Quantization and Factorization Algebras

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Lecture 1: Geometry of QFT

1. Quantum field theory and Renormalization
2. Observables and Factorization algebras
3. Examples: a first geometric look

1. Quantum field theory and Renormalization

"Anyone who is not shocked by quantum theory has not understood it."—Niels Bohr

"I think I can safely say that nobody understands quantum mechanics."—Richard Feynman

Quantum field theory deals with "∞-dimensional geometry", which lies behind many of its nontrivial consequences and predictions.

Typically (but not always) a physics system is described by a map

$$S : \mathcal{E} \to \mathbb{R}.$$

- $\mathcal{E}$: *space of fields*.
- S: *action functional*.

Typical examples

- Scalar field theory

$$\mathcal{E} = C^\infty(X)$$

- Gauge theory

$$\mathcal{E} = \{\text{connections on } V \to X\}$$

- σ-model

$$\mathcal{E} = \mathrm{Map}(\Sigma, X)$$

- Gravity

$$\mathcal{E} = \{\text{metrics on } X\}$$

Path integral

- Classical physics is decribed by the critical locus (equation of motion, eg: Laplace equation, Yang-Mills equation, etc)

$$\mathrm{Crit}(S) = \{\delta S = 0\}.$$

- One standard approach to Quantum physics is described by Feynman's "path integral"

$$\langle \mathcal{O} \rangle := \int_{\mathcal{E}} \mathcal{O} e^{S/\hbar}$$

$\mathcal{O}$: quantum observable. $\langle \mathcal{O} \rangle$: correlation function.

  - Mathematical challenge for such ∞-dim integral.
  - Asymptotic analysis leads to - renormalization theory.

We are mainly interested in “integrals”

$$\int f$$

We rarely compute integrals by definition (Riemann/Lebesgue).
Instead, we use symmetries and differential equations.

Gaussian integral

Gaussian integral

$$\int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}}\ e^{-\frac{1}{2}x^2} = 1$$

or more generally

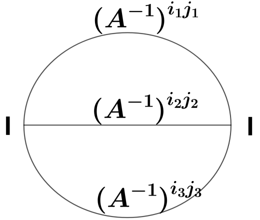
$$\int_{\mathbb{R}^n} \prod_i \frac{dx^i}{\sqrt{2\pi}}\ e^{-\frac{1}{2}A(X)} = \frac{1}{\sqrt{\det A}}, \quad \text{where} \quad A(x) = \sum_{i,j} A_{ij} x^i x^j$$

$A = (A_{ij})$ is a positive definite matrix.

Feynman diagram expansion

$$\int_{\mathbb{R}^n} \prod_{i=1}^{n} \frac{dx^i}{\sqrt{2\pi}} e^{-\frac{1}{2}A(x)+I(x)} \sim \frac{1}{\sqrt{\det(A)}} \exp\left(\sum_{\Gamma:\text{conn}} \frac{W_\Gamma}{|Aut(\Gamma)|} \right)$$

W_Γ :



Combinatorial formula via the inverse matrix A^{-1} and I.

A^{-1} : propagator

In quantum field theory, we can use Feynman's formula to model the ∞-dim integral asymptotically.

Example (ϕ^4-theory)

$$\int_{\mathcal{E}=C^\infty(X)} [D\phi]\ e^{-\frac{1}{\hbar}S[\phi]}, \qquad S[\phi] = \frac{1}{2}\int_X \phi\Delta\phi + \lambda \int_X \phi^4.$$

where Δ is the Laplacian operator. The inverse Δ^{-1} is

$$\text{Green's function} \qquad G(x,y) \sim \Delta^{-1}$$

The index i, j is replaced by points x, y on X.

The Green's function is singular along the diagonal

$$G(x, y) \sim \frac{1}{|x-y|^{d-2}}, \qquad x \to y.$$

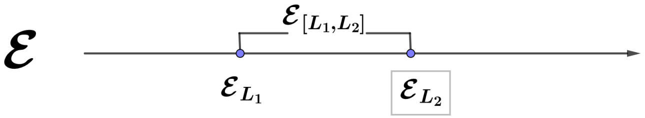
In Feynman diagrams, we will encounter integrals where we multiply many G's together. They are divergent in general!

This is called the UV divergence in QFT, due to the nature of ∞-many degrees of freedom.

Renormalization group flow

Basic idea of renormalization (we use Wilson's viewpoint): we set a scale and cut the full degrees of freedom

$$\mathcal{E} = \bigcup \mathcal{E}_L, \qquad \mathcal{E}_{L_2} = \mathcal{E}_{L_1} \oplus \mathcal{E}_{[L_1,L_2]}.$$



On each $\mathcal{E}_L$, we have an effective action S_L. They are related by

$$e^{\frac{i}{\hbar}S_{L_1}} = \int_{\mathcal{E}_{[L_1,L_2]}} e^{\frac{i}{\hbar}S_{L_2}}.$$

Renormalization group flow.

There are many ways we can cut:

- Momentum cut
- Distance cut
- Energy/ Eigenvalue cut
- …

To construct such S_L, we can have

- Scale dependence of the coupling constants
- Running under renormalization group flow
- Renormalizable theories: perturbation computation
- …

Some examples of renormalization method in QFT

Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) approach

- A scheme for subtracting UV divergence in Feynman integral
- Locality of subtractions (divergent counter-terms).
- Normalization conditions (finite counter-terms).

Connes-Kreimer: BPHZ Renormalization as a Birkhoff decomposition over the dual Hopf algebra of Feynman graphs.

Costello: Homotopic renormalization in perturbative BV formalism. Basic idea: homological interpretation of integral

$$\int \quad \Longrightarrow \quad \text{Homology}$$

Renormalization group flow: chain homotopy (in BV formalism).

$$e^{\frac{i}{\hbar} S_{L_1}} = \int_{\mathcal{E}_{[L_1, L_2]}} e^{\frac{i}{\hbar} S_{L_2}}.$$

2. Observables and Factorization algebras

Why QFT has rich structures?

$$\text{Spacetime} : X \quad \Longrightarrow \quad \text{Fields} : \mathcal{E} = \Gamma(X, E).$$

- $\mathcal{E}$ is the space (called fields) where we will do calculus $\int_{\mathcal{E}}$.
- Topology of X leads to new structures in ∞-dim geometry

When $X =$point, $\mathcal{E} = \mathbb{R}^n$. We arrive at the usual calculus.

$$\boxed{\text{Calculus} = \text{0-dim QFT}}.$$

When $\dim X > 0$, the geometry and topology of X come in!

One algebraic structure associated to the topology of X is

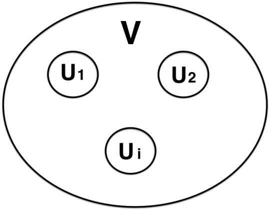
observables=functions on fields

Given an open subset $U \subset X$, we can talk about

$Obs(U) =$ observables supported in U

Example: δ-function.

Observables form an algebraic structure as follows: given disjoint open subset U_i contained in an open V: $\coprod_i U_i \subset V$



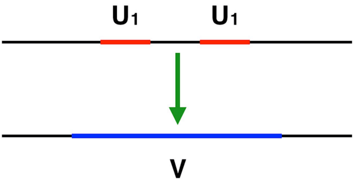
we have a factorization product for observables

$$\bigotimes_i Obs(U_i) \to Obs(V).$$

- Physics: OPE (operator product expansion)
- Mathematics: factorization algebra.
  - Origin: **Beilinson-Drinfeld** in 2d CFT
  - **Costello-Gwilliam**: (perturbative renormalized) QFT.

Example: $\dim X = 1$ (topological quantum mechanics)

QFT in $\dim = 1$ is quantum mechanics.



In the topological case, for any contractible open U, $Obs(U) = A$.
The factorization product doesn't depend on the location and size:

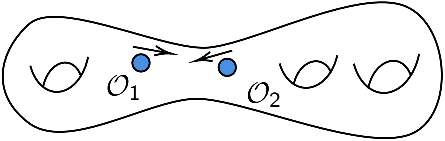
$$A \otimes A \to A.$$

We find an (homotopy) associative algebra.

Example: $\dim X = 2$ (chiral conformal field theory)

The factorization product of 2d chiral theory is holomorphic.

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_n \frac{\mathcal{O}_{1(n)}\mathcal{O}_2(w)}{(z-w)^{n+1}}$$



which is the 2d analogue of "associative product". We find ∞-many binary operations $\mathcal{O}_{1(n)} \cdot \mathcal{O}_2$!

In this case, observable algebra forms a vertex algebra.

An important class of quantities are correlation functions of observables. They capture "global" information of the theory.

- Local correlation

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_i(x_i) \cdots \mathcal{O}_n(x_n) \rangle , \quad x_i \in X.$$

  It is singular when points collide, hence a function on

$$\mathrm{Conf}_n(X) := \{x_1, \cdots, x_n \in X | x_i \neq x_j \text{ for} i \neq j\}.$$

- Non-local correlation

$$\int_{\mathcal{Z} \subset \mathrm{Conf}_n(X)} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_i(x_i) \cdots \mathcal{O}_n(x_n) \rangle$$

  which might be divergent and require further renormalization.

3. Examples: a first geometric look

Example: abelian Chern-Simons and Linking

We consider the abelian Chern-Simons theory on S^3.

$$CS[A] = \frac{1}{2}\int_{S^3} A \wedge dA, \qquad A: \text{ 1-form on } S^3$$

Let C, C' be two disjoint circles inside S^3. Consider

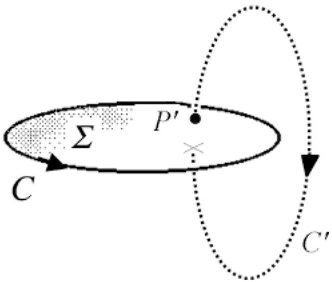
$$\left\langle \int_C A \int_{C'} A \right\rangle = \int [DA] e^{iCS[A]} \left(\int_C A\right)\left(\int_{C'} A\right)$$

The propagator is d^{-1}.

In a suitable interpretation (gauge), this correlation function is

$$\int_{C'} d^{-1}([C]) = \text{fill } C \text{ by a disk } \Sigma \text{ and intersect with } C'$$

$$= \text{Linking number of } C \text{ and } C'.$$



Example: Iterated integral and quantum mechanics

Let $LX = \mathrm{Map}(S^1, X)$ free loop space of X. Consider

$$\begin{array}{ccc} \mathrm{Conf}_n(S^1) \times LX & \xrightarrow{ev} & X^n \\ \downarrow{\scriptstyle \pi} & & \\ LX & & \end{array}$$

where ev sends $(p_1, \cdots, p_n) \times \gamma \to (\gamma(p_1), \cdots, \gamma(p_n))$. Then

$$\pi_* ev^* = \int_{\mathrm{Conf}_n(S^1)} ev^*(-) : (\Omega(X))^{\otimes n} \to \Omega(LX)$$

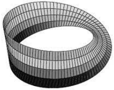
defines a quasi-isomorphism [**K.T. Chen**]

$$\mathrm{Hochschild}(\Omega(X)) \quad \to \quad \Omega(LX).$$

This can be viewed as correlation functions in quantum mechanical model. It will lead to Index theorem as we will show.

Example: σ-model and Geometry enhanced by QFT

One main object in geometry and topology is the vector bundle

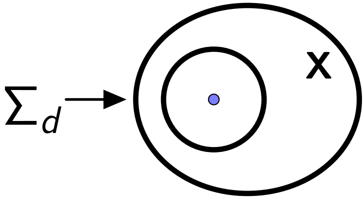


$$\begin{array}{c} E \\ \downarrow \\ X \end{array}$$

This is fibered by $\mathbb{R}^n$, which can be viewed as 0-dim QFT.

$$\begin{array}{c} QFT_0 \\ \downarrow \\ X \end{array}$$

In generel, a QFT of σ-model $\Sigma_d \to X$

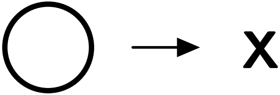


will produce a geometry of

$$\begin{array}{c} QFT_d \\ \downarrow \\ X \end{array}$$

We get a large class of new geometries enhanced by QFT.

Example: topological quantum mechanics



They glue [**Fedosov**] on X to give a bundle of Weyl algebras

$$\begin{array}{c}\mathcal{W}(X)\\ \downarrow\\ X\end{array}$$

- [**Grady-Li-L**, **Gui-L-Xu**]: Quantization of TQM. Correlation function of non-local observables $\int_{\mathrm{Conf}(S^1)}$ gives

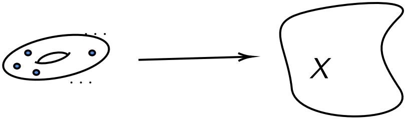
$$\langle 1\rangle = \int_X e^{\omega_\hbar/\hbar}\hat{A}(X).$$

This is the simplest version of algebraic index theorem which was first formulated by **Fedosov** and **Nest-Tsygan** as the algebraic analogue of Atiyah-Singer index theorem.

Example: 2d Chiral CFT

A chiral σ-model

$$\varphi : E \to X$$



will produce a bundle $\mathcal{V}(X)$ of chiral vertex algebras

$$\begin{array}{c}\mathcal{V}(X) \\ \downarrow \\ X\end{array}$$

The quantization/renormalization leads to a flat gluing [**L**].

Correlation function of non-local observables

$$\int_{E^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle.$$

$\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle$ is very singular along diagonals and this integral requires renormalization.

- Geometric renormalization by regularized integrals [**L-Zhou**].
- Elliptic chiral algebraic index.

Algebraic Index vs Elliptic Chiral Index

1d TQM	2d Chiral CFT
Associative algebra	Vertex operator algebra
Hochschild homology	Chiral homology
QME: $(\hbar\Delta + b)\langle - \rangle_{1d} = 0$	QME: $(\hbar\Delta + d_{ch})\langle - \rangle_{2d} = 0$
n-point correlator: $\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{1d}$ =integrals on the compactified configuration spaces of S^1	n-point correlator: $\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d}$ =regularized integrals of singular forms on Σ^n
Algebraic Index theory	Elliptic Chiral Algebraic Index

Joint with **Zhengping Gui**.

Example: $\dim X = 4$ (holomorphic theory)

We consider 4d holomorphic theory on $X = \mathbb{C}^2$. The algebraic structures that lie behind the factorization products will contain

$$\mathrm{H}^{\bullet}_{\bar{\partial}}(\mathbb{C}^2 - \{0\}) = \mathrm{H}^0_{\bar{\partial}} \oplus \mathrm{H}^1_{\bar{\partial}}.$$

By Hartogs's extension theorem, $\mathrm{H}^0_{\bar{\partial}} = \mathbb{C}[z_1, z_2]$ while

$$\mathrm{H}^1_{\bar{\partial}} = \mathbb{C}\left[\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right].$$

So it will predict degree one OPEs indexed by $\mathrm{H}^1_{\bar{\partial}}$.

What are they in physics?

Example: Mirror symmetry

Mirror symmetry is about a duality between

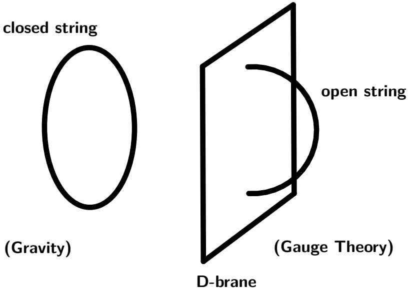
$$\boxed{\text{symplectic geometry}}\ (\text{A-model}) \iff \boxed{\text{complex geometry}}\ (\text{B-model})$$

$$\begin{array}{ccc}
\int_{\mathrm{Map}(\Sigma_g, X)} (\text{A-model}) & \overset{\text{Fourier transform}}{=\!=\!=\!=\!=} & \int_{\mathrm{Map}(\Sigma_g, X')} (\text{B-model}) \\
\big\downarrow \text{localize} & & \text{localize} \big\downarrow \\
\int_{\text{Holomorphic maps}(\Sigma_g, X)} & \leftarrow - - - - \rightarrow & \int_{\text{Constant maps}(\Sigma_g, X')} \ ??? \\
\Big\Downarrow & & \Big\downarrow \\
\text{Gromov-Witten Theory} & & \text{Hodge theory}
\end{array}$$

The B-model can be viewed as a suitable mysterious way to "count constant surfaces", which will be related to the variation of Hodge structures and its quantization [**BCOV**,**Costello-L**].

Example: Gauge/Gravity duality

Gauge theory at large N $\Longrightarrow$ Dynamics of Gravity



[**Costello-L**]: In topological strings: Quantization theory for

- Twisted supergravity (twist by SUSY ghost)
- Open-closed string field theory in the large N.

Lecture 2: Quantization and Index

1. Localization and Index theorem
2. Batalin-Vilkovisky (BV) Quantization formalism
3. Example: Topological Quantum Mechanics (TQM)
4. Example: Chiral deformation of 2d Conformal Field Theory

1: Localization and Index theorem

The Marriage with SUSY

Infinite dim geometry		Finite dim geometry
	$\Longrightarrow$	
QFT		*Math*

Typically, one starts with a path integral in quantum field theory

$$\int_{\mathcal{E}} e^{iS/\hbar}$$

In good situations (e.g. when supersymmetry exists), the ill-defined path integral is localized to a well-defined integral

$$\int_{\mathcal{E}} e^{iS/\hbar} = \int_{\mathcal{M}} (-)$$

over a finite dim $\mathcal{M} \subset \mathcal{E}$. $\mathcal{M}$ is some interesting moduli space.

Example: Topological quantum mechanics (TQM)

TQM leads to a path integral on the loop space

$$\int_{\mathrm{Map}(S^1,X)} e^{-S/\hbar} \quad \overset{\hbar\to 0}{\Longrightarrow} \quad \int_X (\text{curvatures})$$

Topological nature implies the exact semi-classical limit $\hbar \to 0$, which localizes the path integral to constant loops.

- LHS= the analytic index expressed in physics
- RHS= the topological index.

This is the physics “derivation” of Atiyah-Singer Index Theorem.

Example: Witten's "Index Theorem" on loop space

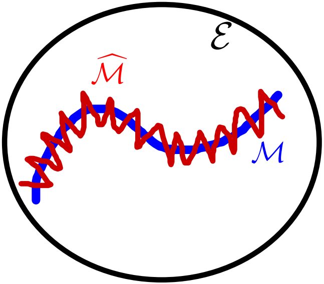
Replace S^1 by an elliptic curve E:

$$\int_{\mathrm{Map}(E,X)} e^{-S/\hbar} \quad \stackrel{\hbar\to 0}{\Longrightarrow}$$

Intuitively, if we view

$$\mathrm{Map}(E,X) = \mathrm{Map}(S^1, LX)$$

as defining a quantum mechanics on LX, then this leads to **Witten**'s proposal for index of dirac operators on loop space.



Let $\widehat{\mathcal{M}}$ be the formal neighborhood of $\mathcal{M}$ inside $\mathcal{E}$. Then intuitively

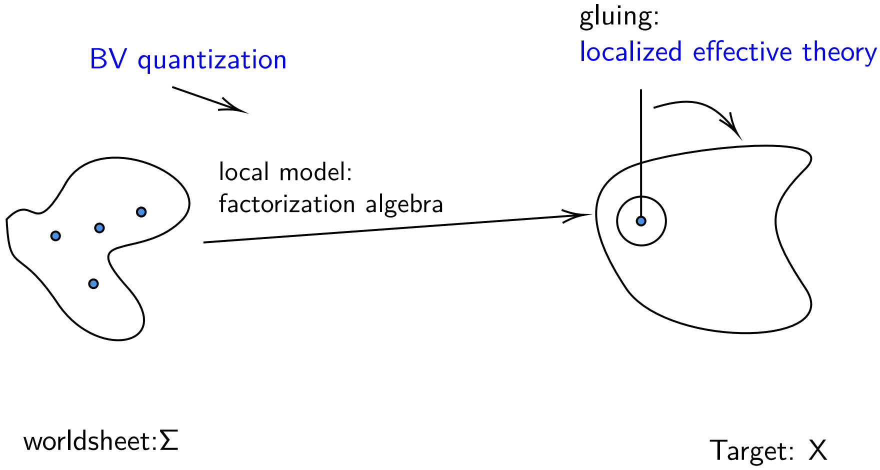
$$\int_{\mathcal{E}} e^{iS/\hbar} = \int_{\widehat{\mathcal{M}}} e^{iS^{eff}/\hbar} = \int_{\mathcal{M}} (-) \,.$$

The pair $(\widehat{\mathcal{M}}, S^{eff})$ will be called the localized effective theory, which usually have an exact geometric description.

Ref: [**Gui-L-Xu**, 2020] *Geometry of Localized Effective Theories, Exact Semi-classical Approximation and the Algebraic Index.*

We will be interested in σ-models about the mapping space

$$\varphi : \Sigma \to X$$



Example: Deformation quantization and algebraic index

Let (X, ω) be a symplectic manifold. $C^\infty(X)$ is a Poisson algebra

$$\{f, g\} = \sum_{i,j} \omega^{ij}(\partial_i f)(\partial_j g).$$

A deformation quantization is defined to be an $\hbar$-linear associative product $\star$ (usually called the star product) on $C^\infty(X)[\![\hbar]\!]$ satisfying

- **Locality**: $\star$ is represented by bi-differential operators
- **Classical limit**: $\forall f, g \in C^\infty(X)$

$$f \star g = fg + O(\hbar)$$

- **1st-order noncommutativity**: $\forall f, g \in C^\infty(X)$

$$\frac{1}{\hbar}(f \star g - g \star f) = \{f, g\} + O(\hbar).$$

Given a deformation quantization, there exists a unique linear map

$$\mathrm{Tr} : C^\infty(X)[\![\hbar]\!] \to \mathbb{R}(\!(\hbar)\!)$$

satisfying

- Trace property: $\mathrm{Tr}(f \star g) = \mathrm{Tr}(g \star f)$
- Normalization condition.

The Algebraic Index Theorem [**Fedosov**, **Nest-Tsygan**] says that

$$\mathrm{Tr}(1) = \int_X e^{\omega_\hbar/\hbar} \hat{A}(X).$$

I will explain by an example how to use topological quantum mechanics to approach such index theorem, making the physics argument into rigorous math realization.

2: Batalin-Vilkovisky (BV) Quantization formalism

Homological methods (such as BRST-BV) arises in physics as a general method to quantize theories with gauge symmetries. We want to emphasize the philosophy

$$\boxed{\text{Integration}} \qquad \Longrightarrow \qquad \boxed{\text{Homology}}$$

$$\boxed{\text{Observable algebra}} \qquad \Longrightarrow \qquad \boxed{\text{Homological algebra}}$$

Calculus $\Longrightarrow$ BRST-BV

Let X be a compact oriented manifold of dimension n. Let $(\Omega^\bullet(X), d)$ be the de Rham complex of smooth differential forms.

$$\int_X : \Omega^\bullet(X) \to \mathbb{R}, \quad \alpha \in \Omega^n(X) \to \int_X \alpha \in \mathbb{R}.$$

Observe that $H^n_{dR}(X) = H^n(\Omega^\bullet(X), d) \simeq \mathbb{R}$. Hence

$$\boxed{\int = H^n_{dR}} : \quad \Omega^n(X) \to H^n_{dR}(X) \simeq \mathbb{R}$$

$$\alpha \to [\alpha].$$

Question: how to take $n \to \infty$ for H^n_{dR}?

BV approach

Let us consider the smooth polyvector fields

$$\mathrm{PV}^\bullet(X) := \Gamma(X, \wedge^\bullet T_X)$$

Let Ω be a fixed volume form on X. It induces identifications

$$\mathrm{PV}^\bullet(X) \xleftrightarrow{\lrcorner\Omega} \Omega^{n-\bullet}(X)$$

$$\Delta \qquad\qquad\qquad d$$

- The induced differential Δ (divergence operator) from the de Rham d is an example of BV operator.
- $\mathrm{PV}(X)$ carries a shifted Poisson structure (Schouten–Nijenhuis bracket). The symbol of Δ is the Poisson kernel.

$$
\begin{array}{ccccccccc}
\Omega^0 & \xrightarrow{d} & \cdots & & \Omega^{n-1} & \xrightarrow{d} & \Omega^n & \xrightarrow{\int = H^n} & \mathbb{R} \\
\updownarrow & & & & \updownarrow & & \updownarrow & \nearrow_{\int_{\mathrm{BV}} = H^0} & \\
\mathrm{PV}^n & \xrightarrow{\Delta} & \cdots & & \mathrm{PV}^1 & \xrightarrow{\Delta} & \mathrm{PV}^0 & &
\end{array}
$$

The BV philosophy of integration is to consider

$$\int_{\mathrm{BV}} : \mathrm{PV}^\bullet(X) \to \mathbb{R}. \qquad \boxed{\int_{BV} = H^0.}$$

Remarks:

- 0 doesn't depend on n! Better for $n \to \infty$ philosophically.
- The challenge is to construct Δ in the ∞-dim setting.

BV algebra

A **Batalin-Vilkovisky** (BV) algebra is a pair $(\mathcal{A}, \Delta)$ where

- $\mathcal{A}$ is a $\mathbb{Z}$-graded commutative associative unital algebra.
- $\Delta : \mathcal{A} \to \mathcal{A}$ is a linear operator of degree 1 such that $\Delta^2 = 0$.
- The **BV bracket** $\{-,-\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ by

$$\{a, b\} := \Delta(ab) - (\Delta a)b - (-1)^{|a|} a \Delta b, \ \ a, b \in \mathcal{A}.$$

It measures the failure of Δ being a derivation.

$\{-,-\}$ satisfies the following graded Leibniz rule

$$\{a, bc\} := \{a, b\}c + (-1)^{(|a|+1)|b|} b\{a, c\}, \quad a, b, c \in \mathcal{A}.$$

Example: polyvector fields

The space of smooth polyvector fields with a divergence operator

$$(\mathrm{PV}^\bullet(X) = \Gamma(X, \wedge^\bullet T_X), \quad \Delta = \text{divergence})$$

is a BV algebra.

Quantum master equation

Let $(C_\bullet, d)$ be a chain complex over $\mathbb{C}[[\hbar]]$. A $\mathbb{C}[[\hbar]]$-linear map

$$\langle - \rangle : C_\bullet \to \mathcal{A}((\hbar))$$

is said to satisfy quantum master equation (QME) if

$$\boxed{(d + \hbar\Delta)\langle - \rangle = 0.}$$

We will usually have a BV integration (a choice of "gauge fixing")

$$\int_{BV} : \mathcal{A} \to \mathbb{C}, \qquad \int_{BV} \Delta(-) = 0.$$

Then $\langle - \rangle$ leads to a chain map

$$\int_{BV} \langle - \rangle : C_\bullet \to \mathbb{C}((\hbar)).$$

Example of QME

Let $(C_\bullet, d) = (\mathbb{C}[[\hbar]], 0)$ and $I = I_0 + I_1\hbar + \cdots \in \mathcal{A}[[\hbar]]$. The $\hbar$-linear map (in a suitable sense)

$$c \to c e^{I/\hbar}, \qquad c \in C_\bullet$$

satisfies QME if and only if $I \in \mathcal{A}[[\hbar]]$ satisfies

$$\boxed{\hbar \Delta I + \frac{1}{2}\{I, I\} = 0}$$

The leading equation (by sending $\hbar \to 0$) is given by

$$\{I_0, I_0\} = 0.$$

This is called the classical master equation.

Example of BV-$\int$: Singularity theory

Let $f(z^i)$ be a polynomial in n variables with an isolated critical point at the origin. We consider $(\mathcal{A}, \Delta)$ where

- $\mathcal{A} = \mathbb{C}[z^i, \theta_i]$, where $\theta_i\theta_j = -\theta_j\theta_i$ are anticommuting.
- $\Delta = \sum_i \frac{\partial}{\partial z^i}\frac{\partial}{\partial \theta_i}$.
- $f(z)$ gives a solution of QME in $\mathcal{A}[[\hbar]]$. $\Delta f = \{f, f\} = 0$.
- BV integration models the oscillatory integral

$$\int_{BV} \langle \mathcal{O} \rangle = \int_{\mathcal{L}} \mathcal{O} e^{f/\hbar}.$$

- $\hbar$ is related to Hodge filtration.

QFT can be viewed as a ∞-dim analogue of Hodge theory.

BV formalism

Roughly speaking, BV quantization in QFT leads to

- Factorization algebra Obs of observables. [**Costello-Gwilliam**]
- $(C_\bullet(\mathrm{Obs}), d)$: a chain complex via algebraic structures of Obs.
- A BV algebra $(\mathcal{A}, \Delta)$ with a BV-$\int$ map $\int_{BV} : \mathcal{A} \to \mathbb{C}$.
- A $\mathbb{C}[[\hbar]]$-linear map (correlation function),

$$\langle - \rangle : C_\bullet(\mathrm{Obs}) \to \mathcal{A}((\hbar))$$

  satisfies QME , which means it is a chain map

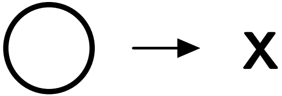
$$(d + \hbar\Delta)\langle - \rangle = 0.$$

- Partition function: $\mathrm{Index} = \int_{BV} \langle 1 \rangle.$

3: Example: Topological Quantum Mechanics (TQM)

One way to formulate TQM is to consider the mapping space

$$\varphi : S^1_{dR} \to (X, \omega).$$



Here (X, ω) is a symplectic manifold. S^1_{dR} is the supermanifold

$$S^1_{dR} = (S^1, \Omega^\bullet_{S^1})$$

with underlying topology S^1 and the structure ring the sheaf of de Rham complex $\Omega^\bullet_{S^1}$.

Local Model

We first study the local model

$$\varphi : S^1_{dR} \to \mathbb{R}^{2n}, \quad \omega = \sum_{i=1}^{n} dp_i \wedge dq^i.$$

Such φ can be represented by $\varphi = (\mathbb{P}_i, \mathbb{Q}^i)$ where $\mathbb{P}_i, \mathbb{Q}^i \in \Omega^\bullet_{S^1}$

$$\mathrm{Map}\left(S^1_{dR}, \mathbb{R}^{2n}\right) = \Omega^\bullet_{S^1} \otimes \mathbb{R}^{2n}.$$

The action is the free one

$$S_{free}[\varphi] = \int_{S^1} \mathbb{P}_i d\mathbb{Q}^i.$$

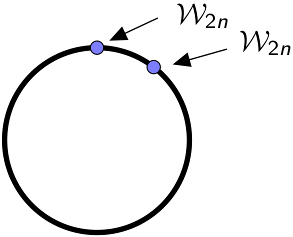
$$\text{Equation of motion} \Longrightarrow d\mathbb{P}_i = d\mathbb{Q}^i = 0.$$

Local observables on S^1 form the Weyl algebra

$$\mathcal{W}_{2n} = \left(\mathbb{C}[\![p_i, q^i]\!][\![\hbar]\!], \star\right)$$

where $\star$ is the Moyal-Weyl product

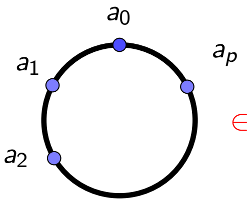
$$(f \star g)(p,q) := f(p,q) e^{\hbar\left(\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i} - \frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}\right)} g(p,q).$$



Hochschild chain complex

Let A be an associative algebra and $\overline{A} := A/\mathbb{C} \cdot 1$. Define

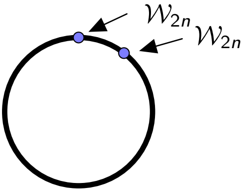
$$C_p(A) := A \otimes \overline{A}^{\otimes p}, \quad \text{cyclic } p\text{-chains}.$$



$\in C_p(A)$

It carries a natural Hochschild differential $b : C_\bullet(A) \to C_{\bullet-1}(A)$

$$b(a_0 \otimes \cdots \otimes a_p)$$
$$= \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_p + (-1)^p a_p a_0 \otimes \cdots \otimes a_{p-1}.$$



- Local observables: Weyl algebra

$$\mathrm{Obs}_{1d} = \mathcal{W}_{2n} = \left(\mathbb{C}[\![p_i, q^i]\!][\![\hbar]\!], \star\right)$$

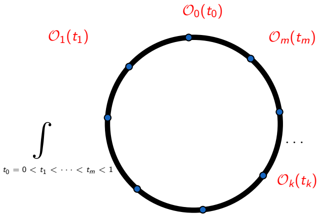
- $(C_\bullet(\mathrm{Obs}_{1d}), b)$ = the Hochschild chain complex.

- BV algebra $(\mathcal{A}_{1d}, \Delta) = (\widehat{\Omega}^\bullet(\mathbb{R}^{2n}), \mathcal{L}_\Pi)$. Here Π = Poisson tensor. In physics, this describes the geometry of zero modes.

- $\langle - \rangle_{1d} : C_\bullet(\mathcal{W}_{2n}) \to \mathcal{A}_{1d}((\hbar))$ where

$$\langle \mathcal{O}_0 \otimes \mathcal{O}_1 \cdots \otimes \mathcal{O}_m \rangle_{1d} \qquad \mathcal{O}_i \in \mathcal{W}_{2n}$$

$$= \left\langle \int_{t_0=0<t_1<\cdots<t_m<1} \mathcal{O}_0(\varphi(t_0))\mathcal{O}_1(\varphi(t_1)) \cdots \mathcal{O}_m(\varphi(t_m)) \right\rangle_{free}$$



It satisfies

$$\text{QME} \qquad (b + \hbar\Delta)\langle - \rangle_{1d} = 0$$

Here b is the Hochschild differential.

Ref: [**L-Xu-Gui**, 2020]

This construction can be glued on a symplectic target X

$$\begin{array}{c} W(X) := Fr(X) \times_{Sp_{2n}} \mathcal{W}_{2n} \\ \downarrow \\ X \end{array}$$

which carries a flat connection (Fedosov connection)

$$D = d + \frac{1}{\hbar}[\gamma, -]_\star, \quad D^2 = 0.$$

Here $\gamma \in \Omega^1(X, W(X))$. Fedosov connection is the geometric interpretation of quantum master equation [**Grady-Li-L** 2017].

$\langle - \rangle_{1d}$ leads to a trace map on deformation quantized algebra, as explicitly described by [**Feigin-Felder-Shoikhet**, 2003].

We can develop the method of exact semi-classical approximation in BV formalism (**Grady-Li-L** 2017; **Gui-L-Xu** 2020) to compute

$$\mathrm{Index} = \mathrm{Tr}(1) = \int_X e^{\omega_\hbar/\hbar}\hat{A}(X).$$

This proves the algebraic index theorem [**Fedosov**, **Nest-Tsygan**].

4: Example: Chiral deformation of 2d Conformal Field Theory

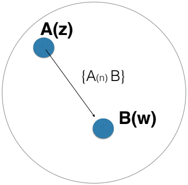
Vertex operator algebras

A *vertex algebra* is a vector space $\mathcal{V}$ with the structure of state-field correspondence (and other axioms like vacuum, locality, etc.)

$$\mathcal{V} \to End(\mathcal{V})[[z, z^{-1}]]$$
$$A \to A(z) = \sum_n A_{(n)} z^{-n-1}$$

We ofter write $Y(A, z)$ for $A(z)$ for the corresponding operator.
It defines the operator product expansion (OPE)

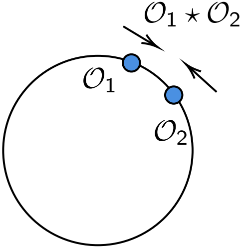
$$A(z)B(w) = \sum_{n \in \mathbb{Z}} \frac{(A_{(n)} \cdot B)(w)}{(z-w)^{n+1}}$$



Free CFT's give rise to examples of vertex algebras $\mathcal{V}$.

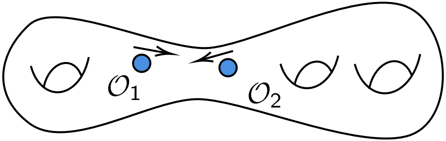
1d TQM	2d Chiral CFT
S^1	Σ
Associative algebra	Vertex operator algebra

Associative product



Operator product expansion

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_n \frac{\mathcal{O}_{1(n)}\mathcal{O}_2(w)}{(z-w)^{n+1}}$$



A chiral σ-model

$$\varphi : \Sigma \to X$$

will produce a bundle $\mathcal{V}(X)$ of chiral vertex operator algebras

$$\begin{array}{c} \mathcal{V}(X) \\ \downarrow \\ X \end{array}$$

This is the chiral analogue of Weyl bundle in TQM.

Theorem (L, 2016)

The quantization of the 2d chiral model is equivalent to solving a flat connection on the vertex algebra bundle $\mathcal{V}(X)$

$$D = d + \frac{1}{\hbar}\left[\oint \mathcal{L}, -\right], \quad D^2 = 0$$

where $\mathcal{L} \in \Omega^1(X, \mathcal{V}(X))$ and $\oint \mathcal{L}$ is the associated chiral vertex operator fiberwise.

- This is the chiral analogue of Fedosov connection.
- The quantization is formulated in the BV formalism.
- BRST reduction of chiral models falls into this setup

$$\oint \mathcal{L} = \text{BRST operator.}$$

Ref: [**L**, 2016] Vertex algebras and quantum master equation.