vanishing cycles γ in $F^{-1}(0)$ and $H^3 = H^3(F^{-1}(0), \mathbb{C})$ is the dual space. For simplicity, let us consider the case with no mass parameters. Then H^3 has a natural symplectic structure induced dually from the intersection pairing. This allows choices of an electro-magnetic charge lattice from H_3.

Seiberg-Witten differential

The low energy effective theory of Coulomb branch is described by Seiberg-Witten (SW) geometry.

Question

What is the Seiberg-Witten differential associated to singularities?

Seiberg-Witten differential

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Naive guess: the SW differential is the family of 3-forms

$$\frac{\xi}{dF}, \qquad \text{where} \qquad \xi = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

The $N = 2$ prepotential comes from the corresponding period map. This is indeed true in the case of ADE singularities ($\hat{c}_f < 1$).

If we go beyond ADE singularities, this no longer holds.

Seiberg-Witten differential

Solution [L-Xie-Yau, 2018]: for 3-fold singularity

$$\boxed{\xi \text{ primitive form} \Longrightarrow \frac{\xi}{dF} \text{ SW differential}}$$

where ξ is K. Saito's primitive form.

The main support for the connection between primitive form and SW differential is about the integrability condition

$$\langle d\mathcal{P}, d\mathcal{P} \rangle = 0$$

for the existence of $N = 2$ prepotential arising from SW period map. The verification of such integrability requires a connection between the intersection pairing for vanishing homology and the period map. Primitive form provides precisely such a relationship [K.Saito, 1982].

Curve v.s. 3-fold geometry

This result also generalizes to curve geometry

$$\boxed{\xi \text{ primitive form} \Longrightarrow \frac{\xi^{(-1)}}{dF} \text{ SW differential}}$$

The SW differential for three-fold geometry picks up ξ instead of $\xi^{(-1)}$ for curve geometry, by the reason of shift of Hodge theory arising from the shift of dimension.

There is also a 5d hypersurface singularity example. The analogue discussion implies that the SW differential is expected to be

$$\frac{\xi^{(1)}}{dF}.$$

Examples of primitive forms: ADE type ($\hat{c}_f < 1$)

We consider

$$f(x) = x_1^2 + x_2^2 + x_3^k + x_4^N, \quad \frac{1}{k} + \frac{1}{N} > \frac{1}{2}.$$

This example can be reduced to curve geometry. The primitive form is trivial in this case and doesn't depend on the deformation parameter

$$\xi = dx_1 \wedge \cdots \wedge dx_4.$$

The 3-fold SW differential is given by

$$\lambda = \frac{\xi}{dF}$$

Simple elliptic singularity ($\hat{c}_f = 1$)

We consider

$$f(x) = x_1^3 + x_2^3 + x_3^3 + x_4^2$$

The miniversal deformation is

$$F = f + \lambda_1 + \lambda_2 x_1 + \lambda_3 x_2 + \lambda_4 x_3 + \lambda_5 x_1 x_2 + \lambda_6 x_2 x_3 + \lambda_7 x_3 x_1 + \lambda_8 x_1 x_2 x_3.$$

Primitive form of this example is nontrivial and is not unique. They depend only on the marginal parameter λ_8 described as follows:

$$\xi = \frac{dx_1 \wedge \cdots \wedge dx_4}{P(\lambda_8)}$$

where $P(\lambda_8)$ is a period on the cubic elliptic curve

$$\{x_1^3 + x_2^3 + x_3^3 + \lambda_8 x_1 x_2 x_3 = 0\} \subset \mathbb{P}^2.$$

General singularities ($\hat{c}_f > 1$)

For general singularities with $\hat{c}_f > 1$, there exists a highly nontrivial mixing between relevant and irrelevant deformations. The close formula of primitive form is unknown. Here is one example

$$f = x_1^2 + x_2^2 + x_3^3 + x_4^7.$$

This is type E_{12} of the unimodular exceptional singularities. The miniversal deformation is the following

$$F(x,\lambda) = f + \lambda_1 + \lambda_2 x_4 + \lambda_3 x_4^2 + \lambda_4 x_3 + \lambda_5 x_4^3 + \lambda_6 x_3 x_4 \\ + \lambda_7 x_4^4 + \lambda_8 x_3 x_4^2 + \lambda_9 x_4^5 + \lambda_{10} x_3 x_4^3 + \lambda_{11} x_3 x_4^4 + \lambda_{12} x_3 x_4^5.$$

Here $\lambda_1, \cdots, \lambda_{11}$ are relevant deformations, and λ_{12} is an irrelevant deformation.

There is a recursive formula to compute general primitive forms perturbatively [L-Li-Saito, 2015]. For E_{12}, it gives (up to order 10)

$$\zeta = (\varphi(x,\lambda) + O(\lambda^{11}))dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

where

$$\begin{aligned}
\varphi(x,\lambda) = 1 &+ \frac{4}{3\cdot 7^2}\lambda_{11}{\lambda_{12}}^2 - \frac{64}{3\cdot 7^4}\lambda_{11}^2{\lambda_{12}}^4 - \frac{76}{3^2\cdot 7^4}\lambda_{10}{\lambda_{12}}^5 + \frac{937}{3^3\cdot 7^5}\lambda_9{\lambda_{12}}^6 + \frac{218072}{3^4\cdot 5\cdot 7^6}\lambda_{11}^3{\lambda_{12}}^6 \\
&+ \frac{1272169}{3^4\cdot 5\cdot 7^7}\lambda_{10}\lambda_{11}{\lambda_{12}}^7 + \frac{28751}{3^4\cdot 7^7}\lambda_8{\lambda_{12}}^8 - \frac{1212158}{3^4\cdot 7^8}\lambda_9\lambda_{11}{\lambda_{12}}^8 - \frac{38380}{3^3\cdot 7^8}\lambda_7{\lambda_{12}}^9 \\
&+ \Big(\frac{1}{7^2}{\lambda_{12}}^3 - \frac{101}{5\cdot 7^4}\lambda_{11}{\lambda_{12}}^5 + \frac{1588303}{3^4\cdot 5\cdot 7^7}\lambda_{11}^2{\lambda_{12}}^7 + \frac{378083}{3^4\cdot 5\cdot 7^7}\lambda_{10}{\lambda_{12}}^8 - \frac{108144}{3\cdot 7^8}\lambda_9{\lambda_{12}}^9\Big)x_3 \\
&+ \Big(\frac{1447}{3^3\cdot 7^6}{\lambda_{12}}^7 - \frac{71290}{3^3\cdot 7^8}\lambda_{11}{\lambda_{12}}^9\Big)x_4 - \frac{45434}{3^4\cdot 7^8}{\lambda_{12}}^{10}x_3x_4 \\
&- \Big(\frac{53}{3^2\cdot 7^4}{\lambda_{12}}^6 - \frac{46244}{3^3\cdot 7^7}\lambda_{11}{\lambda_{12}}^8\Big)x_3^2 + \frac{22054}{3^4\cdot 7^7}{\lambda_{12}}^9 x_3^3.
\end{aligned}$$

In particular, the Seiberg-Witten differential of this example is not given by a rescaling of the trivial form and depends on the coupling constants in a nontrivial way.

Thank you!