INTRODUCTION TO ALGEBRAIC TOPOLOGY
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CONTENTS

1 CATEGORY AND FUNCTOR

Category

In category theory, we will encounter many presentations in terms of diagrams. Roughly speaking, a **diagram** is a collection of 'objects' denoted by $A, B, C, X, Y, \cdots$, and 'arrows' between them denoted by $f, g, \cdots$, as in the examples

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, "g_1"'] & Y \arrow[d, "f_2"] \\
Z \arrow[r, "g_2"] & W
\end{tikzcd}
$$

We will always have an operation $\circ$ to compose arrows. The diagram is called **commutative** if all the composite paths between two objects ultimately compose to give the same arrow. For the above examples, they are commutative if

$$h = g \circ f \qquad f_2 \circ f_1 = g_2 \circ g_1.$$

Definition 1.1. A **category** $\mathcal{C}$ consists of

1°. A class of **objects**: $\mathrm{Obj}(\mathcal{C})$ (a category is called *small* if its objects form a set).
 We will write both $A \in \mathrm{Obj}(\mathcal{C})$ and $A \in \mathcal{C}$ for an object A in $\mathcal{C}$.

2°. A set of **morphisms**: $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for each $A, B \in \mathrm{Obj}(\mathcal{C})$. An element $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ will be called a morphism from X to Y, and denoted by

$$A \xrightarrow{f} B \quad \text{or} \quad f : A \to B.$$

 When $\mathcal{C}$ is clear from the context, we will simply write $\mathrm{Hom}(A, B)$ for $\mathrm{Hom}_{\mathcal{C}}(A, B)$.

3°. A **composition** operation $\circ$ between morphisms

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C), \quad \text{for each} \quad A, B, C \in \mathrm{Obj}(\mathcal{C})$$
$$f \times g \to g \circ f,$$

which will be denoted in terms of a diagram by

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}
$$

These are subject to the following axioms:

1°. **Associativity**: $h \circ (g \circ f) = (h \circ g) \circ f$ holds, and will be denoted by $h \circ g \circ f$ without ambiguity. This property can be expressed in terms of the following commutative diagram

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"'] \arrow[dr, "h\circ g"] & \\
& C \arrow[r, "h"] & {}
\end{tikzcd}
$$

2°. **Identity**: for each $A \in \mathrm{Obj}(\mathcal{C})$, there exists $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ called the identity element such that

$$f \circ 1_A = f = 1_B \circ f, \quad \forall A \xrightarrow{f} B,$$

i.e. we have the following commutative diagrams

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[dr, "f"'] & A \arrow[d, "f"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f"'] & B \arrow[d, "1_B"] \\ & B \end{tikzcd}$$

Definition 1.2. A **subcategory** $\mathcal{C}'$ of $\mathcal{C}$ (denoted by $\mathcal{C}' \subset \mathcal{C}$) is a category such that

1°. $\mathrm{Obj}(\mathcal{C}') \subset \mathrm{Obj}(\mathcal{C})$
2°. $\mathrm{Hom}_{\mathcal{C}'}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B), \quad \forall A, B \in \mathrm{Obj}(\mathcal{C}')$
3°. compositions in $\mathcal{C}'$ coincide with that in $\mathcal{C}$ under the above inclusion.

$\mathcal{C}'$ is called a **full subcategory** of $\mathcal{C}$ if $\mathrm{Hom}_{\mathcal{C}'}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B), \quad \forall A, B \in \mathrm{Obj}(\mathcal{C}')$.

Definition 1.3. A morphism $f\colon A \to B$ is called **an isomorphism** (or **invertible**) if there exists $g\colon B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$, i.e. we have the following commutative diagram

$$\begin{tikzcd} A \arrow[loop left, "1_A"] \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \arrow[loop right, "1_B"] \end{tikzcd}$$

Two objects A, B are called **isomorphic** if there exists an isomorphism $f : A \to B$.

Example 1.4. We will frequently use the following categories.

1°. $\mathcal{C} =$ **Set**, the category of sets:

$$\mathrm{Obj}(\mathcal{C}) = \{\text{set}\}, \quad \mathrm{Hom}_{\mathcal{C}}(A, B) = \{\text{set map } A \to B\}.$$

2°. $\mathcal{C} =$ **Vect**$_k$, the category of vector spaces over a field k:

$$\mathrm{Obj}(\mathcal{C}) = \{\text{k-vector space}\}, \quad \mathrm{Hom}_{\mathcal{C}}(A, B) = \{\text{k-linear map } A \to B\}.$$

Vect$_k$ is a subcategory of **Set**, but not a full subcategory.

3°. $\mathcal{C} =$ **Group**, the category of groups:

$$\mathrm{Obj}(\mathcal{C}) = \{\text{group}\}, \quad \mathrm{Hom}_{\mathcal{C}}(A, B) = \{\text{group homomorphism } A \to B\}.$$

It has a full subcategory

Ab, the category of abelian groups.

4°. $\mathcal{C} =$ **Ring**, the category of rings:

$$\mathrm{Obj}(\mathcal{C}) = \{\text{ring}\}, \quad \mathrm{Hom}_{\mathcal{C}}(A, B) = \{\text{ring homomorphism } A \to B\}.$$

Ring is a subcategory of **Ab**, but not a full subcategory. **Ring** has a full subcategory

CRing, the category of commutative rings.

The main object of our interest is

$$\boxed{\underline{\mathbf{Top}} := \text{the category of topological spaces}}$$

- whose objects are topological spaces and
- whose morphisms $f : X \to Y$ are continuous maps.

Example 1.5. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We can construct a new category $\mathcal{C} \times \mathcal{D}$, called the **product** of $\mathcal{C}$ and $\mathcal{D}$, as follows.

- An object of $\mathcal{C} \times \mathcal{D}$ is a pair (X, Y) of objects $X \in \mathcal{C}$ and $Y \in D$.
- A morphism $(f, g) : (X_1, Y_1) \to (X_2, Y_2)$ is a pair of $f \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2), g \in \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_2)$.
- Compositions are componentwise.

Quotient category and homotopy

Definition 1.6. Let $\mathcal{C}$ be a category. Let $\simeq$ be an equivalence relation defined on each $\mathrm{Hom}_{\mathcal{C}}(A, B), A, B \in \mathrm{Obj}(\mathcal{C})$ and compatible with the composition in the following sense

$$f_1 \simeq f_2, \quad g_1 \simeq g_2 \Longrightarrow g_1 \circ f_1 \simeq g_2 \circ f_2.$$

The compatibility can be represented by the following diagram

$$\begin{tikzcd} A \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\simeq"] & B \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\simeq"] & C \arrow[r, Rightarrow] & A \arrow[r, bend left, "g_1 \circ f_1"] \arrow[r, bend right, "g_2 \circ f_2"'] \arrow[r, phantom, "\simeq"] & C \end{tikzcd}$$

We say $\simeq$ defines an equivalence relation on $\mathcal{C}$. The **quotient category** $\mathcal{C}' = \mathcal{C}/\simeq$ is defined by

- $\mathrm{Obj}(\mathcal{C}') = \mathrm{Obj}(\mathcal{C}')$
- $\mathrm{Hom}_{\mathcal{C}'}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)/\simeq, \quad \forall A, B \in \mathrm{Obj}(\mathcal{C}')$.

Exercise 1.7. Check the definition above is well-defined.

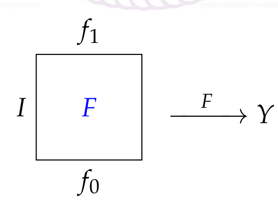
One of the most important equivalence relations in algebraic topology is the homotopy relation.

Let $I = [0, 1]$. Let $X \times Y$ denote the topological product of $X, Y \in \underline{\mathbf{Top}}$.

Definition 1.8. Two morphisms $f_0, f_1 : X \to Y$ in $\underline{\mathbf{Top}}$ are said to be **homotopic**, denoted by $f_0 \simeq f_1$, if

$$\exists F : X \times I \to Y \text{ such that } F|_{X\times\{0\}} = f_0 \text{ and } F|_{X\times\{1\}} = f_1.$$

We will also write $F : f_0 \simeq f_1$ or $f_0 \overset{F}{\simeq} f_1$ to specify the homotopy F. This can be illustrated as



Let $f \colon X \to Y$ be a morphism in $\underline{\mathbf{Top}}$. We define its homotopy class

$$[f] \colon \{g \in \mathrm{Hom}(X, Y) \mid g \simeq f\}.$$

We denote

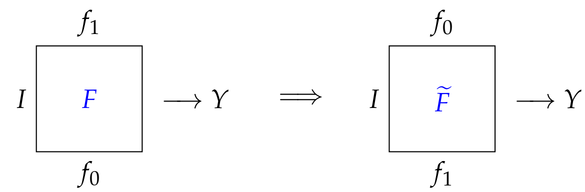
$$[X, Y] \colon = \mathrm{Hom}(X, Y)/\simeq.$$

Theorem 1.9. *Homotopy defines an equivalence relation on* $\underline{\mathbf{Top}}$.

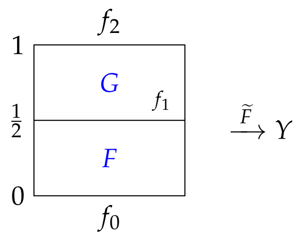
Proof. We first check that $\simeq$ defines an equivalence relation on morphisms.

- Reflexivity: Take F such that $F|_{X\times t} = f$ for any $t \in I$.

- Symmetry: Assume we have a homotopy $F : f_0 \simeq f_1$. Then reversing I as



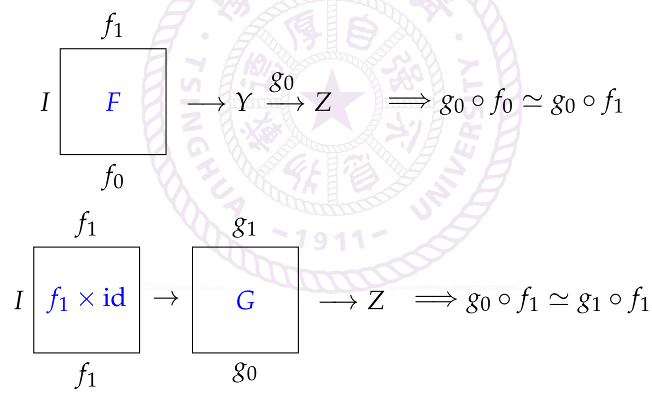
  i.e. taking $\widetilde{F}(x,t) = F(x, 1-t) \colon X \times I \to Y$, gives $f_1 \simeq f_0$ as required.
- Transitivity: Assume we have two homotopies $F : f_0 \simeq f_1$ and $G : f_1 \simeq f_2$, then putting them together gives $\widetilde{F} \colon f_0 \simeq f_2$ as



$$\widetilde{F}(x,t) = \begin{cases} G(x, 2t-1), & 1/2 \le t \le 1; \\ F(x, 2t), & 0 \le t \le 1/2. \end{cases}$$

We next check $\simeq$ is compatible with compositions.

Let $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$. Assume $f_0 \overset{F}{\simeq} f_1$ and $g_0 \overset{G}{\simeq} g_1$. Then



By transitivity, we have proved the compatibility $g_0 \circ f_0 \simeq g_0 \circ f_1 \simeq g_1 \circ f_1$. □

We denote the quotient category of $\underline{\mathbf{Top}}$ under homotopy relation $\simeq$ by

$$\underline{\mathrm{h}\mathbf{Top}} = \underline{\mathbf{Top}} / \simeq$$

with morphisms $\mathrm{Hom}_{\underline{\mathrm{h}\mathbf{Top}}}(X, Y) = [X, Y]$.

Definition 1.10. Two topological spaces X, Y are said to have the **same homotopy type** (or homotopy equivalent) if they are isomorphic in $\underline{\mathrm{h}\mathbf{Top}}$.

Example 1.11. $\mathbb{R}$ and $\mathbb{R}^2$ are homotopy equivalent, but not homeomorphic. In other words, they are isomorphic in $\underline{\mathrm{h}\mathbf{Top}}$, but not isomorphic in $\underline{\mathbf{Top}}$. As we will see, $\mathbb{R}^1$ and S^1 are not homotopy equivalent.

There is also a relative version of homotopy as follows.

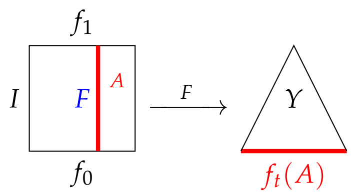
Definition 1.12. Let $A \subset X \in \underline{\textbf{Top}}$ and $f_0, f_1 : X \to Y$ such that $f_0|_A = f_1|_A : A \to Y$. We say f_0 is homotopic to f_1 relative to A, denoted by

$$f_0 \simeq f_1 \ \text{rel}\, A$$

if there exists $F : X \times I \to Y$ such that

$$F|_{X\times\{0\}} = f_0, \quad F|_{X\times\{1\}} = f_1, \quad F|_{A\times t} = f_0|_A, \quad \forall t \in I.$$

We will also write $F : f_0 \simeq f_1 \ \text{rel}\, A$ or $f_0 \overset{F}{\simeq} f_1 \ \text{rel}\, A$ to specify the homotopy F.



Functor

Definition 1.13. Let $\mathcal{C}, \mathcal{D}$ be two categories. A **covariant functor** (resp. **contravariant functor**) $F : \mathcal{C} \to \mathcal{D}$ consists of

1°. $F : \text{Obj}(\mathcal{C}) \to \text{Obj}(\mathcal{D})$, $A \to F(A)$

2°. $\text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{D}}(F(A), F(B)), \forall A, B \in \text{Obj}(\mathcal{C})$. We denote by

$$A \xrightarrow{f} B \Longrightarrow F(A) \xrightarrow{F(f)} F(B)$$

(resp. $\text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{D}}(F(B), F(A)), \quad \forall A, B \in \text{Obj}(\mathcal{C})$, denoted by

$$A \xrightarrow{f} B \Longrightarrow F(B) \xrightarrow{F(f)} F(A). \quad)$$

satisfying

1°. $F(g \circ f) = F(g) \circ F(f)$ (resp. $F(g \circ f) = F(f) \circ F(g)$) for any composable morphisms f, g

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[dr, "F(g)\circ F(f)"'] & F(B) \arrow[d, "F(g)"] \\ & F(C) \end{tikzcd}$$

(resp. reversing all arrows in the diagram on the right).

2°. $F(1_A) = 1_{F(A)}, \quad \forall A \in \text{Obj}(\mathcal{C})$.

F is called **faithful** (or **full**) if $\text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{D}}(F(A), F(B))$ is injective (or surjective) $\forall A, B \in \text{Obj}(\mathcal{C})$.

Example 1.14. The identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ maps

$$1_{\mathcal{C}}(A) = A, \quad 1_{\mathcal{C}}(f) = f$$

for any object A and morphism f.

Example 1.15. $\forall X \in \text{Obj}(\mathcal{C})$,

$$\begin{aligned} \text{Hom}(X, -) : \mathcal{C} &\to \underline{\textbf{Set}}, \\ A &\mapsto \text{Hom}(X, A) \end{aligned}$$

defines a covariant functor and

$$\begin{aligned} \mathrm{Hom}(-,X)\colon \mathcal{C} &\to \underline{\mathbf{Set}}, \\ A &\mapsto \mathrm{Hom}(A,X) \end{aligned}$$

defines a contravariant functor.

Functors of these two types are called **representable** (by X).

Example 1.16. The forgetful functor $\underline{\mathbf{Group}} \to \underline{\mathbf{Set}}$ (mapping a group to its set of group elements) is representable by the free group with one generator.

Example 1.17. Let G be an abelian group. Given $X \in \underline{\mathbf{Top}}$, we will study its n-th cohomology $\mathrm{H}^n(X;G)$ with coefficients in G. It defines a contravariant functor

$$\mathrm{H}^n(-;G) : \underline{\mathrm{h}\mathbf{Top}} \to \underline{\mathbf{Set}}, \quad X \to \mathrm{H}^n(X;G).$$

We will see that this functor is representable by the Eilenberg-Maclane space $K(G,n)$ if we work with the subcategory of CW-complexes.

Example 1.18. We define a contravariant functor

$$\mathrm{Fun} : \underline{\mathbf{Top}} \to \underline{\mathbf{Ring}}, \quad X \to \mathrm{Fun}(X) = \mathrm{Hom}_{\underline{\mathbf{Top}}}(X,\mathbb{R}).$$

$\mathrm{Fun}(X)$ are continuous real functions on X. A classical result of Gelfand-Kolmogoroff says that two compact Hausdorff spaces X, Y are homeomorphic (i.e. isomorphic in $\underline{\mathbf{Top}}$) if and only if $\mathrm{Fun}(X)$ and $\mathrm{Fun}(Y)$ are ring isomorphism (i.e. isomorphic in $\underline{\mathbf{Ring}}$).

Proposition 1.19. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Suppose $f\colon A \to B$ is an isomorphism in $\mathcal{C}$, then $F(f) : F(A) \to F(B)$ is an isomorphism in $\mathcal{D}$.*

Proof. Exercise. □

Natural transformation

Definition 1.20. Let $\mathcal{C}, \mathcal{D}$ be two categories and $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A **natural transformation** $\tau : F \Rightarrow G$ consists of morphisms

$$\tau = \{\tau_A : F(A) \to G(A) | \forall A \in \mathrm{Obj}(\mathcal{C})\}$$

such that the following diagram commutes for any $A, B \in \mathrm{Obj}(\mathcal{C})$ (here $f : A \to B$ if F, G are covariant and $f : B \to A$ if F, G are contravariant)

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\tau_A"'] & F(B) \arrow[d, "\tau_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

τ is called a **natural isomorphism** if τ_A is an isomorphism for any $A \in \mathrm{Obj}(\mathcal{C})$ and we write $F \simeq G$.

Example 1.21. We consider the following two functors

$$\mathrm{GL}_n, (-)^\times : \underline{\mathbf{CRing}} \to \underline{\mathbf{Group}}.$$

Given a commutative ring $R \in \underline{\mathbf{CRing}}$, $\mathrm{GL}_n(R)$ is the group of invertible $n \times n$ matrices with entries in R, and $R^\times$ is the multiplicative group of invertible elements of R. We can identity $(-)^\times = \mathrm{GL}_1$.

The determinant defines a natural transformation

$$\det : \mathrm{GL}_n \to (-)^\times$$

where $\det_R : \mathrm{GL}_n(R) \to R^\times$ is the determinant of the matrix. The naturality of det is rooted in the fact that the formula for determinant is the same for any coefficient ring. In this way, we can say precisely that taking the determinant of a matrix is a natural operation.

Example 1.22. Let $A, B \in \mathcal{C}$ and $f : A \to B$. We have

- A natural transformation
$$f_* : \mathrm{Hom}(-, A) \Rightarrow \mathrm{Hom}(-, B)$$
for (contravariant) representable functors $\mathrm{Hom}(-, A), \mathrm{Hom}(-, B) : \mathcal{C} \to \underline{\mathbf{Set}}$.
- A natural transformation
$$f^* : \mathrm{Hom}(B, -) \Rightarrow \mathrm{Hom}(A, -)$$
for (covariant) representable functors $\mathrm{Hom}(A, -), \mathrm{Hom}(B, -) : \mathcal{C} \to \underline{\mathbf{Set}}$.

Example 1.23. The above example is a special case of the following construction. Let $A \in \mathcal{C}$.

- Let $F : \mathcal{C} \to \underline{\mathbf{Set}}$ be a contravariant functor. Then any $\varphi \in F(A)$ induces a natural transformation
$$\mathrm{Hom}(-, A) \Rightarrow F$$
by assigning $f \in \mathrm{Hom}(B, A)$ to $F(f)(\varphi) \in F(B)$.
- Let $G : \mathcal{C} \to \underline{\mathbf{Set}}$ be a covariant functor. Then any $\varphi \in G(A)$ induces a natural transformation
$$\mathrm{Hom}(A, -) \Rightarrow G$$
by assigning $f \in \mathrm{Hom}(A, B)$ to $G(f)(\varphi) \in G(B)$.

Definition 1.24. Let $F, G, H : \mathcal{C} \to \mathcal{D}$ be functors and $\tau_1 : F \Rightarrow G, \tau_2 : G \Rightarrow H$ be two natural transformations. The composition $\tau_2 \circ \tau_1$ is a natural transformation from F to H defined by

$$(\tau_2 \circ \tau_1)_A : F(A) \xrightarrow{\tau_1} G(A) \xrightarrow{\tau_2} H(A), \quad \forall A \in \mathrm{Obj}(\mathcal{C}).$$

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left=60, "F"] \arrow[rr, "G" description] \arrow[rr, bend right=60, "H"'] & {} \arrow[phantom, u, shift left=0, "\Downarrow\tau_1"] \arrow[phantom, d, "\Downarrow\tau_2"] & D \end{tikzcd} \quad \text{compose to} \quad \begin{tikzcd} \mathcal{C} \arrow[rr, bend left=60, "F"] \arrow[rr, bend right=60, "H"'] \arrow[rr, phantom, "\Downarrow\tau_2\circ\tau_1"] & & D \end{tikzcd}$$

Definition 1.25. Two categories $\mathcal{C}, \mathcal{D}$ are called **isomorphic** if $\exists F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G = 1_\mathcal{D}, G \circ F = 1_\mathcal{C}$. They are called **equivalent** if $\exists F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G \simeq 1_\mathcal{D}, G \circ F \simeq 1_\mathcal{C}$. In this case, we say $F : \mathcal{C} \to \mathcal{D}$ gives an isomorphism/equivalence of categories.

In applications, isomorphism is a too strong condition to impose for most interesting functors. Equivalence is more realistic and equally good essentially. The following proposition is very useful in practice.

Proposition 1.26. *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. Then F is fully faithful.*

Functor Category

Definition 1.27. Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ be a category. We define the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$

- Objects: functors from $\mathcal{C}$ to $\mathcal{D}$

$$F : \mathcal{C} \to \mathcal{D}.$$

- Morphisms: natural transformations between two functors (which is indeed a set since $\mathcal{C}$ is small).

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{D} \end{tikzcd}$$

The following Yoneda Lemma plays a fundamental role in category theory and applications.

Theorem 1.28 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. Denote the two functors*

$$h_A = \mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C} \to \underline{\mathbf{Set}}, \quad h^A = \mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \underline{\mathbf{Set}}.$$

1°. ***Contravariant version****: Let $F : \mathcal{C} \to \underline{\mathbf{Set}}$ be a contravariant functor. Then there is an isomorphism of sets*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \underline{\mathbf{Set}})}(h_A, F) \cong F(A).$$

This isomorphism is functorial in A.

2°. ***Covariant version****: Let $G : \mathcal{C} \to \underline{\mathbf{Set}}$ be a covariant functor. Then there is an isomorphism of sets*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \underline{\mathbf{Set}})}\left(h^A, G\right) \cong G(A).$$

This isomorphism is functorial in A.

The precise meaning of functoriality in A is that we have isomorphisms of functors $\mathcal{C} \to \underline{\mathbf{Set}}$

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \underline{\mathbf{Set}})}\left(h_{(-)}, F\right) \cong F(-), \quad \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \underline{\mathbf{Set}})}\left(h^{(-)}, G\right) \cong G(-).$$

The required isomorphisms in the above Yoneda Lemma are those maps described in Example 1.23.

Duality

Many concepts and statements in category theory have dual descriptions. It is worthwhile to keep eyes on such dualities. Roughly speaking, the dual of a category-theoretical expression is the result of reversing all the arrows for morphisms, changing each reference to a domain to refer to the target (and vice versa), and reversing the order of composition.

For example, let $\mathcal{C}$ for a category. We can define its **opposite category** $\mathcal{C}^{\mathrm{op}}$ by declaring

- $\mathrm{Obj}(\mathcal{C}^{oop}) = \mathrm{Obj}(\mathcal{C})$;
- $f : A \to B$ is a morphism in $\mathcal{C}^{\mathrm{op}}$ if and only if $f : B \to A$ is a morphism in $\mathcal{C}$;
- the composition of two morphisms $g \circ f$ in $\mathcal{C}^{\mathrm{op}}$ is the same as the composite $f \circ g$ in $\mathcal{C}$.

A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is the same as a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. With this help, we can work entirely with covariant functors or contravariant functors. For example, the two statements in Yoneda Lemma are actually the same if we consider opposite categories.

An another example, we will often consider the **lifting problem** by finding a map $F : X \to E$ such that the following diagram is commutative

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "F"] \arrow[r, "f"'] & B \end{tikzcd}$$

The dual problem is the **extension problem** by finding a map G such that the dual diagram is commutative

$$\begin{tikzcd} & E \arrow[dl, dashed, "G"'] \\ X & B \arrow[l, "g"] \arrow[u, "j"'] \end{tikzcd}$$

Adjunction

Let $\mathcal{C}, \mathcal{D}$ be two categories, and let $L : \mathcal{C} \to \mathcal{D}, R : \mathcal{D} \to \mathcal{C}$ be two (covariant) functors. The rules

$$(A, B) \to \mathrm{Hom}_{\mathcal{D}}(L(A), B), \quad (A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, R(B)), \qquad A \in \mathrm{Obj}(\mathcal{C}), B \in \mathrm{Obj}(\mathcal{D})$$

define two functors

$$\mathrm{Hom}_{\mathcal{D}}(L(-), -), \mathrm{Hom}_{\mathcal{C}}(-, R(-)) : \quad \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \underline{\mathbf{Set}}\,.$$

We say L and R are **adjoint** to one another (more precisely, L is the **left adjoint**, R is the **right adjoint**), if there is a natural isomorphism

$$\tau : \mathrm{Hom}_{\mathcal{D}}(L(-), -) \cong \mathrm{Hom}_{\mathcal{C}}(-, R(-));$$

that is, for each $A \in \mathrm{Obj}(\mathcal{C}), B \in \mathrm{Obj}(\mathcal{D})$, we have a set isomorphism

$$\tau_{A,B} : \mathrm{Hom}_{\mathcal{D}}(L(A), B) \cong \mathrm{Hom}_{\mathcal{C}}(A, R(B))$$

and this isomorphism is functorial both in A and in B. We sometimes write adjoint functors as

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R.$$

Example 1.29 (Free vs Forget)**.** Let X be a set, and $F(X) = \bigoplus_{x \in X} \mathbb{Z}$ denote the free abelian group generated by X. This defines a functor

$$F : \underline{\mathbf{Set}} \to \underline{\mathbf{Ab}}, \quad X \to F(X).$$

Forgetting the group structure defines a functor (such functor is often called a forgetful functor)

$$G : \underline{\mathbf{Ab}} \to \underline{\mathbf{Set}}, \quad A \to A.$$

These two functors are adjoint to each other

$$F : \underline{\mathbf{Ab}} \rightleftarrows \underline{\mathbf{Set}} : G.$$

In fact, many "free constructions" in mathematics are left adjoint to certain forgetful functors.

Proposition 1.30. *Let* $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ *be adjoint functors. Then there are natural transformations*

$$1_{\mathcal{C}} \Rightarrow R \circ L \qquad L \circ R \Rightarrow 1_{\mathcal{D}}.$$

Proof. Given $A \in \mathcal{C}$, the required morphism $A \to RL(A)$ corresponds to the identity $1_{L(A)} : L(A) \to L(A)$ under adjoint. The construction of $L \circ R \Rightarrow 1_{\mathcal{D}}$ is similar. $\square$

2 FUNDAMENTAL GROUPOID

Path connected component π_0

Definition 2.1. Let $X \in \underline{\mathbf{Top}}$.

- A map $\gamma : I \to X$ is called a path from $\gamma(0)$ to $\gamma(1)$.
- We denote γ^{-1} be the path from $\gamma(1)$ to $\gamma(0)$ defined by $\gamma^{-1}(t) = \gamma(1-t)$
- We denote $i_{x_0} : I \to X$ be the constant map to $x_0 \in X$.

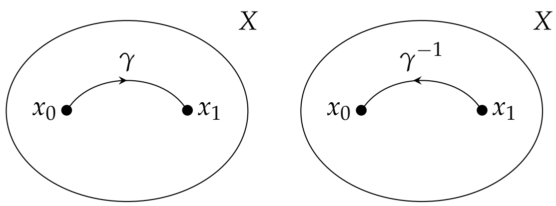


FIGURE 1. A path γ in a topological space X and its inverse

Let us introduce an equivalence relation on X by

$$x_0 \sim x_1 \iff \exists \text{ a path from } x_0 \text{ to } x_1.$$

Remark. Check this is an equivalence relation.

We denote the quotient space

$$\pi_0(X) = X/\sim$$

which is the set of path connected components of X.

Theorem 2.2. $\pi_0 \colon \underline{\mathrm{h}\mathbf{Top}} \to \underline{\mathbf{Set}}$ *defines a covariant functor.*

Proof. Exercise. ☐

Corollary 2.3. *If X, Y are homotopy equivalent, then $\pi_0(X) \cong \pi_0(Y)$.*

Proof. Applying Proposition 1.19 to the functor $\pi_0 : \underline{\mathrm{h}\mathbf{Top}} \to \underline{\mathbf{Set}}$. ☐

Path category / fundamental groupoid

Definition 2.4. Let $\gamma : I \to X$ be a path. We define the path class of γ by

$$[\gamma] = \{\tilde{\gamma} : I \to X | \gamma \simeq \tilde{\gamma} \ \operatorname{rel} \partial I = \{0, 1\}\}.$$

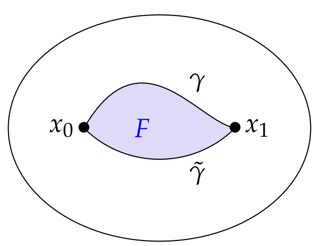


FIGURE 2. In a path class, $F\colon \gamma \simeq \tilde{\gamma} \ \operatorname{rel} \partial I$

$[\gamma]$ is the class of all paths that can be continuously deformed to γ while fixing the endpoints.

Definition 2.5. Let $\gamma_1, \gamma_2 : I \to X$ such that $\gamma_1(1) = \gamma_2(0)$. We define the composite path
$$\gamma_2 \star \gamma_1 : I \to X$$
by
$$\gamma_2 \star \gamma_1(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2 \\ \gamma_2(2t-1) & 1/2 \le t \le 1, \end{cases}$$
cf. Figure 2.5.

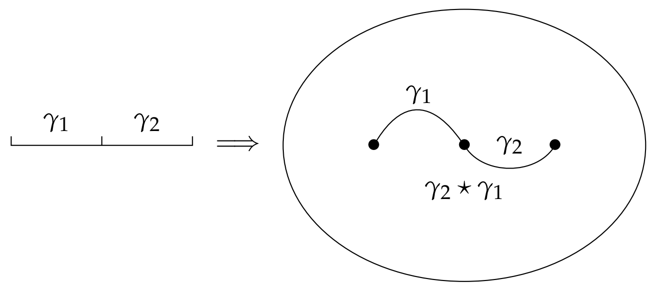
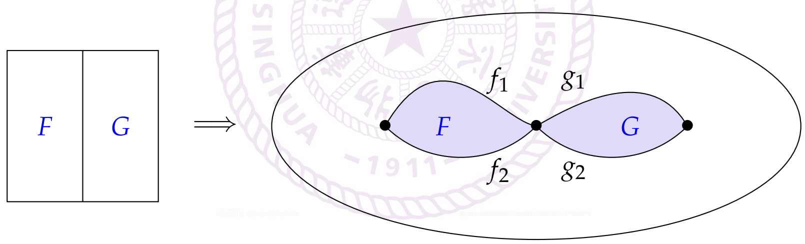


FIGURE 3. Composition of paths

Proposition 2.6. *Let f_1, f_2, g_1, g_2 be paths, such that $f_i(1) = g_i(0)$, $[f_1] = [f_2]$, $[g_1] = [g_2]$. Then*
$$[g_1 \star f_1] = [g_2 \star f_2].$$

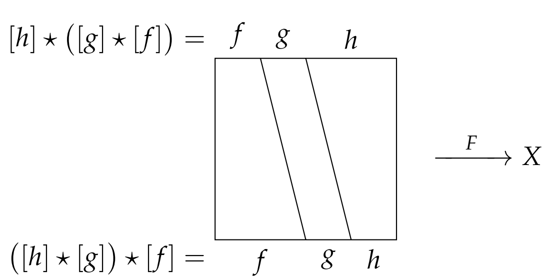
Proof. We illustrate the proof as the following, where $F\colon f_1 \simeq f_2$ and $G\colon g_1 \simeq g_2$. □



We conclude that $\star$ is well-defined for path classes:
$$[g \star f] = [g] \star [f].$$

Proposition 2.7 (Associativity)**.** *Let $f, g, h\colon I \to X$ with $f(1) = g(0)$ and $g(1) = h(0)$. Then*
$$\big([h] \star [g]\big) \star [f] = [h] \star \big([g] \star [f]\big).$$

Proof. We illustrate the proof as follows

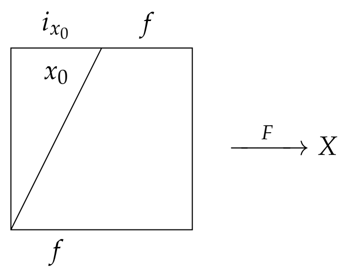


□

Proposition 2.8. *Let $f\colon I \to X$ with endpoints $f(0) = x_0$ and $f(1) = x_1$. Then*

$$[f] \star [i_{x_0}] = [f] = [i_{x_1}] \star [f].$$

Proof. We only show the first equality, which follows from the figure below.



□

Definition 2.9. Let $X \in$ **Top**. We define a category $\Pi_1(X)$ as follows:

- $\mathrm{Obj}(\Pi_1(X)) = X$.
- $\mathrm{Hom}_{\Pi_1(X)}(x_0, x_1)$=path classes from x_0 to x_1.
- $1_{x_0} = i_{x_0}$.

The propositions above imply $\Pi_1(X)$ is a well-defined category. $\Pi_1(X)$ is called the **path category** or **fundamental groupoid** of X.

Groupoid

Definition 2.10. A category where all morphisms are isomorphisms is called a **groupoid**. All groupoids form a category **Groupoid**.

Example 2.11. A group G can be regarded as a groupoid $\underline{G}$ with

- $\mathrm{Obj}(\underline{G}) = \{\star\}$ consists of a single object.
- $\mathrm{Hom}_{\underline{G}}(\star, \star) = G$ and composition is group multiplication.

Thus we have a fully faithful functor **Group** $\to$ **Groupoid**.

Let $\mathcal{C}$ be a groupoid, and define the set

$$\Pi_0(\mathcal{C}) = \mathrm{Obj}(\mathcal{C}) / \sim,$$

where $A \sim B$ if and only if $\exists\, f\colon A \to B$ in $\mathcal{C}$. We can view $\Pi_0(\mathcal{C})$ as a (discrete) category whose objects are its elements with only identity morphisms. Then $\mathcal{C} \to \Pi_0(\mathcal{C})$ is a functor (path connected component). We say $\mathcal{C}$ is **path connected** if $\Pi_0(\mathcal{C})$ is one point.

Lemma 2.12. *X is path connected if and only if $\Pi_0(X)$ is path connected.*

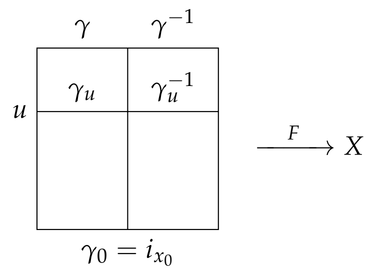
Recall that γ^{-1} is the inverse of γ.

Theorem 2.13. *Let $\gamma\colon I \to X$ with endpoints $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Then*

$$[\gamma] \star [\gamma^{-1}] = [1_{x_1}], \quad \textit{and} \quad [\gamma^{-1}] \star [\gamma] = [1_{x_0}].$$

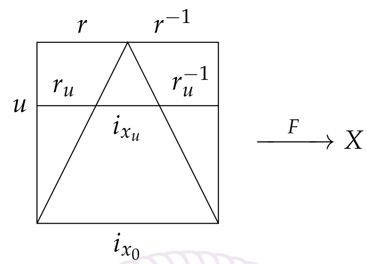
In other words, all morphism in $\Pi_1(X)$ are isomorphisms and thus $\boxed{\Pi_1(X) \textit{ is a groupoid.}}$

Proof. Let $\gamma_u \colon I \to X$ such that $\gamma_u(t) = \gamma(tu)$. The following figure gives the homotopy $\gamma^{-1} \star \gamma \simeq 1_{x_0}$:



$\square$

Exercise 2.14. Use the following figure to give another homotopy $\gamma^{-1} \star \gamma \simeq 1_{x_0}$ for Theorem 2.13.



Definition 2.15. Let $\mathcal{C}$ be a groupoid. Let $A \in \mathrm{Obj}(\mathcal{C})$, we define its automorphism group by

$$\mathrm{Aut}_{\mathcal{C}}(A) := \mathrm{Hom}_{\mathcal{C}}(A, A).$$

Note that this indeed forms a group.

For any $f : A \to B$, it induces a group isomorphism

$$\begin{aligned} \mathrm{Ad}_f : \mathrm{Aut}_{\mathcal{C}}(A) &\to \mathrm{Aut}_{\mathcal{C}}(B) \\ g &\to f \circ g \circ f^{-1}. \end{aligned}$$

Here is a figure to illustrate

$$\mathrm{Ad}_f : \text{maps} \quad g \circlearrowright A \quad \text{to} \quad g \circlearrowright A \underset{f^{-1}}{\overset{f}{\rightleftarrows}} B$$

This naturally defines a functor

$$\mathcal{C} \to \underline{\mathbf{Group}} \quad \text{by assigning} \quad A \mapsto \mathrm{Aut}_{\mathcal{C}}(A), \quad f \mapsto \mathrm{Ad}_f\,.$$

Specialize this to topological spaces, we find a functor

$$\boxed{\Pi_1(X) \to \underline{\mathbf{Group}}}.$$

Definition 2.16. Let $x_0 \in X$, the group

$$\pi_1(X, x_0) := \mathrm{Aut}_{\Pi_1(X)}(x_0)$$

is called the fundamental group of the pointed space (X, x_0).

Theorem 2.17. *Let X be path connected. Then for $x_0, x_1 \in X$, we have a group isomorphism*

$$\pi_1(X, x_0) \cong \pi_1(X, x_1).$$

Proof. Consider the functor $\Pi_1(X) \to \underline{\mathbf{Group}}$ described above. Since X is path connected and $\Pi_1(X)$ is a groupoid, any two points x_0 and x_1 are isomorphic in $\Pi_1(X)$. By Proposition 1.19, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. □

In the path connected case, we will simply denote by $\pi_1(X)$ the fundamental group without mentioning the reference point.

Let $f : X \to Y$ be a continuous map. It defines a functor

$$\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y) \quad \text{by assigning} \quad x \mapsto f(x), \quad [\gamma] \mapsto [f \circ \gamma].$$

Proposition 2.18. Π_1 *defines a functor*

$$\boxed{\Pi_1 : \underline{\mathbf{Top}} \to \underline{\mathbf{Groupoid}}},$$

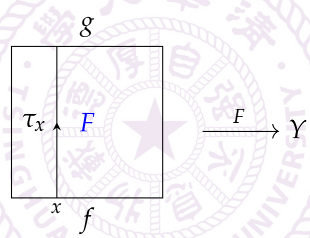
that sends X to $\Pi_1(X)$.

Proof. Exercise. □

Proposition 2.19. *Let $f, g : X \to Y$ be maps which are homotopic by $F : X \times I \to Y$. Let us define path classes*

$$\tau_x = [F|_{x \times I}] \in \mathrm{Hom}_{\Pi_1(Y)}(f(x), g(x)),$$

cf.



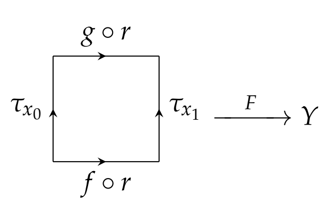
Then τ defines a natural transformation

$$\tau_F : \Pi_1(f) \Longrightarrow \Pi_1(g).$$

Proof. Let $r : I \to X$ with $r(t) = x_t$. We only need to show that the following diagram is commutative at the level of path classes:

$$\begin{tikzcd}
f(x_0) = & \Pi_1(f)(x_0) \arrow[r, "f \circ r"] \arrow[d, "\tau_{x_0}"'] & \Pi_1(f)(x_1) \arrow[d, "\tau_{x_1}"] & = f(x_1) \\
g(x_0) = & \Pi_1(g)(x_0) \arrow[r, "g \circ r"'] & \Pi_1(g)(x_1) & = g(x_1)
\end{tikzcd}$$

The composition $F \circ (r \times I)$ gives the following diagram:



which implies that $[g \circ r] \star [\tau_{x_0}] = [\tau_{x_1}] \star [f \circ r]$ as required. □

This proposition can be pictured by the following diagram

$$\begin{tikzcd} \underline{\mathbf{Top}} \arrow[r, "\Pi_1"] & \underline{\mathbf{Groupoid}} \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d, bend right=70, "f"'] \arrow[d, bend left=70, "g"] \arrow[d, phantom, "\xRightarrow[F]{\text{homotopy}}"] \\ Y \end{tikzcd} \qquad\qquad \begin{tikzcd} \Pi_1(X) \arrow[d, bend right=70, "\Pi_1(f)"'] \arrow[d, bend left=70, "\Pi_1(g)"] \arrow[d, phantom, "\xRightarrow[\tau_F]{\text{nat. trans.}}"] \\ \Pi_1(Y) \end{tikzcd}$$

The following theorem is a formal consequence of the above proposition

Theorem 2.20. *Let $f : X \to Y$ be a homotopy equivalence. Then*

$$\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y)$$

is an equivalence of categories. In particular, it induces a group isomorphism

$$\pi_1(X, x_0) \cong \pi_1(Y, f(x_0)),$$

Proof. Let $g\colon Y \to X$ represents the inverse of f in $\mathrm{h}\underline{\mathbf{Top}}$. Applying Π_1 to $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$, we find $\Pi_1(f) \circ \Pi_1(g)$ and $\Pi_1(g) \circ \Pi_1(f)$ are naturally equivalent to identity functors. Thus the first statement follows. The second statement follows from Proposition 1.26. □

Proposition 2.21. *Let $X, Y \in \underline{\mathbf{Top}}$. Then we have a canonical isomorphism of categories*

$$\Pi_1(X \times Y) \cong \Pi_1(X) \times \Pi_1(Y).$$

In particular, for any $x_0 \in X, y_0 \in Y$, we have a group isomorphism

$$\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Example 2.22. For a point $X = \mathrm{pt}$, $\pi_1(\mathrm{pt}) = 0$ is trivial. It it not hard to see that $\mathbb{R}^n$ is homotopy equivalent to a point. It follows that

$$\pi_1(\mathbb{R}^n) = 0 \quad n \geq 0.$$

Example 2.23. As we will see,

$$\pi_1(S^1) = \mathbb{Z}, \quad \text{and} \quad \pi_1(S^n) = 0, \forall n > 1.$$

Example 2.24. Let $T^n = (S^1)^n$ be the n-dim torus. Then

$$\pi_1(T^n) = \mathbb{Z}^n.$$

Example 2.25 (Braid groups)**.** Artin's braid group Br_n of n strings can be realized as mapping class group (symmetry group) of a disk of n punctures. It has the following finite presentation:

$$\begin{aligned} \mathrm{Br}_n = \langle b_1, \ldots, b_{n-1} \mid \quad & b_i b_j b_i = b_j b_i b_j \quad \forall |j-i| = 1, \\ & b_j b_i = b_i b_j \qquad \forall\ |j-i| > 1 \rangle. \end{aligned}$$

Braid groups can be also realized as fundamental groups.

Let $X \in \underline{\mathbf{Top}}$, the n^{th} (ordered) configuration space of X is the set of n pairwise distinct points in X:

$$\mathrm{Conf}_n(X) \colon = \{\underline{x} = (x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j,\ \forall i \neq j\}.$$

There is a natural action of the permutation group S_n on $\mathrm{Conf}_n(X)$ given by

$$\begin{aligned} S_n \times \mathrm{Conf}_n(X) &\longrightarrow \mathrm{Conf}_n(X) \\ (\sigma, \underline{x}) &\longmapsto \sigma(\underline{x}) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}). \end{aligned}$$

The unordered configuration space of X is the orbit space of this action:

$$\mathrm{UConf}_n(X) = \mathrm{Conf}_n(X)/S_n.$$

A classical result says

$$\mathrm{Br}_n \cong \pi_1(\mathrm{UConf}_n(\mathbb{R}^2)) \cong \pi_1(\mathrm{UConf}_n(D^2)).$$

Moreover, elements in this (fundamental) group can be visulized as braids in $\mathbb{R}^3$ as follows. Fix n distinct points $Z_1, \cdots, Z_n$ in $\mathbb{R}^2$. A geometric braid is an n-tuple $\Psi = (\psi_1, \ldots, \psi_n)$ of paths

$$\psi_i \colon [0,1] \to \mathbb{R}^2 \times I \subset \mathbb{R}^3$$

such that

- $\psi_i(0) = Z_i \times \{0\}$;
- $\psi_i(1) = Z_{\nu(i)} \times \{1\}$ for some permutation ν of $\{1, \ldots, n\}$;
- $\{\psi_1(t), \ldots, \psi_n(t)\}$ are distinct points in $\mathbb{R}^2 \times \{t\}$, for $0 < t < 1$.

The product of geometric braids follows the same way of products of paths (in the fundamental group setting). The isotopy class of all braids on $\mathbb{R}^3$ with the product above form the braid group. See Figure 4.

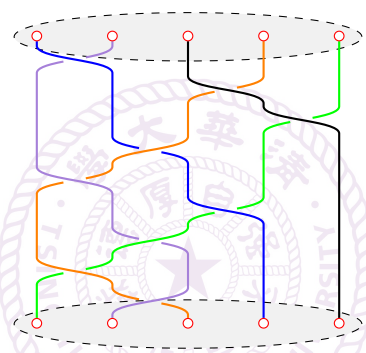


FIGURE 4. Classical braids

3 COVERING AND FIBRATION

Covering and Lifting

Definition 3.1. Let $p : E \to B$ be in $\underline{\mathbf{Top}}$. A *trivialization* of p over an open set $U \subset B$ is a homeomorphism $\varphi : p^{-1}(U) \to U \times F$ over U, i.e. , the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

p is called *locally trivial* if there exists an open cover $\mathcal{U}$ of B such that p has a trivialization over each open $U \in \mathcal{U}$. Such p is called a **fiber bundle**, F is called the **fiber** and B is called the **base**. We denote it by

$$F \to E \to B$$

where there is no ambiguity from the context. If we can find a trivialization of p over the whole B, then E is homeomorphic to $F \times B$ and we say p is a **trivial fiber bundle**.

Example 3.2. The projection map

$$\mathbb{R}^{m+n} \to \mathbb{R}^n, \quad (x_1, \cdots, x_n, \cdots, x_{n+m}) \mapsto (x_1, \cdots, x_n)$$

is a trivial fiber bundle with fiber $\mathbb{R}^m$.

Example 3.3. A real vector bundle of rank n over a manifold is a fiber bundle with fiber $\simeq \mathbb{R}^n$.

Example 3.4. We identify S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$ parametrized by

$$S^{2n+1} = \{z_0, z_1, \cdots, z_n \in \mathbb{C}^{n+1} | |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1\}.$$

There is a natural S^1-action on S^{2n+1} given by

$$e^{i\theta} : (z_0, \cdots, z_n) \mapsto (e^{i\theta} z_0, \cdots, e^{i\theta} z_n), \quad e^{i\theta} \in S^1.$$

This action is free, and the orbit space can be identified with the n-dim complex projective space $\mathbb{CP}^n$

$$S^{2n+1}/S^1 \cong \mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*.$$

Then the projection map $S^{2n+1} \to \mathbb{CP}^n$ is a fiber bundle with fiber S^1. It is a nontrivial fact that they are not trivial fiber bundles. The case when $n = 1$ gives the Hopf fibratioin

$$S^1 \to S^3 \xrightarrow{p} S^2 = \mathbb{CP}^1$$

which is particularly interesting. In this case, the projection sends $(z_0, z_1) \in S^3 \subset \mathbb{C}^2$ to $z_0/z_1 \in S^2 = \mathbb{C} \cup \{\infty\}$. In polar coordinates, we have $z_j = r_j e^{\mathbf{i}\theta_j}$ for $r_0^2 + r_1^2 = 1$ and $p(z_0, z_1) = (r_0/r_1) e^{\mathbf{i}(\theta_0 - \theta_1)}$. For a fix $\rho = r_0/r_1$, we obtain a torus T_ρ in S^3. When identifying S^3 with the compatification of $\mathbb{R}^3$ (or considering the stereographic projection $S^3 \to \mathbb{R}^3$), we have the Figure 5 to visualize the foliation of $\mathbb{R}^3$ by these tori T_ρ, where T_0 degenerates to the unit circle on xy-plane of $\mathbb{R}^3$ and T_∞ degenerates to z-axis. Each S^1-fiber is a slope 1 simple closed curve on one of the tori T_ρ, and the image of the projection is exactly the compatification S^2 of the xy-plane of $\mathbb{R}^3$.

Definition 3.5. A **covering (space)** is a locally trivial map $p : E \to B$ with discrete fiber F (cf. Figure 6). A covering map which is a trivial fiber bundle is also called a **trivial covering**. If we would like to specify the fiber, we call it a F-covering. If the fiber F has n points, we also call it a n**-fold covering**.

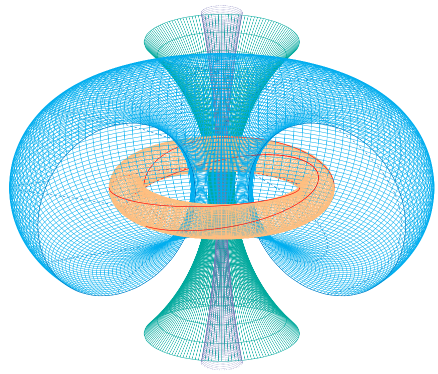


FIGURE 5. A visualization of Hopf fibration

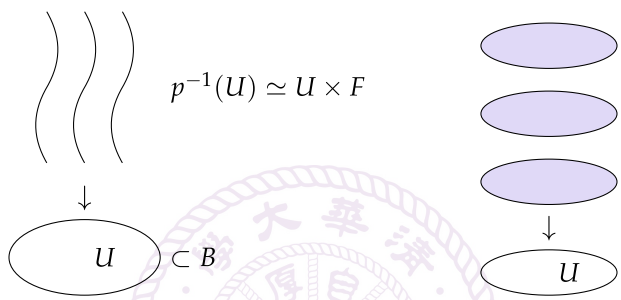


FIGURE 6. Trivialization (left) and covering (right)

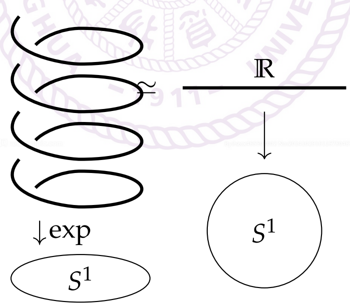


FIGURE 7. The $\mathbb{Z}$-covering of S^1

Example 3.6. The map $\exp : \mathbb{R}^1 \to S^1, \quad t \to e^{2\pi \mathbf{i} t}$ is a $\mathbb{Z}$-covering, cf. Figure 7. If $U = S^1 - \{-1\}$, then

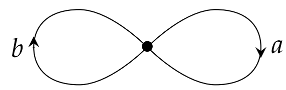
$$\exp^{-1}(U) = \bigsqcup_{n \in \mathbb{Z}} (n - \frac{1}{2}, n + \frac{1}{2}).$$

Example 3.7. The map $S^1 \to S^1, e^{2\pi \mathbf{i}\theta} \mapsto e^{2\pi \mathbf{i} n\theta}$ is an $|n|$-fold covering, for $n \in \mathbb{Z} - \{0\}$.

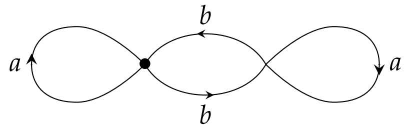
Example 3.8. The map $\mathbb{C} \to \mathbb{C}, z \mapsto z^n$, is not a covering (why?). But

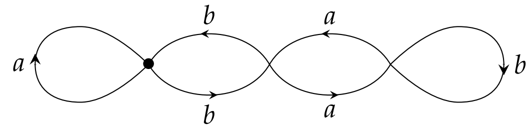
- the map $\mathbb{C}^* \to \mathbb{C}^*, z \mapsto z^n$, is an $|n|$-covering, where $\mathbb{C}^* = \mathbb{C} - \{0\}$and $n \in \mathbb{Z} - \{0\}$.
- the map $\exp : \mathbb{C} \to \mathbb{C}^*, z \mapsto e^{2\pi i z}$ is a $\mathbb{Z}$-covering.

Example 3.9 (From Hatcher)**.** The figure-8



has two coverings as follows (the left is a 2-fold (or double) covering and the right is a 3-fold covering).





The 4-regular tree is its universal cover (a covering which is simply connected), see Figure 8.

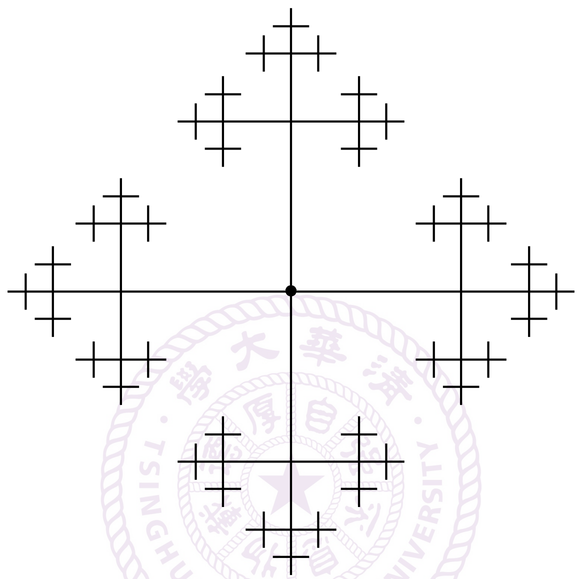


FIGURE 8. 4-regular tree

Example 3.10. Recall that the number of holes (genus) and number of boundary components determine the homeomorphism type of a closed oriented topological surface. Denote by $S_{g,b}$ the surface with genus g and b boundary components.

- The surface $S_{4,0}$ admits a 7-fold covering from $S_{22,0}$, cf. Figure 9.
- In general, $S_{g,b}$ admits a m-fold covering from $S_{mg-m+1,mb}$.

Example 3.11. Denote by $\mathbb{RP}^n$ the real projective space of dimension n, i.e.

$$\mathbb{RP}^n = \mathbb{R}^{n+1} - \{0\}/(\underline{x} \sim t\underline{x}), \quad \forall t \in \mathbb{R} - \{0\}, \underline{x} \in \mathbb{R}^{n+1} - \{0\}.$$

Let S^n be the n-sphere. Then there is a natural double cover $S^n \to \mathbb{RP}^n$.

Example 3.12 (Branched double cover)**.** Figure 10 shows a branched double cover of a disk

$$\iota \colon \Sigma_n \to D^2,$$

branching at n points. Namely, when deleting those n (red) points (denoted by Δ) from both Σ_n and D^2, we obtain a 2-fold covering:

$$\iota \colon \Sigma_n \backslash \Delta \xrightarrow{2:1} D^2 \backslash \Delta$$

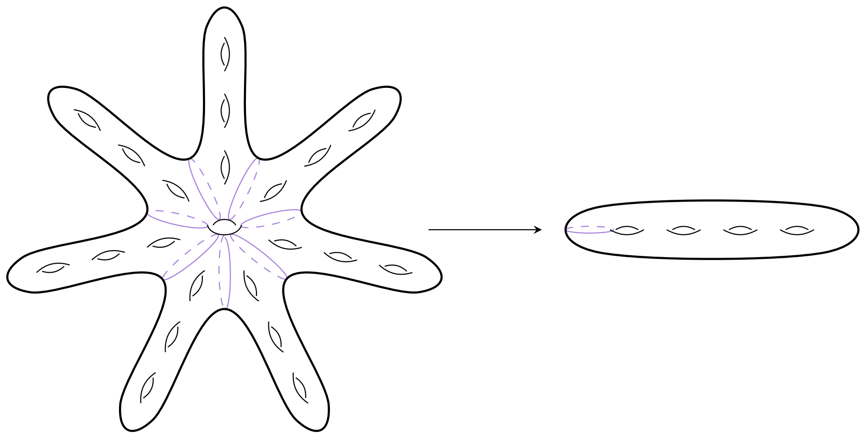


FIGURE 9. A 7-covering

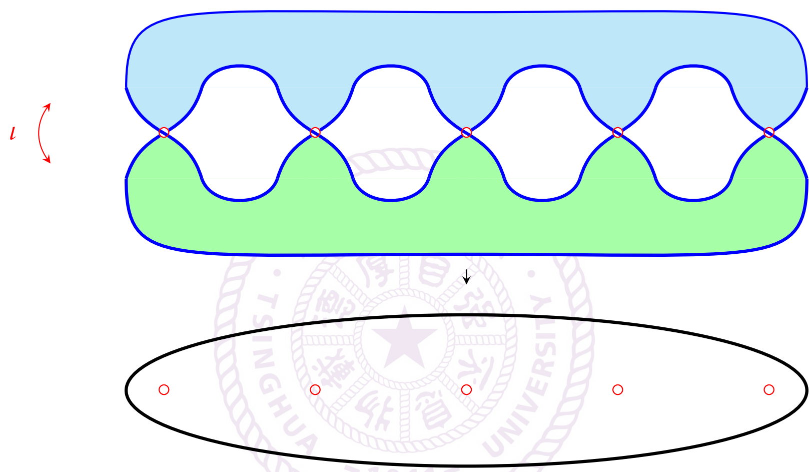


FIGURE 10. The Birman-Hilden double cover via the twisted surface

with a bijection $\iota\colon \Delta \to \Delta$ on Δ. This can be used to show the homeomorphism in Figure **??** (it is a special case of Figure 10 for $n = 3$. In general, it has genus $g = \lfloor \frac{n}{2} \rfloor - 1$ with $b = n - 2g$ boundary components.)

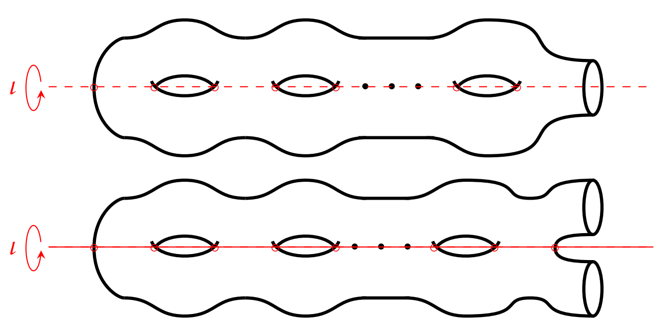


FIGURE 11. The normal view of the branched double cover of the punctured disk

Definition 3.13. Let $p : E \to B, f : X \to B$. A **lifting** of f along p is a map $F : X \to E$ such that $p \circ F = f$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "F"] \arrow[r, "f"'] & B \end{tikzcd}$$

Lemma 3.14. *Let $p : E \to B$ be a covering. Let*

$$D = \{(x, x) \in E \times E | x \in E\}$$
$$Z = \{(x, y) \in E \times E | p(x) = p(y)\}.$$

Then $D \subset Z$ is both open and closed.

Proof. Exercise. □

Theorem 3.15 (Uniqueness of lifting)**.** *Let $p : E \to B$ be a covering. Let $F_0, F_1 : X \to E$ be two liftings of f. Suppose X is connected and F_0, F_1 agree somewhere. Then $F_0 = F_1$.*

Proof. Let D, Z be defined in Lemma 3.14. Consider the map $\tilde{F} = (F_0, F_1) : X \to Z \subset E \times E$. By assumption, we have $\tilde{F}(X) \cap D \neq \emptyset$. Moreover, Lemma 3.14 implies that $\tilde{F}^{-1}(D)$ is both open and closed. Since X is connected, we find $\tilde{F}^{-1}(D) = X$ which is equivalent to $F_0 = F_1$.

$$\begin{tikzcd} X \arrow[dr, "\tilde{F}"] \arrow[drr, bend left, "F_0"] \arrow[ddr, bend right, "F_1"'] & & \\ & Z \arrow[r] \arrow[d] & E \arrow[d] \\ & E \arrow[r] & B \end{tikzcd}$$

□

Fibration

Definition 3.16. A map $p : E \to B$ is said to have the **homotopy lifting property** (HLP) with respect to X if for any maps $\tilde{f} : X \to E$ and $F : X \times I \to B$ such that $p \circ \tilde{f} = F|_{X \times \{0\}}$, there exists a lifting $\tilde{F}$ of F along p such that $\tilde{F}|_{X \times \{0\}} = \tilde{f}$, i.e., the following diagram is commutative

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists \tilde{F}"] & B \end{tikzcd}$$

Definition 3.17. A map $p : E \to B$ is called a **fibration** (or **Hurwitz fibration**) if p has HLP for any space.

Theorem 3.18. $\boxed{\text{\textit{A covering is a fibration}}}$.

Proof. Let $p : E \to B, f : X \to B, \tilde{f} : X \to E, F : X \times I \to B$ be the data as in Definition 3.16. We only need to show the existence of $\tilde{F}_x$ for some neighbourhood N_x of any given point $x \in X$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ N_x \times I \arrow[ur, dashed, "\tilde{F}_x"] \arrow[r, "F"'] & B \end{tikzcd}$$

In fact, for any two such neighbourhoods N_x and N_y with $N_x \cap N_y = N_0 \neq \emptyset$, we have $\tilde{F}_x \mid_{N_0}$ and $\tilde{F}_y \mid_{N_0}$ agree at some point on $\tilde{f} \mid_{N_0}$ and hence agree everywhere in N_0 by the uniqueness of lifting (Theorem 3.15). Thus $\{\tilde{F}_x \mid x \in X\}$ glue to give the required lifting $\tilde{F}$.

Next, we proceed to prove the existence. Since I is compact, given $x \in X$ we can find a neighbourhood N_x and a partition

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

such that p has a trivialization over open sets $U_i \supset F(N_x \times [t_i, t_{i+1}])$. Now we construct the lifting $\tilde{F}_x$ on $N_x \times [t_0, t_k]$, for $1 \leq k \leq m$, by induction on k.

- For $k = 1$, the lifting $\tilde{F}_x$ on $N_x \times [t_0, t_1]$ to one of the sheets of $p^{-1}(U_1)$ is determined by $\tilde{f} \mid_{N_x \times \{0\}}$:

$$\begin{tikzcd} & p^{-1}(U_1) = \bigsqcup_\alpha \widetilde{(U_1)}^\alpha \arrow[d, "p"] \\ N_x \times [t_0, t_1] \arrow[ur, dashed, "\tilde{F}_x"] \arrow[r, "F"'] & U_1 \end{tikzcd}$$

- Assume that we have constructed $\tilde{F}_x$ on $N_x \times [t_0, t_k]$ for some k. Now, the lifting of $\tilde{F}_x$ on $N_x \times [t_k, t_{k+1}]$ to one of the sheets of $p^{-1}(U_k)$ is determined by $\tilde{f} \mid_{N_x \times \{t_k\}}$, which can be glued to the lifting on $N_x \times [t_0, t_k]$ by the uniqueness of lifting again. This finishes the inductive step.

We obtain a lifting $\tilde{F}_x$ of F on $N_X \times I$ as required. □

Corollary 3.19. *Let $p : E \to B$ be a covering. Then for any path $\gamma : I \to B$ and $e \in E$ such that $p(e) = \gamma(0)$, there exists a unique path $\tilde{\gamma} : I \to E$ which lifts γ and $\tilde{\gamma}(0) = e$.*

Proof. Apply HLP to $X = \text{pt}$.

$$\begin{tikzcd} 0 \arrow[r, "e"] \arrow[d, hook] & E \arrow[d, "p"] \\ I \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & B \end{tikzcd}$$

□

Corollary 3.20. *Let $p : E \to B$ be a covering. Then $\Pi_1(E) \to \Pi_1(B)$ is a faithful functor. In particular, the induced map $\pi_1(E, e) \to \pi_1(B, p(e))$ is injective.*

Proof. Let $\tilde{\gamma}_i \colon I \to E$ be two paths and $[\tilde{\gamma}_i] \in \mathrm{Hom}_{\Pi_1(E)}(e_1, e_2)$. Let $\gamma_i = p \circ \tilde{\gamma}_i$. Suppose $[\gamma_1] = [\gamma_2]$ and we need to show that $[\tilde{\gamma}_1] = [\tilde{\gamma}_2]$.

Let $F\colon \gamma_1 \simeq \gamma_2$ be a homotopy. Consider the following commutative diagram with the lifting $\tilde{F}$ by HLP

$$\begin{tikzcd} I \times \{0\} \arrow[r, "\tilde{\gamma}_1"] \arrow[d, hook] & E \arrow[d, "p"] \\ I \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists \tilde{F}"] & B \end{tikzcd}$$

Then the uniqueness of lifting implies $\tilde{F}\mid_{I\times\{1\}} = \tilde{\gamma}_2$. Thus, $\tilde{F}\colon \tilde{\gamma}_1 \simeq \tilde{\gamma}_2$. □

Transport functor

Let $p : E \to B$ be a covering. Let $\gamma : I \to B$ be a path in B from b_1 to b_2. It defines a map

$$T_\gamma : p^{-1}(b_1) \to p^{-1}(b_2)$$
$$e_1 \mapsto \tilde{\gamma}(1)$$

where $\tilde{\gamma}$ is a lift of γ with initial condition $\tilde{\gamma}(0) = e_1$.

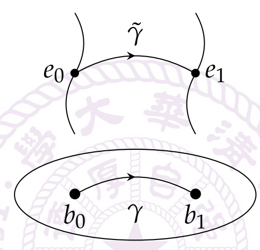


FIGURE 12. The transportation

Assume $[\gamma_1] = [\gamma_2]$ in B. HLP implies that $T_{\gamma_1} = T_{\gamma_2}$. We find a well-defined map:

$$T : \mathrm{Hom}_{\Pi_1(B)}(b_1, b_2) \to \mathrm{Hom}_{\underline{\mathbf{Set}}}(p^{-1}(b_1), p^{-1}(b_2))$$
$$[\gamma] \mapsto T_{[\gamma]}$$

This leads to the following definition (check the functor property!).

Definition 3.21. The following data

$$T : \Pi_1(B) \to \underline{\mathbf{Set}}$$
$$b \to p^{-1}(b)$$
$$[\gamma] \mapsto T_{[\gamma]}.$$

define a functor, called the **transport functor**. In particular, we have a well-defined map

$$\pi_1(B, b) = \mathrm{Aut}_{\Pi_q(B)}(b) \to \mathrm{Aut}_{\underline{\mathbf{Set}}}(p^{-1}(b)).$$

Example 3.22. Consider the covering map

$$\mathbb{Z} \to \mathbb{R}^1 \overset{\exp}{\to} S^1.$$

Consider the following path representing an element of $\pi_1(S^1)$

$$\gamma_n : I \to S^1, \quad t \to \exp(nt) = e^{2\pi int}, \quad n \in \mathbb{Z}.$$

Start with any point $m \in \mathbb{Z}$ in the fiber, γ_n lifts to a map to $\mathbb{R}^1$

$$\tilde{\gamma}_n : I \to R^1, \quad t \mapsto m + nt.$$

We find $T_{[\gamma_n]}(m) = \tilde{\gamma}(1) = m + n$. Therefore $T_{[\gamma_n]} \in \mathrm{Aut}_{\underline{\mathbf{Set}}}(\mathbb{Z})$ is

$$T_{[\gamma_n]} : \mathbb{Z} \to \mathbb{Z}, \quad m \mapsto m + n.$$

Proposition 3.23. *Let $p : E \to B$ be a covering, E be path connected. Let $e \in E, b = p(e) \in B$. Then the action of $\pi_1(B, b)$ on $p^{-1}(b)$ is transitive, whose stabilizer at e is $\pi_1(E, e)$. In other words,*

$$p^{-1}(b) \cong \pi_1(B, b)/\pi_1(E, e)$$

as a coset space, i.e. we have the following ***short exact sequence***

$$\begin{aligned} 1 \to \pi_1(E, e) \to \pi_1(B, b) &\xrightarrow{\partial_e} p^{-1}(b) \to 1. \\ [\gamma] &\mapsto T_\gamma(e) \end{aligned}$$

Proof. For any point $e' \in p^{-1}(b)$, let $\tilde{\gamma}\colon e \to e'$ be a path in E and $\gamma = p \circ \tilde{\gamma}$. Then $e' = \partial_e(\gamma)$. This shows the surjectivity of ∂_e.

HLP implies that $p_* : \pi_1(E, e) \to \pi_1(B, b)$ is injective and we can view $\pi_1(E, e)$ as a subgroup of $\pi_1(B, b)$. By definition, for $\tilde{\gamma} \in \pi_1(E, e)$, we have $\partial_e([p \circ \tilde{\gamma}]) = \tilde{\gamma}(1) = e$, i.e. $\pi_1(E, e) \subset \mathrm{Stab}_e(\pi_1(B, b))$. On the other hand, if $T_\gamma(e) = e$, then the lift $\tilde{\gamma}$ of γ is a loop, i.e. $\tilde{\gamma} \in \pi_1(E, e)$. Therefore, $\pi_1(E, e) \supset \mathrm{Stab}_e(\pi_1(B, b))$. This implies $\pi_1(E, e) = \mathrm{Stab}_e(\pi_1(B, b))$, which finishes the proof. □

Lifting Criterion

Theorem 3.24 (Lifting Criterion)**.** *Let $p : E \to B$ be a covering. Consider a continuous map $f : X \to B$, where X is path connected and locally path connected. Let $e_0 \in E, x_0 \in X$ such that $f(x_0) = p(e_0)$. Then there exists a lift F of f with $F(x_0) = e_0$ if and only if*

$$f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0)).$$

Proof. If such F exists, then

$$f_*(\pi_1(X, x_0)) = p_*\Big(F_*(\pi_1(X, x_0))\Big) \subset p_*(\pi_1(E, e_0)).$$

Conversely, let

$$\tilde{E} = \{(x, e) \in X \times E | f(x) = p(e)\} \subset X \times E$$

and consider the following commutative diagram

$$\begin{tikzcd} \tilde{E} \arrow[r] \arrow[d, "\tilde{p}"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B. \end{tikzcd}$$

The projection $\tilde{p}$ is also a covering. We have an induced commutative diagram of functors

$$\begin{tikzcd} \Pi_1(X) \arrow[r] \arrow[dr, "T"'] & \Pi_1(B) \arrow[d, "T"] \\ & \underline{\mathbf{Set}} \end{tikzcd}$$

which induces natural group homomorphisms

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(B, b_0) \to \mathrm{Aut}(p^{-1}(b_0)) = \mathrm{Aut}(\tilde{p}^{-1}(x_0)), \quad b_0 = f(x_0) = p(e).$$

Let $\tilde{e}_0 = (x_0, e_0) \in \tilde{E}$. The condition $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0))$ says that $\pi_1(X, x_0)$ stabilizes $\tilde{e}_0$. By Proposition 3.23, this implies we have a group isomorphism

$$\tilde{p}_* : \pi_1(\tilde{E}, \tilde{e}_0) \cong \pi_1(X, x_0).$$

Since X is locally path connected, $\tilde{E}$ is also locally path connected. Then path connected components and connected components of $\tilde{E}$ coincide. Let $\tilde{X}$ be the (path) connected component of $\tilde{E}$ containing $\tilde{e}_0$, then $\pi_1(\tilde{E}, \tilde{e}) \cong \pi_1(X, x_0)$ implies that $\tilde{p} : \tilde{X} \to X$ is a covering with fiber a single point, hence a homeomorphism. Its inverse defines a continuous map $X \to \tilde{E}$ whose composition with $\tilde{E} \to E$ gives F.

$$\begin{tikzcd}
\tilde{E} \arrow[r] \arrow[d, "\tilde{p}"', shift right] & E \arrow[d, "p"] \\
X \arrow[r, "f"'] \arrow[u, bend right, shift right] & B.
\end{tikzcd}$$

∎

G-principal covering

Definition 3.25. Let G be a discrete group. A continuous action $G \times X \to X$ is called **properly discontinuous** if for any $x \in X$, there exists an open neighborhood U of x such that

$$g(U) \cap U = \emptyset, \quad \forall g \neq 1 \in G.$$

We define the orbit space $X/G = X/\sim$ where $x \sim g(x)$ for any $x \in X, g \in G$.

Proposition 3.26. *Assume G acts properly discontinuously on X, then the quotient map $X \to X/G$ is a covering.*

Proof. For any $x \in X$, let U be the neighbourhood satisfying $g(U) \cap U = \emptyset, \forall g \neq 1 \in G$. Then

$$p^{-1}(p(U)) = \bigsqcup_{g \in G} gU$$

is a disjoint union of open sets. Thus, p is locally trivial with discrete fiber G, hence a covering. ∎

Definition 3.27. A left (right) G-**principal covering** is a covering $p : E \to B$ with a left (right) properly discontinuous G-action on E over B

$$\begin{tikzcd}
E \arrow[rr, "g"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\
& B &
\end{tikzcd}, \quad \forall g \in G$$

such that the induced map $E/G \to B$ is a homeomorphism.

Example 3.28. $\exp\colon \mathbb{R}^1 \to S^1$ is a $\mathbb{Z}$-principal covering for the action $n : t \to t + n, \forall n \in \mathbb{Z}$.

Example 3.29. $S^n \to \mathbb{R}P^n \cong S^n/\mathbb{Z}_2$ is a $\mathbb{Z}_2$-principal covering.

Proposition 3.30. *Let $p : E \to B$ be a G-principal covering. Then the transport is G-equivariant, i.e.,*

$$T_{[\gamma]} \circ g = g \circ T_{[\gamma]}, \quad \forall g \in G, \gamma \text{ a path in } B.$$

Proof. Let $\gamma\colon b_0 \to b_1$ and $e_0 \in p^{-1}(b_0)$. Then $\tilde{\gamma}\colon e_0 \to e_1 = T_{[\gamma]}(e_0)$ for some $e_1 \in p^{-1}(b_1)$. If we apply the transformation g to the path $\tilde{\gamma}$, we find another lift of γ but with endpoints $g(e_0)$ and $g(e_1)$. Therefore

$$T_{[\gamma]}(g(e_0)) = g(e_1).$$

It follows that $T_{[\gamma]}(g(e_0)) = g(e_1) = g(T_{[\gamma]}(e_0))$. This proves the proposition.

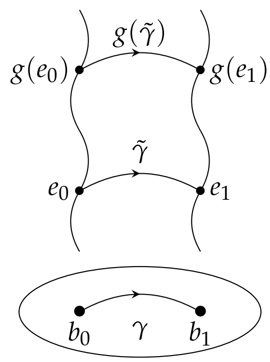


FIGURE 13. Transport commutes with G-action

$\square$

Theorem 3.31. *Let $p: E \to B$ be a G-principal covering, E path connected, $e \in E, b = p(e)$. Then we have an exact sequence of groups*

$$1 \to \pi_1(E,e) \to \pi_1(B,b) \to G \to 1.$$

In other words, $\pi_1(E,e)$ is a normal subgroup of $\pi_1(B,b)$ and $G = \pi_1(B,b)/\pi_1(E,e)$.

Proof. Let $F = p^{-1}(b)$. The previous proposition implies that $\pi_1(B,b)$-action and G-action on F commute. It induces a $\pi_1(B,b) \times G$-action on F. Consider its stabilizer at e and two projections

$$\begin{tikzcd} & \mathrm{Stab}_e(\pi_1(B,b) \times G) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \pi_1(B,b) & & G \end{tikzcd}$$

pr_1 is an isomorphism and pr_2 is an epimorphism with $\ker(\mathrm{pr}_2) = \mathrm{Stab}_e(\pi_1(B,b)) = \pi_1(E,e)$. $\square$

Apply this theorem to the covering $\exp\colon \mathbb{R}^1 \to S^1$, we find a group isomorphism

$$\deg : \boxed{\pi_1(S^1) \xrightarrow{\cong} \mathbb{Z}}$$

which is called the **degree map**. An example of degree n map is

$$S^1 \to S^1, \quad e^{i\theta} \mapsto e^{in\theta}.$$

Applications

Definition 3.32. Let $i : A \subset X$ be an inclusion. A continuous map $r : X \to A$ is called a **retraction** if $r \circ i = 1_A$. It is called a **deformation retraction** if furthermore we have a homotopy $i \circ r \simeq 1_X$ rel A. We say A is a (deformation) retract of X if such a (deformation) retraction exists.

Proposition 3.33. *If $i : A \subset X$ is a retract, then $r_* : \pi_1(A) \to \pi_1(X)$ is injective.*

Corollary 3.34. *Let D^2 be the unit disk in $\mathbb{R}^2$. Then its boundary S^1 is not a retract of D^2.*

Proof. Since D^2 is contractible, we have $\pi_1(D^2) = 1$. But $\pi_1(S^1) = \mathbb{Z}$. Then the corollary follows from the proposition above. $\square$

Theorem 3.35 (Brouwer fixed point Theorem)**.** *Let $f : D^2 \to D^2$. Then there exists $x \in D^2$ such that $f(x) = x$.*

Proof. Assume f has no fixed point. Let l_x be the ray starting from $f(x)$ pointing toward x. Then

$$D^2 \to S^1, \quad x \mapsto l_x \cap \partial D^2$$

is a retraction of $\partial D^2 = S^1 \subset D^2$. Contradiction. $\square$

Theorem 3.36 (Fundamental Theorem of Algebra)**.** *Let $f(x) = x^n + c_1 x^{n-1} + \cdots + c_n$ be a polynomial with $c_i \in \mathbb{C}, n > 0$. Then there exists $a \in \mathbb{C}$ such that $f(a) = 0$.*

Proof. Assume f has no root in $\mathbb{C}$. Define a homotopy of maps from S^1 to S^1

$$F : S^1 \times I \to S^1, \quad F(e^{i\theta}, t) = \frac{f(\tan(\frac{\pi t}{2})e^{i\theta})}{|f(\tan(\frac{\pi t}{2})e^{i\theta})|}.$$

On one hand, $\deg(F|_{S^1 \times 0}) = 0$. On the other hand, $\deg(F|_{S^1 \times 1}) = n$. But they are homotopic hence representing the same element in $\pi_1(S^1)$. Contradiction. $\square$

Proposition 3.37 (Antipode)**.** *Let $f\colon S^1 \to S^1$ be an antipode-preserving map, i.e. $f(-x) = f(-x)$. Then $\deg(f)$ is odd. In particular, f is NOT null homotopic.*

Proof. Let $\sigma\colon S^1 \to S^1$ be the antipode map, with $\sigma(x) = -x$. Then $\deg(\sigma) = -1$. Let

$$F\colon \mathbb{R}^1 \to \mathbb{R}^1$$
$$F(x+1) = F(x) + \deg(f),$$

be a lifting of f. Since f is antipode-preserving, $F(x + 1/2) = F(x) + m$ for $m \in \mathbb{Z} + 1/2$. So $F(x+1) = F(x) + 2m$ which implies $\deg(f) = 2m$ is odd. $\square$

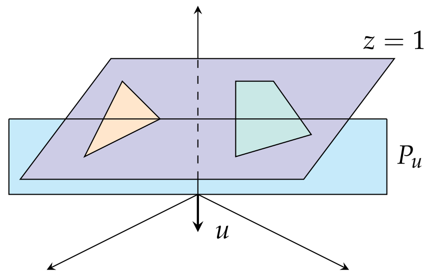
Theorem 3.38 (Borsuk-Ulam)**.** *Let $f : S^2 \to \mathbb{R}^2$. Then there exists $x \in S^2$ such that $f(x) = f(-x)$.*

Proof. Assume $f(x) \neq f(-x), \forall x \in S^2$. Define

$$\rho : S^2 \to S^1, \quad \rho(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

Let D^2 be the upper hemi-sphere of S^2. It defines a homotopy between constant map and $\rho|_{\partial D^2} : S^1 \to S^1$, hence $\deg(\rho|_{\partial D^2}) = 0$. On the other hand, $\rho|_{\partial D^2}$ is antipode-preserving: $\rho|_{\partial D^2}(-x) = -\rho|_{\partial D^2}(x)$, hence $\deg(\rho|_{\partial D^2})$ is odd. Contradiction. $\square$

Corollary 3.39 (Ham Sandwich Theorem)**.** *Let A_1, A_2 be two bounded regions of positive areas in $\mathbb{R}^2$. Then there exists a line which cuts each A_i into half of equal areas.*



Proof. Let $A_1, A_2 \subset \mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$.

Given $u \in S^2$, let P_u be the plane passing the origin and perpendicular to the unit vector u. Let $A_i(u) = \{p \in A_i | p \cdot u \leq 0\}$. Define the map

$$f : S^2 \to \mathbb{R}^2, \quad f_i(u) = \text{Area}(A_i(u)).$$

By Borsuk-Ulam, $\exists u$ such that $f(u) = f(-u)$. The intersection $\mathbb{R}^2 \times \{1\} \cap P_u$ gives the required line since

$$f(u) = f(-u) \Longleftrightarrow f_i(u) = \frac{1}{2}(A_i).$$

□

4 CLASSIFICATION OF COVERING

Definition 4.1. The universal cover of B is a covering map $p : E \to B$ with E simply connected.

The universal cover is unique (if exists) up to homeomorphism. This follows from the lifting criterion and the unique lifting property of covering maps. We left it as an exercise to readers.

Definition 4.2. A space is semi-locally simply connected if for any $x_0 \in X$, there is a neighbourhood U_0 such that the image of the map $i_* \colon \pi_1(U_0, x_0) \to \pi_1(X, x_0)$ is trivial.

We recall the following theorem from point-set topology.

Theorem 4.3 (Existence of the universal cover)**.** *Assume B is path connected and locally path connected. Then universal cover of B exists if and only if B is semi-locally simply connected.*

Definition 4.4. We define the category $\mathrm{Cov}(B)$ of coverings of B where

- an object is a covering map $p : E \to B$;
- a morphism between two coverings $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ is a map $f : E_1 \to E_2$ such that the following diagram is commutative

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd} .$$

Definition 4.5. Let B be connected. We define $\mathrm{Cov}_0(B) \subset \mathrm{Cov}(B)$ to be the subcategory whose objects consist of coverings of B which are connected spaces.

Proposition 4.6. *Let B be connected and locally path connected. Then any morphism in $Cov_0(B)$ is a covering map.*

Proof. Exercise. □

Definition 4.7. We define the category G-**Set**, where

- an object is a set S with G-action and
- morphisms are G-equivariant set maps, i.e. $f \colon S_1 \to S_2$ such that $f \circ g = g \circ f$, for any $g \in G$.

Given a covering $p : E \to B$, $b \in B$, the transport functor implies that

$$p^{-1}(b) \in \pi_1(B, b)\text{-}\mathbf{Set}.$$

Lemma 4.8. *Let B be path connected. Then $\pi_1(B, b)$ acts transitively on $p^{-1}(b)$ if and only if E is path connected.*

Proof. The "if" part follows from Proposition 3.23. We prove the "only if" part.

Let us fix a point $e_0 \in p^{-1}(b)$. Assume $\pi_1(B, b)$ acts transitively on $p^{-1}(b)$. This implies that any point in $p^{-1}(b)$ is connected to e_0 by a path. Given any point $e \in E$, let γ be a path in B that connects $p(e)$ to b. The transport functor $T_{[\gamma]}$ gives a path connecting e to some point in $p^{-1}(b)$. This further implies that e is path connected to e_0. This proves the "only if" part. □

Corollary 4.9. *Let B be path connected, $p : E \to B$ be a covering. Then there is a one-to-one correspondence between path connected components of E and $\pi_1(B, b)$-orbits in $p^{-1}(b)$.*

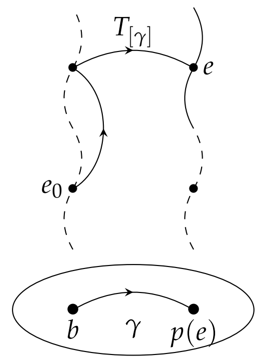


FIGURE 14. Transitivity v.s. path connectedness

Example 4.10. Let $p : \tilde{B} \to B$ be the universal covering. By Proposition 3.23, the fiber $p^{-1}(b)$ can be identified with $\pi_1(B, b)$ itself.

Proposition 4.11. *Assume B is path connected and locally path connected. Let $p_1, p_2 \in Cov(B)$. Then*

$$\mathrm{Hom}_{Cov(B)}(p_1, p_2) \cong \mathrm{Hom}_{\pi_1(B,b)\text{-}\underline{\mathbf{Set}}}(p_1^{-1}(b), p_2^{-1}(b))$$

for any $b \in B$.

Proof. Let $f \in \mathrm{Hom}_{\mathrm{Cov}(B)}(p_1, p_2)$, i.e.

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd} .$$

It induces a map by restricting f to the fiber $p^{-1}(b)$

$$f_b \colon p_1^{-1}(b) \to p_2^{-1}(b).$$

By the same argument as in Proposition 3.30, we find f_b is $\pi_1(B, b)$-equivariant. Thus we obtain a map

$$\begin{aligned} \Phi \colon \mathrm{Hom}_{\mathrm{Cov}(B)}(p_1, p_2) &\to \mathrm{Hom}_{\pi_1(B,b)\text{-}\underline{\mathbf{Set}}}(p_1^{-1}(b), p_2^{-1}(b)) \\ f &\to f_b \end{aligned}$$

The injectivity of Φ comes from the uniqueness of the lifting.

To prove surjectivity, we can assume E_1 is path connected, and $\pi_1(B, b)$ acts transitively on $p_1^{-1}(b)$ (see the Corollary above). Given $f_b \colon p_1^{-1}(b) \to p_2^{-1}(b)$, let us fix two points $e_i \in p_i^{-1}(b)$ such that $f(e_1) = e_2$. The $\pi_1(B, b)$-equivariance of f_b gives rise to the homomorphism

$$\begin{array}{ccc} \mathrm{Stab}_{e_1}(\pi_1(B, b)) & \longrightarrow & \mathrm{Stab}_{e_2}(\pi_1(B, b)) \\ = \pi_1(E_1, e_1) & & = \pi_1(E_2, e_2). \end{array}$$

By Lifting Criterion (Theorem 3.24), we obtain a morphism $f \colon E_1 \to E_2$ as required. ☐

Theorem 4.12. *Assume B is path connected, locally path connected and semi-locally simply connected. $b \in B$. Then there exists an equivalence of categories*

$$\boxed{Cov(B) \simeq \pi_1(B, b)\text{-}\underline{\mathbf{Set}}.}$$

Proof. Let us denote $\pi_1 = \pi_1(B, b)$. Let $\tilde{p} : \widetilde{B} \to B$ be a fixed universal cover of B and $\tilde{b} \in \pi^{-1}(b)$ chosen.

We will define the following functors

$$\begin{tikzcd} \mathrm{Cov}(B) \arrow[r, shift left, "F"] & \pi_1\text{-}\underline{\mathbf{Set}}. \arrow[l, shift left, "G"] \end{tikzcd}$$

Let $p : E \to B$ be a covering, we define

$$F(p) := p^{-1}(b).$$

Let $S \in \pi_1\text{-}\underline{\mathbf{Set}}$, we define

$$G(S) := \widetilde{B} \times_{\pi_1} S = \widetilde{B} \times S/\sim,$$

where $(e \cdot g, s) \sim (e, g \cdot s)$ for any $e \in \widetilde{B}, s \in S, g \in \pi_1$. Note that here $e \cdot g$ represents the (right) π_1-action on $\widetilde{B}$. Then we have natural isomorphisms

$$F \circ G \overset{\eta}{\simeq} 1, \quad G \circ F \overset{\tau}{\simeq} 1.$$

Here η is the natural isomorphism

$$\eta_S \in \mathrm{Hom}_{\pi_1\text{-}\underline{\mathbf{Set}}}(F \circ G(S), S), \quad \eta_S(e, s) = g \cdot s \quad \text{if } e = \tilde{b} \cdot g.$$

τ is the natural isomorphism

$$\tau_p \in \mathrm{Hom}_{\mathrm{Cov}(B)}(p', p) \cong \mathrm{Hom}_{\pi_1\text{-}\underline{\mathbf{Set}}}(p^{-1}(b), p^{-1}(b)), \quad p' = G \circ F(p) : \widetilde{B} \times_{\pi_1} p^{-1}(b) \to B,$$

which is determined by the identity map in $\mathrm{Hom}_{\pi_1\text{-}\underline{\mathbf{Set}}}(p^{-1}(b), p^{-1}(b))$. $\square$

Definition 4.13. Let B be path connected and $p : E \to B$ be a connected covering. A **deck transformation** (or **covering transformation**) of p is a homeomorphism $f : E \to E$ such that $p \circ f = p$.

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

Let $\mathrm{Aut}(p)$ denote the group of deck transformations.

Note that $\mathrm{Aut}(p)$ acts freely on E by the Uniqueness of Lifting.

Proposition 4.14. *Let B be path connected and $p : E \to B$ be a connected covering. Then $\mathrm{Aut}(p)$ acts properly discontinuously on E.*

Proof. Exercise. $\square$

Corollary 4.15. *Assume B is path connected, locally path connected. Let $p : E \to B$ be a connected covering, $e \in E, b = p(e) \in B$, $G = \pi_1(B, b), H = \pi_1(E, e)$. Then*

$$\mathrm{Aut}(p) \cong N_G(H)/H$$

where

$$N_G(H) := \{r \in G \mid rHr^{-1} = H\}$$

is the normalizer of H in G.

Proof. By the above proposition,

$$\mathrm{Aut}(p) \cong \mathrm{Hom}_{G\text{-}\underline{\mathbf{Set}}}(G/H, G/H) = N_G(H)/H.$$

$\square$

Definition 4.16. We define the orbit category $\mathrm{Orb}(G)$:

- objects consist of (left) coset G/H, where H is a subgroup of G;
- morphisms are G-equivariant maps: $G/H_1 \to G/H_2$.

Note G/H_1 and G/H_2 are isomorphic in $\mathrm{Orb}(G)$ if and only if H_1 and H_2 are conjugate subgroups of G.

If we restrict Theorem 4.12 to connected coverings, we find

Theorem 4.17. *Assume B is path connected, locally path connected and semi-locally simply connected. $b \in B$. Then there exists an equivalence of categories*

$$Cov_0(B) \simeq \mathrm{Orb}(\pi_1(B, b)).$$

The universal cover $\widetilde{B} \to B$ corresponds to the orbit $\pi_1(B, b)$. For the orbit $\pi_1(B, b)/H$, it corresponds to

$$E = \widetilde{B}/H \to B.$$

This can be illustrated by the following correspondence

$$\begin{tikzcd} \pi_1(B,b) \arrow[rr] \arrow[dr] & & \tilde{\pi}_1(B,b)/H \arrow[dl] & \Longleftrightarrow & \widetilde{B} \arrow[rr, "f"] \arrow[dr] & & \widetilde{B}/H \arrow[dl] \\ & 1 & & & & B & \end{tikzcd}$$

A more intrinsic formulation is as follows. Given a covering $p : E \to B$, we obtain a transport functor

$$T_p : \Pi_1(B) \to \underline{\mathbf{Set}}.$$

Given a commutative diagram

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

we find a natural transformation

$$\tau : T_{p_1} \Longrightarrow T_{p_2}, \quad \tau = \{f : p_1^{-1}(b) \to p_2^{-1}(b) | b \in B\}.$$

The above structure can be summarized by a functor

$$\boxed{T : \mathrm{Cov}(B) \to \mathrm{Fun}(\Pi_1(B), \underline{\mathbf{Set}})}.$$

Theorem 4.18. *Assume B is path connected, locally path connected and semi-locally simply connected. Then*

$$T : Cov(B) \to \mathrm{Fun}(\Pi_1(B), \underline{\mathbf{Set}})$$

is an equivalence of categories.

5 LIMIT AND COLIMIT

Many constructions in algebraic topology are described by their universal properties. There are two important ways to define new objects of such types, called the **limit** and **colimit**, which are dual to each other. In this section, we give a brief discussion of these two notions.

Let $\mathcal{I}$ be a small category (i.e. objects form a set). Let $\mathcal{C}$ be a category. Recall that we have a functor category (Definition 1.27)

$$\mathrm{Fun}(\mathcal{I}, \mathcal{C}),$$

where objects are functors from $\mathcal{I}$ to $\mathcal{C}$, and morphisms are natural transformations. We also write

$$\mathcal{C}^{\mathcal{I}} := \mathrm{Fun}(\mathcal{I}, \mathcal{C}).$$

Definition 5.1. We define the diagonal (or constant) functor

$$\Delta : \mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C}),$$

which assigns $X \in \mathcal{C}$ to the functor $\Delta(X) : \mathcal{I} \to \mathcal{C}$ that sends all objects in $\mathcal{I}$ to X and all morphisms to 1_X.

Diagram

Let $\mathcal{I}$ be a diagram, with vertices and arrows. We can define a category still denoted by $\mathcal{I}$

- $\mathrm{Obj}(\mathcal{I})$ = vertices in the diagram $\mathcal{I}$
- morphisms are composites of all given arrows as well as additional "identity arrows" that compose like identity maps.

Example 5.2. The following diagram

$$\begin{tikzcd} \bullet \arrow[r] & \circ \arrow[d] \\ & \star \end{tikzcd}$$

defines a category with three objects $\bullet, \circ, \star$. There is only one morphism from $\bullet$ to $\circ$, one from $\circ$ to $\star$, and one from $\bullet$ to $\star$ which is the composite of the previous two.

Given an object $A \in \mathcal{C}$, the constant functor $\Delta(A) : \mathcal{I} \to \mathcal{C}$ can be represented by the following data

$$\begin{tikzcd} A \arrow[r, "1_A"] & A \arrow[d, "1_A"] \\ & A \end{tikzcd}$$

Example 5.3. The following diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[dr] & \circ \arrow[d] \\ & \star \end{tikzcd}$$

defines a category with three objects $\bullet, \circ, \star$. There is only one morphism from $\bullet$ to $\circ$ and one from $\circ$ to $\star$. There are two morphisms from $\bullet$ to $\star$, one of them is the composite of the previous two morphisms, and the other one is represented by the arrow $\bullet \to \star$.

Example 5.4. The following diagram

$$\begin{tikzcd} \bullet \arrow[r, shift left] & \circ \arrow[l, shift left] \end{tikzcd}$$

defines a category with two objects $\bullet, \circ$. Morphisms from $\bullet$ to $\bullet$ contains the identity $1_\bullet$, the composition of $\bullet \to \circ$ and $\circ \to \bullet$ and so on.

Given a diagram $\mathcal{I}$, a functor $F : \mathcal{I} \to \mathcal{C}$ is determined by assigning vertices and arrows the corresponding objects and morphisms in $\mathcal{C}$. For example, the following data

$$X \xrightarrow{f} Y \xleftarrow{g} Z \ , \qquad X, Y, Z \in \mathcal{C}$$

define a functor from $\bullet \to \circ \leftarrow \star$ to $\mathcal{C}$. Such a data will be also called a **$\mathcal{I}$-shaped diagram** in $\mathcal{C}$.

Limit

Definition 5.5 (Limit)**.** Let $F : \mathcal{I} \to \mathcal{C}$. A **limit** for F is an object P in $\mathcal{C}$ together with a natural transformation

$$\tau : \Delta(P) \Rightarrow F$$

such that for every object Q of $\mathcal{C}$ and every natural transformation $\eta : \Delta(Q) \Rightarrow F$, there exists a unique map $f : Q \to P$ such that $\tau \circ \Delta(f) = \eta$. In other words, the following diagram is commutative.

$$\begin{tikzcd}
\Delta(Q) \arrow[r, Rightarrow, dashed, "\exists!\Delta(f)"] \arrow[dr, Rightarrow, "\eta"'] & \Delta(P) \arrow[d, Rightarrow, "\tau"] \\
& F
\end{tikzcd} .$$

For example, consider the following $\mathcal{I}$-shaped diagram in $\mathcal{C}$ which represents a functor $F : \mathcal{I} \to \mathcal{C}$

$$\begin{tikzcd}
& Y \arrow[d] \\
X \arrow[r] & Z
\end{tikzcd} .$$

Then its limit is an object $P \in \mathcal{C}$ that fits into the commutative diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & Z
\end{tikzcd} .$$

Moreover for any other object Q fitting into the same commutative diagram, there exists a unique $f : Q \to A$ to making the following diagram commutative

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& P \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \arrow[r] & Z
\end{tikzcd} .$$

Proposition 5.6. *Let $F : \mathcal{I} \to \mathcal{C}$ and P_1, P_2 be two limits of F with natural transformations $\tau_i : \Delta(A_i) \Rightarrow F$. Then there exists a unique isomorphism $P_1 \to P_2$ in $\mathcal{C}$ which makes the following diagram commutative*

$$\begin{tikzcd}
\Delta(P_1) \arrow[rr, Rightarrow] \arrow[dr, Rightarrow, "\tau_1"'] & & \Delta(P_2) \arrow[dl, Rightarrow, "\tau_2"] \\
& F &
\end{tikzcd} .$$

The above proposition says that if the limit of F exists, then it is unique up to a canonical isomorphism.

Definition 5.7. We denote the limit of $F : \mathcal{I} \to \mathcal{C}$ by $\mathbf{lim}\, F$ (if exists).

The universal property of the limit gives the adjunction

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{I},\mathcal{C})}(\Delta(X), F) = \operatorname{Hom}_{\mathcal{C}}(X, \mathbf{lim}\, F).$$

This immediately leads to the following theorem.

Theorem 5.8. *Let $\mathcal{C}$ be a category. Then the following are equivalent*

(1) *Every $F : \mathcal{I} \to \mathcal{C}$ has a limit*
(2) *The constant functor $\Delta : \mathcal{C} \to \operatorname{Fun}(\mathcal{I}, \mathcal{C})$ has a right adjoint.*

$$\Delta : \mathcal{C} \rightleftarrows \operatorname{Fun}(\mathcal{I}, \mathcal{C}) : \mathbf{lim} \ .$$

In this case, the right adjoint of the constant functor is the limit.

Example 5.9 (Pullback)**.** The limit of the following diagram $X \to Y \leftarrow Z$ gives

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd},$$

which is called the **pullback**.

In the category **Set**, the pull-back exists and is given by the subset of $X \times Y$

$$P = \{(x, y) \in X \times Y | f(x) = g(y)\} \subset X \times Y.$$

Example 5.10 (Tower and inverse limit)**.** We consider the following category $\mathbb{N}$:

- Objects of $\mathbb{N}$ are positive integers.
- Given $m, n \in \mathbb{N}$, the morphism set $\operatorname{Hom}_{\mathbb{N}}(m, n)$ is empty if $m > n$ and is a single point if $m \leq n$.

Let $\mathbb{N}^{\text{op}}$ be the opposite of $\mathbb{N}$. A functor $F : \mathbb{N}^{\text{op}} \to \mathcal{C}$ is represented by the **tower diagram**

$$\cdots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \ .$$

The limit of tower diagram is also called the **inverse limit** of the tower and written as $\varprojlim X_i$.

$$\begin{tikzcd} & & Q \arrow[d, dashed, "\exists!"] \arrow[ddll] \arrow[ddrrr] & & & \\ & & \varprojlim X_i \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\ \cdots \arrow[r] & X_{n+1} \arrow[r] & X_n \arrow[r] & \cdots \arrow[r] & X_2 \arrow[r] & X_1 \end{tikzcd}.$$

Theorem 5.11. *Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be adjoint functors. Assume the limit of $F : \mathcal{I} \to \mathcal{D}$ exists. Then the limit of $R \circ F : \mathcal{I} \to \mathcal{C}$ also exists and is given by*

$$\mathbf{lim}(R \circ F) = R(\mathbf{lim}\, F).$$

In other words, right adjoint functors preserve limit.

Proof. Let $A \in \mathcal{C}$. Assume we have a natural transformation

$$\tau : \Delta(A) \Rightarrow R \circ F.$$

By adjunction, this is equivalent to a natural transformation $\Delta(L(A)) \Rightarrow F$.

By the universal property of limit, there exists a unique map $L(A) \to \mathbf{lim}\, F$ factorizing $\Delta(L(A)) \Rightarrow F$

$$\Delta(L(A)) \Rightarrow \mathbf{lim}\, F \Rightarrow F.$$

By adjunction again, this is equivalent to natural transformations

$$\Delta(A) \Rightarrow R(\mathbf{lim}\, F) \Rightarrow R \circ F.$$

This implies that $R(\mathbf{lim}\, F)$ is the limit of $R \circ F$. □

Remark 5.12. A functor is called **continuous** if it preserves all limits. This theorem says if a functor has a left adjoint, then it is continuous. Under certain conditions, the reverse is also true (Adjoint Functor Theorem).

Corollary 5.13. *The forgetful functor* Forget : $\underline{\mathbf{Top}} \to \underline{\mathbf{Set}}$ *preserves limit.*

Proof. Forget : $\underline{\mathbf{Top}} \to \underline{\mathbf{Set}}$ has a left adjoint

$$\mathrm{Discrete} : \underline{\mathbf{Set}} \rightleftarrows \underline{\mathbf{Top}} : \mathrm{Forget}\ ,$$

where Discrete associates a set X with discrete topology. □

Example 5.14. Consider the following diagram in $\underline{\mathbf{Top}}$

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd} .$$

We would like to understand the pull-back P of the above diagram in $\underline{\mathbf{Top}}$. By Example 5.9 and Corollary 5.13, we know that the underlying set for P (if exists) is

$$\mathrm{Forget}(P) = \{(x, y) \in X \times Y | f(x) = g(y)\} \subset X \times Y.$$

It is not hard to see that if we assign P the subspace topology of the topological product $X \times Y$, then P is indeed the pull-back in $\underline{\mathbf{Top}}$. In particular, pull-back exists in $\underline{\mathbf{Top}}$. Fibrations behave well under pull-back.

Proposition 5.15. *Let $p : Y \to Z$ be a fibration, and $f : X \to Z$ be continuous. Consider the pull-back diagram*

$$\begin{tikzcd} Q \arrow[r, "q"'] \arrow[d] & Y \arrow[d, "p"] \\ X \arrow[r, "f"'] & Z \end{tikzcd} .$$

Then $q : Q \to X$ is also a fibration. In other words, the pull-back of a fibration is a fibration.

Colimit

The notion of colimit is dual to limit.

Definition 5.16 (Colimit)**.** Let $F : \mathcal{I} \to \mathcal{C}$. A **colimit** for F is an object P in $\mathcal{C}$ together with a natural transformation

$$\tau : F \Rightarrow \Delta(P)$$

such that for every object Q of $\mathcal{C}$ and every natural transformation $\eta : F \Rightarrow \Delta(Q)$, there exists a unique map $f : P \to Q$ such that $\Delta(f) \circ \tau = \eta$. In other words, the following diagram is commutative

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\tau"] \arrow[dr, Rightarrow, "\eta"'] & \Delta(P) \arrow[d, dashed, "\exists! \Delta(f)"] \\ & \Delta(Q) \end{tikzcd} .$$

The colimit, if exists, is unique up to a unique isomorphism, and will be denoted by **colim** F.

The following theorems are dual to the limit case as well and can be proved dually.

Theorem 5.17. *Let $\mathcal{C}$ be a category. Then the following are equivalent*

(1) *Every $F : \mathcal{I} \to \mathcal{C}$ has a limit*
(2) *The constant functor $\Delta : \mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ has a left adjoint.*

$$\mathbf{colim} : \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \rightleftarrows \mathcal{C} : \Delta$$

In this case, the left adjoint of the constant functor is the colimit.

Theorem 5.18. *Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be adjoint functors. Assume the colimit of $F : \mathcal{I} \to \mathcal{C}$ exists. Then the colimit of $L \circ F : \mathcal{I} \to \mathcal{D}$ also exists and is given by*

$$\mathbf{colim}(L \circ F) = L(\mathbf{colim}\, F).$$

In other words, left adjoint functors preserve colimit.

Remark 5.19. A functor is called **co-continuous** if it preserves all colimits. This theorem says if a functor has a right adjoint, then it is co-continuous. Under certain conditions, the reverse is also true (Adjoint Functor Theorem).

Corollary 5.20. *The forgetful functor* $\mathrm{Forget} : \underline{\mathbf{Top}} \to \underline{\mathbf{Set}}$ *preserves colimit.*

Proof. $\mathrm{Forget} : \underline{\mathbf{Top}} \to \underline{\mathbf{Set}}$ has a right adjoint

$$\mathrm{Forget} : \underline{\mathbf{Top}} \rightleftarrows \underline{\mathbf{Set}} : \mathrm{Triv} \ ,$$

where Triv associates a set X with trivial topology (only open subsets are $\emptyset$ and X). ☐

Example 5.21 (Pushout)**.** The colimit of the following diagram $X \leftarrow Y \to Z$ gives

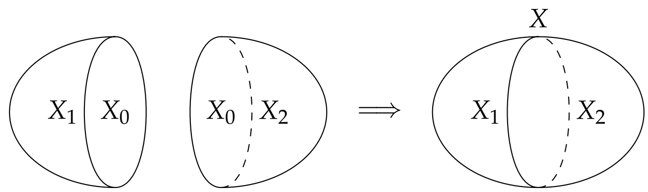
$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & P \end{tikzcd} .$$

This colimit is called the **pushout**. This is a dual notion to pullback. It has the following universal property

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \arrow[ddr, bend left] & \\ X \arrow[r] \arrow[drr, bend right] & P \arrow[dr, dashed, "\exists!"] & \\ & & Q \end{tikzcd} .$$

Here are some examples.

- Let $j_1 : X_0 \to X_1, j_2 : X_0 \to X_2$ in **Top**. Their pushout is the quotient of the disjoint union $X_1 \coprod X_2$ by identifying $j_1(y) \sim j_2(y), y \in X_0$. It glues X_1, X_2 along X_0 using j_1, j_2. For instance:



- Let $\rho_1 : H \to G_1, \rho_2 : H \to G_2$ be two morphisms in **Group**, then their pushout is

$$(G_1 * G_2)/N,$$

  where $G_1 * G_2$ is the free product and N is the normal subgroup generated by $\rho_1(h)\rho_2^{-1}(h), h \in H$.

Example 5.22 (Telescope and direct limit)**.** A functor $F : \mathbb{N} \to \mathcal{C}$ is represented by the **telescope diagram**

$$X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \cdots .$$

The colimit of telescope diagram is also called the **direct limit** of the telescope and written as $\varinjlim X_i$.

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[drrr] \arrow[ddrrr] & X_2 \arrow[r] \arrow[drr] & \cdots \arrow[r] \arrow[dr] & X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[dl] & \cdots \arrow[dll] \arrow[ddll] \\
& & & \varinjlim X_i \arrow[d, dashed, "\exists!"] & & \\
& & & Q & &
\end{tikzcd} .$$

Product

Definition 5.23. Let $\mathcal{C}$ be a category, $\{A_\alpha\}_{\alpha \in I}$ be a set of objects in $\mathcal{C}$. Their **product** is an object A in $\mathcal{C}$ together with $\pi_\alpha : A \to A_\alpha$ satisfying the following universal property: for any X in $\mathcal{C}$ and $f_\alpha : X \to A_\alpha$, there exists a unique morphism $f : X \to A$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, dashed, "\exists! f"] \arrow[dr, "f_\alpha"'] & A \arrow[d, "\pi_\alpha"] \\
& A_\alpha
\end{tikzcd} .$$

For product of two objects, we have the following diagram

$$\begin{tikzcd}
& A_\times A_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
A_1 & & A_2 \\
& X \arrow[ul, "f_1"] \arrow[ur, "u_2"'] \arrow[uu, dashed, "\exists! f"'] &
\end{tikzcd}$$

The product is a limit. In fact, let us equip the index set I with the category structure such that it has only identity morphisms. Then the data $\{A_\alpha\}_{\alpha\in I}$ is the same as a functor $F : I \to \mathcal{C}$. Their product is precisely $\mathbf{lim}\, F$. In particular, the product is unique up to isomorphism if it exists. We denote it by

$$\prod_{\alpha\in I} A_\alpha.$$

A useful consequence is that the product is preserved under right adjoint functors (like forgetful functors).

Example 5.24.

- Let $S_\alpha \in \underline{\mathbf{Set}}$. $\prod\limits_\alpha S_\alpha = \{(s_\alpha) | s_\alpha \in S_\alpha\}$ is the Cartesian product.
- Let $X_\alpha \in \underline{\mathbf{Top}}$. Then $\prod\limits_\alpha X_\alpha$ is the Cartesian product with induced product topology. Namely, we have $X \xrightarrow{f} \prod\limits_\alpha X_\alpha$ is continuous if and only if $\{X \xrightarrow{f_\alpha} X_\alpha\}$ are continuous for any α.
- Let $G_\alpha \in \underline{\mathbf{Group}}$. Then $\prod\limits_\alpha G_\alpha$ is the Cartesian product with induced group structure, i.e.

$$\prod_\alpha G_\alpha = \{(g_\alpha) \mid g_\alpha \in G_\alpha\}$$

  with $(g_\alpha) \cdot (g'_\alpha) = (g_\alpha \cdot g'_\alpha)$.

Coproduct

Definition 5.25. Let $\mathcal{C}$ be a category, $\{A_\alpha\}_{\alpha\in I}$ be a set of objects in $\mathcal{C}$. Their **coproduct** is an object A in $\mathcal{C}$ together with $i_\alpha : A_\alpha \to A$ satisfying the following universal property: for any X in $\mathcal{C}$ and $f_\alpha : A_\alpha \to X$, there exists a unique morphism $f : A \to X$ such that the following diagram commutes

$$\begin{tikzcd}
X & A \arrow[l, dashed, "\exists! f"'] \\
 & A_\alpha \arrow[ul, "f_\alpha"] \arrow[u, "\pi_\alpha"']
\end{tikzcd}$$

The coproduct is a colimit. As in the discussion of product, the data $\{A_\alpha\}_{\alpha\in I}$ defines a functor $F : I \to \mathcal{C}$. Their coproduct is precisely $\mathbf{colim}\, F$, which is unique up to isomorphism if it exists. We denote it by

$$\coprod_{\alpha\in I} A_\alpha.$$

A useful consequence is that the coproduct is preserved under left adjoint functors (like free constructions).

Example 5.26.

- Let $S_\alpha \in \underline{\mathbf{Set}}$. $\coprod\limits_\alpha S_\alpha = \{(s_\alpha) | s_\alpha \in S_\alpha\}$ is the disjoint union of sets.
- Let $X_\alpha \in \underline{\mathbf{Top}}$. Then $\coprod\limits_\alpha X_\alpha$ is the disjoint union of topological spaces. Clearly, continuous maps $\{X_\alpha \xrightarrow{f_\alpha} Y\}$ uniquely extends to $\coprod\limits_\alpha X_\alpha \to Y$.
- Let $G_\alpha \in \underline{\mathbf{Group}}$. Then $\coprod\limits_\alpha G_\alpha$ is the free product of groups. More precisely, we have

$$\coprod_\alpha G_\alpha :\ = \{\text{word of finite length: } x_1 x_2 \cdots x_n \mid x_i \in G_{\alpha_i}\} / \sim,$$

  where

$$x_1 \cdots x_i x_{i+1} \cdots x_n \sim x_1 \cdots (x_i \cdot x_{i+1}) \cdots x_n$$

if $x_i, x_{i+1} \in G_\alpha$ and $(x_i \cdot x_{i+1})$ is the group production in G_α. The group structure in $\coprod_\alpha G_\alpha$ is

$$(x_1 \cdots x_n) \cdot (y_1 \cdots y_m) := x_1 \cdots x_n y_1 \cdots y_m.$$

Given group homomorphisms $G_\alpha \xrightarrow{f_\alpha} H$, it uniquely determines the group homomorphism

$$\begin{aligned} f \colon \coprod_\alpha G_\alpha &\to H \\ x_q \cdots x_n &\mapsto f_{\alpha_1}(x_1) \cdots f_{\alpha_n}(x_n). \end{aligned}$$

This is precisely the coproduct property. When there are only finitely many G_α, we will write

$$\coprod_\alpha G_\alpha =: G_1 \star G_2 \star \cdots \star G_n.$$

Wedge and smash product

Definition 5.27. We define the category $\underline{\mathbf{Top}_\star}$ of pointed topological space where

- an object (X, x_0) is a topological space X with a based point $x_0 \in X$
- morphisms are based continuous maps that map based point to based point.

Given a space X, we can define a pointed space X_+ by adding an extra point

$$X_+ = X \coprod \star, \quad \text{with basepoint } \star.$$

This defines a functor

$$()_+ : \underline{\mathbf{Top}} \to \underline{\mathbf{Top}_\star}.$$

On the other hand, we have a forgetful functor by forgetting the base point

$$\text{Forget} : \underline{\mathbf{Top}_\star} \to \underline{\mathbf{Top}}.$$

They form an adjoint pair

$$()_+ : \underline{\mathbf{Top}} \rightleftarrows \underline{\mathbf{Top}_\star} : \text{Forget}$$

This implies that the limit in $\underline{\mathbf{Top}_\star}$ will be the same as the limit in $\underline{\mathbf{Top}}$. In particular, the product of pointed spaces $\{(X_i, x_i)\}$ in $\underline{\mathbf{Top}_\star}$ is the topological product

$$\prod_i X_i, \quad \text{with base point } \{x_i\}.$$

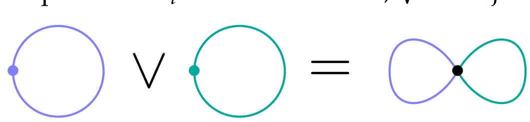
In $\underline{\mathbf{Top}_\star}$, the coproduct of two pointed spaces X, Y is the **wedge product** $\vee$. Specifically,

$$X \vee Y = X \coprod Y / \sim$$

is the quotient of the disjoint union of X and Y by identifying the base points $x_0 \in X$ and $y_0 \in Y$. The identified based point is the new based point of $X \vee Y$. In general, we have

$$\bigvee_{i \in I} X_i = \coprod_{i \in I} X_i / \sim$$

where $\sim$ again identifies all based points in X_i's. In other words, $\bigvee$ is the joining of spaces at a single point.



Example 5.28. The Figure-8 in Example 3.9 can be identified with $S^1 \vee S^1$.

In **Top**$_\star$, there is another operation, called **smash product** $\wedge$, which will have adjunction property and play an important role in homotopy theory. Specifically,

$$X \wedge Y = X \times Y / \sim$$

is the quotient of the product space $X \times Y$ under the identifications $(x, y_0) \sim (x_0, y)$ for all $x \in X, y \in Y$. The identified point is the new based point of $X \wedge Y$. Note that we can write it as the quotient

$$X \wedge Y = X \times Y / X \vee Y.$$

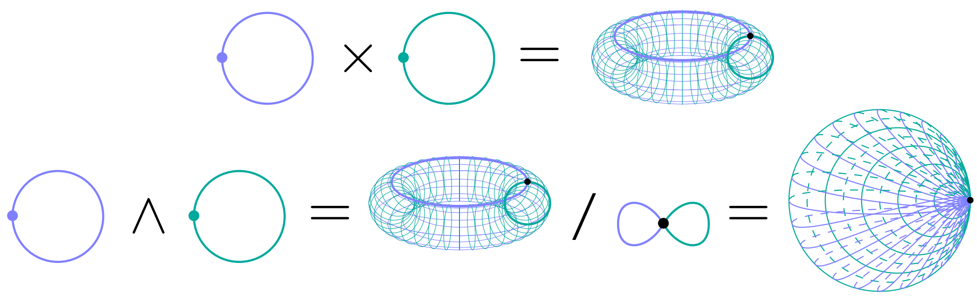


FIGURE 15. Smash product of circles

Example 5.29. There is a natural homeomorphism

$$S^1 \wedge S^n \cong S^{n+1}.$$

This implies that $S^n \wedge S^m \cong S^{n+m}$. For instance, see Figure 15 for $n = 1$ case. In this case, the result, i.e. S^2, can be also realized by cutting the green/purple circles on the torus (where we get a square) and gluing them (the boundary of the square) into one point.

Complete and cocomplete

Definition 5.30. A category $\mathcal{C}$ is called **complete** (**cocomplete**) if for any $F \in \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ with $\mathcal{I}$ a small category, the limit $\mathbf{lim}\, F$ ($\mathbf{colim}\, F$) exists.

Example 5.31. **Set**, **Group**, **Ab**, **Vect**, **Top** are complete and cocomplete.

For example, in **Set**, the limit of $F : I \to \mathbf{Set}$ is given by

$$\mathbf{lim}\, F = \left\{ (x_i)_{i \in I} \in \prod_{i \in I} F(i) \,\middle|\, x_j = F(f)(x_i) \text{ for any } i \xrightarrow{f} j \right\} \subset \prod_{i \in I} F(i)$$

which is a subset of $\prod\limits_{i \in I} F(i)$. The colimit is given by

$$\mathbf{colim}\, F = \coprod_{i \in I} F(i) \Big/ \left\{ x_i \sim F(f)(x_i) \text{ for any } i \xrightarrow{f} j,\ x_i \in F(i) \right\}$$

which is a quotient of $\coprod_{i \in I} F(i)$.

For another example, we consider **Top**. Since the forgetful functor **Top** $\to$ **Set** has both a left adjoint and a right adjoint, it preserves both limits and colimits. Given $F : I \to \mathbf{Top}$, its limit $\mathbf{lim}\, F$ has the same underlying set as that in **Set** above, but equipped with the induced topology from product and subspace. Similarly, the colimit $\mathbf{colim}\, F$ is the quotient of disjoint unions of $F(i)$ with the induced quotient topology.

Initial and terminal object

Definition 5.32. An **initial/universal object** of a category $\mathcal{C}$ is an object $\star$ such that for every object X in $\mathcal{C}$, there exists precisely one morphism $\star \to X$. Dually, a **terminal/final object** $\star$ satisfies that for every object X there exists precisely one morphism $X \to \star$. If an object is both initial and terminal, it is called a **zero object** or **null object**.

The defining universal property implies that the initial object and he terminal object are unique up to isomorphism if they exist.

Example 5.33. The emptyset $\emptyset$ is the initial object in $\underline{\mathbf{Set}}$, and the set with a single point is the terminal object in $\underline{\mathbf{Set}}$. The same is true for $\underline{\mathbf{Top}}$.

The limit of a functor $F : I \to \mathcal{C}$ can be viewed as a terminal object as follows. We define a category $\mathcal{C}_F$

- an object of $\mathcal{C}_F$ is an object $A \in \mathcal{C}$ together with a natural transformation

$$\Delta(A) \Rightarrow F$$

- a morphism in $\mathcal{C}_F$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that the following diagram is commutative

$$\begin{tikzcd} \Delta(A) \arrow[rr, Rightarrow, "\Delta(f)"] \arrow[dr, Rightarrow] & & \Delta(P) \arrow[dl, Rightarrow] \\ & F & \end{tikzcd}$$

Then $\mathbf{lim}\, F$ is the terminal object in $\mathcal{C}_F$. A dual construction says $\mathbf{colim}\, F$ can be viewed as an initial object.

6 SEIFERT-VAN KAMPEN THEOREM

Theorem 6.1 (Seifert-van Kampen Theorem, Groupoid version). *Let $X = U \cup V$ where $U, V \subset X$ are open. Then the following diagram*

$$\begin{tikzcd} \Pi(U \cap V) \arrow[r] \arrow[d] & \Pi(U) \arrow[d] \\ \Pi(V) \arrow[r] & \Pi(X) \end{tikzcd}$$

is a pushout in the category <u>**Groupoid**</u>.

Proof. Let $\mathcal{C}$ be a groupoid fitting into the commutative diagram

$$\begin{tikzcd} \Pi(U \cap V) \arrow[r] \arrow[d] & \Pi(U) \arrow[d] \\ \Pi(V) \arrow[r] & \mathcal{C} \end{tikzcd}$$

and we need to show that

$$\begin{tikzcd} \Pi(U \cap V) \arrow[r] \arrow[d] & \Pi(U) \arrow[d] \arrow[ddr] & \\ \Pi(V) \arrow[r] \arrow[drr] & \Pi(X) \arrow[dr, dashed, "\exists! F"] & \\ & & \mathcal{C} \end{tikzcd}$$

Uniqueness: Let $\gamma\colon I \to X$ be a path in X with $x_t = \gamma(t)$. We subdivide I (by its compactness) into

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

such that $\gamma_i := \gamma(t_{i-1}, t_i)$ lies entirely in U or V. Then

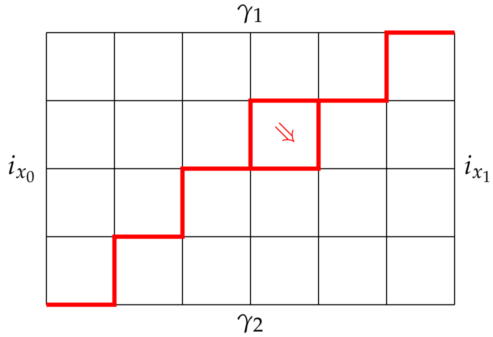
$$F([\gamma]) = F([\gamma_m]) \cdots F([\gamma_1])$$

is determined uniquely in $\mathcal{C}$ as each term is.

Existence: Given a path γ, we can define $F([\gamma])$ using a subdivision of I (or γ), where the result does not depend on the choice of the subdivision. We need to show that this is well-defined on homotopy class. This follows from a refined double subdivision of $I \times I$, as shown in the picture below. Each square represents a homotopy lying entirely in either U or V and combining them together gives the required homotopy.

$$F(\gamma_1) \simeq F(\gamma_1 \star i_{x_0}) \simeq F(i_{x_1} \star \gamma_2) \simeq F(\gamma_2)$$

□



Theorem 6.2 (Seifert-van Kampen Theorem, Group version)**.** *Let $X = U \cup V$ where $U, V \subset X$ are open and $U, V, U \cap V$ are path connected. Let $x_0 \in U \cap V$. Then the following diagram*

$$\begin{tikzcd}
\pi_1(U \cap V, x_0) \arrow[r] \arrow[d] & \pi_1(U, x_0) \arrow[d] \\
\pi_1(V, x_0) \arrow[r] & \pi_1(X, x_0)
\end{tikzcd}$$

is a pushout in the category <u>**Group**</u>.

Proof. Denote by $\underline{G}$ the groupoid with one object that comes from a group G.

For each $x \in X$, we fix a choice of $[\gamma_x] \in \mathrm{Hom}(x_0, x)$ such that γ_x lies entirely in U when $x \in U$ and γ_x lies entirely in V when $x \in V$. Note this implies that γ_x lies entirely in $U \cap V$ when $x \in U \cap V$. Such choice can be achieved because $U, V, U \cap V$ are all path connected. Consider the following functors

$$\begin{aligned}
\Pi_1(U) &\to \underline{\pi_1(U, x_0)} \\
\Pi_1(V) &\to \underline{\pi_1(V, x_0)} \\
\Pi_1(U \cap V) &\to \underline{\pi_1(U \cap V, x_0)} \\
\gamma &\mapsto \gamma_{x_2}^{-1} \star \gamma \star \gamma_{x_1}, \quad \gamma \in \mathrm{Hom}(x_1, x_2).
\end{aligned}$$

These functors are all retracts in <u>**Groupoid**</u>, in other words, the compositions

$$\begin{aligned}
\underline{\pi_1(U, x_0)} \to \Pi_1(U) &\to \underline{\pi_1(U, x_0)} \\
\underline{\pi_1(V, x_0)} \to \Pi_1(V) &\to \underline{\pi_1(V, x_0)} \\
\underline{\pi_1(U \cap V, x_0)} \to \Pi_1(U \cap V) &\to \underline{\pi_1(U \cap V, x_0)}
\end{aligned}$$

are all identity functors.

Suppose there is a group G that fits into the following commutative diagram:

$$\begin{tikzcd}
\Pi_1(U) \arrow[d] & \Pi_1(U \cap V) \arrow[l] \arrow[r] \arrow[d] & \Pi_1(V) \arrow[d] \\
\underline{\pi_1(U, x_0)} \arrow[dr] & \underline{\pi_1(U \cap V, x_0)} \arrow[l] \arrow[r] \arrow[d] & \underline{\pi_1(V, x_0)} \arrow[dl] \\
& \underline{G} &
\end{tikzcd}$$

By Theorem 6.1, we have the following morphism F

$$\begin{tikzcd}
\Pi_1(U) \arrow[d] \arrow[dr] & \Pi_1(U \cap V) \arrow[l] \arrow[r] & \Pi_1(V) \arrow[d] \arrow[dl] \\
\underline{\pi_1(U, x_0)} \arrow[dr] & \Pi_1(X) \arrow[d, dashed, "\exists ! F"] & \underline{\pi_1(V, x_0)} \arrow[dl] \\
 & \underline{G} &
\end{tikzcd}$$

Thus, we obtain a morphism

$$\underline{\pi_1(X, x_0)} \hookrightarrow \Pi_1(X) \xrightarrow{F} \underline{G}$$

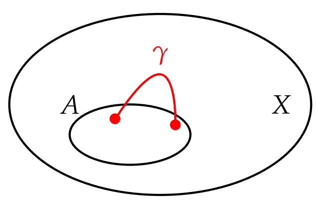
which fits into a commutative diagram

$$\begin{tikzcd}
\underline{\pi_1(U \cap V, x_0)} \arrow[r] \arrow[d] & \underline{\pi_1(U, x_0)} \arrow[d] \arrow[ddr] & \\
\underline{\pi_1(V, x_0)} \arrow[r] \arrow[drr] & \underline{\pi_1(X, x_0)} \arrow[dr] & \\
 & & \underline{G}
\end{tikzcd}$$

Since $\underline{\textbf{Group}}$ is a full subcategory of $\underline{\textbf{Groupoid}}$, the theorem follows. $\square$

We also have the relative version.

Definition 6.3. Let $A \subset X$, we define $\Pi_1(X, A)$ be the full subcategory of $\Pi_1(X)$ consists of objects in A.



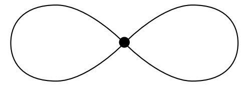
For instance, when $A = \{x_0\}$, we have

$$\Pi_1(X, x_0) = \pi_1(X, x_0).$$

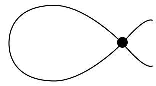
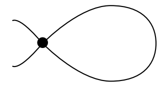
Theorem 6.4. *Let $X = U \cup V$, U, V be open and $A \subset X$ intersects each path connected components of $U, V, U \cap V$. Then we have a pushout*

$$\begin{tikzcd}
\Pi_1(U \cap V, A) \arrow[r] \arrow[d] & \Pi_1(U, A) \arrow[d] \\
\Pi_1(V, A) \arrow[r] & \Pi_1(X, A).
\end{tikzcd}$$

Example 6.5. For the Figure-8 in Example 3.9, which is $S^1 \vee S^1$.

$S^1 \vee S^1 =$ 

It can be decomposed into U, V as follows

$U =$  $V =$ 

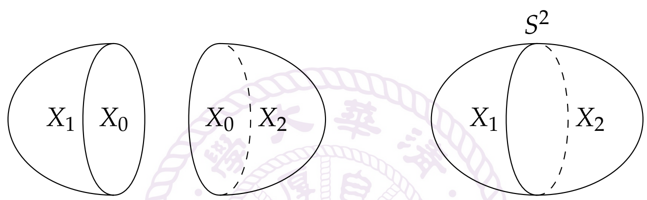
Since U, V are homotopic to S^1, and $U \cap V$ is homotopic to a point, Seifert-van Kampen Theorem implies

$$\pi_1(S^1 \vee S^1) = \pi_1(S^1) \star \pi_1(S^1) = \mathbb{Z} \star \mathbb{Z}.$$

In general, we have

$$\pi_1(\bigvee_{i=1}^{n} S^1) = \underbrace{\mathbb{Z} \star \cdots \star \mathbb{Z}}_{n}.$$

Example 6.6. Consider the 2-sphere $S^2 = D_1 \cup D_2$ where $D_i \cong D^2$ are open disks and $D_0 = D_1 \cap D_2$ is an annulus. Here D_i is an open neighbourhood of X_i for $i = 0, 1, 2$.



Since $\pi_1(D_1) = \pi_1(D_2) = 1, \pi_1(D_0) = \pi_1(S^1) = \mathbb{Z}$, we deduce that

$$\pi_1(S^2) = (1 \star 1)/\mathbb{Z} = 1.$$

Similar argument shows that

$$\pi_1(S^n) = 1, \quad n \geq 2.$$

Example 6.7. Let us identiy $X = S^1$ with the unit circle in $\mathbb{R}^2$. Consider

$$U = \{(x, y) \in S^1 \mid y > -1/2\}, \quad V = \{(x, y) \in S^1 \mid y < 1/2\}$$

and $A = \{(\pm 1, 0)\}$. Then we obtain a pushout by Theorem 6.4

$$\begin{tikzcd}
\Pi_1(U \cap V, A) \arrow[r] \arrow[d] & \Pi_1(U, A) \arrow[d] \\
\Pi_1(V, A) \arrow[r] & \Pi_1(S^1, A).
\end{tikzcd}$$

This implies that the groupoid $\Pi_1(S^1, A)$ contains two objects $a_1 = (1, 0), a_2 = (-1, 0)$ with morphisms

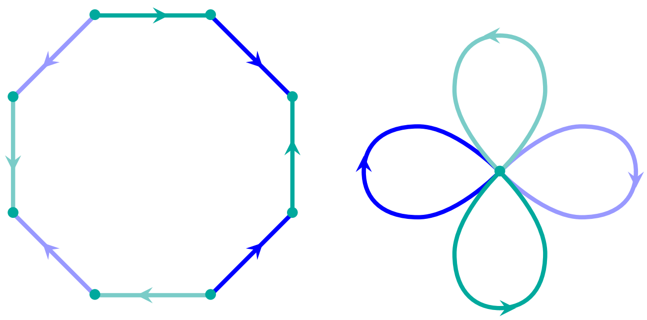
$$\begin{aligned}
\mathrm{Hom}_{\Pi_1(S^1,A)}(a_1, a_1) &= \{(\gamma_- \gamma_+)^n\}_{n \in \mathbb{Z}} \\
\mathrm{Hom}_{\Pi_1(S^1,A)}(a_1, a_2) &= \{(\gamma_+ \gamma_-)^n \gamma_+\}_{n \in \mathbb{Z}} \\
\mathrm{Hom}_{\Pi_1(S^1,A)}(a_2, a_1) &= \{(\gamma_- \gamma_+)^n \gamma_-\}_{n \in \mathbb{Z}} \\
\mathrm{Hom}_{\Pi_1(S^1,A)}(a_2, a_2) &= \{(\gamma_+ \gamma_-)^n\}_{n \in \mathbb{Z}}.
\end{aligned}$$

Here γ_+ represents the semi-circle from $(1, 0)$ to $(-1, 0)$ anti-clockwise, and γ_- represents the semi-circle from $(-1, 0)$ to $(1, 0)$ anti-clockwise.

Example 6.8. Consider the closed orientable surface Σ_g of genus g, which admits a polygon presentation

$$P = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

Here is a figure for $g = 2$.



The edges of the polygon form $V_{2g} = \bigvee_{i=1}^{2g} S^1$. Let U be the interior of the polygon and V be a small open neighbourhood of V_{2g}. Then $U \cap V$ is an annulus, which is homotopic to S_1 with generator P as above. Thus

$$\pi_1(\Sigma_g) = \left(\coprod_{i=1}^{2g} \mathbb{Z}\right) \star 0/\mathbb{Z} = \langle a_i, b_i \mid i = 1, \ldots, g\rangle \,/\, \left(a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}\right).$$

Example 6.9. Using the polygon presentation $P = a^2$, we can similarly compute $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$

The Jordan Curve Theorem

We give an application of Seifert-van Kampen Theorem to prove the Jordan Curve Theorem. This is an example which sounds totally obvious intuitively, but turns out to be very difficult to prove rigorously.

Definition 6.10. A simple closed curve is a subset of $\mathbb{R}^2$ (or S^2) which is homeomorphic to the circle S^1.

Theorem 6.11 (The Jordan Curve Theorem). *Let C be a simple closed curve in the sphere S^2. Then the complement of C has exactly two connected components.*

Proof. We sketch a proof here. Since S^2 is locally path connected, we would not distinguish connected and path connected here. By an arc, we mean a subset of S^2 which is homeomorphic to the interval I.

We first show that:

if A is an arc in S^2, then $S^2 \backslash A$ is connected.

In fact, assume that there are two points $\{a, b\}$ which are disconnected in $S^2 \backslash A$. Let us subdivide $A = A_1 \cup A_2$ into two intervals where $A_1 = [0, 1/2], A_2 = [1/2, 1]$ using the homeomorphism $A \cong [0, 1]$. We argue that a, b are disconnected in either $S^2 \backslash A_1$ or $S^2 \backslash A_2$. Let us choose a set D which contains one point from each connected component of $S^2 \backslash A$ and such that $\{a, b\} \subset D$. Apply Seifert-van Kampen Theorem to $V_1 = S^2 \backslash A_1, V_2 = S^2 \backslash A_2, V_1 \cap V_2 = S^2 \backslash A$, we obtain a pushout in **Groupoid**

$$\begin{tikzcd}
\Pi_1(V_1 \cap V_2, D) \arrow[r] \arrow[d] & \Pi_1(V_2, D) \arrow[d] \\
\Pi_1(V_1, D) \arrow[r] & \Pi_1(Y, D).
\end{tikzcd}$$

Here $Y = V_1 \cup V_2$ is the complement of a point in S^2. If $\{a, b\}$ are connected in both V_1 and V_2, then the pushout implies that there exists a nontrivial morphism via the composition

$$a \xrightarrow{\text{in } V_1} b \overset{\text{in } V_1 \cap V_2}{=\!=\!=} b \xrightarrow{\text{in } V_2} a.$$

But this can not true since Y is contractible. So let us assume a, b are disconnected in $V_1 = S^2 \backslash A_1$. Run the above process replacing A by A_1, and keep doing this, we end up with contradiction in the limit. This proves our claim above for the arc.

Secondly, we show that:

$$\text{the complement of } C \text{ in } S^2 \text{ is disconnected.}$$

Otherwise, assume that $S^2 \backslash C$ is connected. Let us divide $C = A_1 \cup A_2$ into two intervals A_1, A_2 which intersect at two endpoints $\{a, b\}$. Let $U_1 = S^2 \backslash A_1, U_2 = S^2 \backslash A_2, U_1 \cap U_2 = S^2 \backslash C$ and $X = U_1 \cup U_2 = S^2 \backslash \{a, b\}$. Since $U_1, U_2, U_1 \cap U_2$ are all connected, Seifert-van Kampen Theorem leads to a pushout in **Group**

$$\begin{tikzcd}
\pi_1(U_1 \cap U_2) \arrow[r] \arrow[d] & \pi_1(U_2) \arrow[d] \\
\pi_1(U_1) \arrow[r] & \pi_1(X).
\end{tikzcd}$$

Observe $\pi_1(X) = \mathbb{Z}$. We show both $\pi_1(U_i) \to \pi_1(X)$ are trivial. This would lead to a contradiction.

Let us identify $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and assume $a = 0, b = \infty$, so A_1 is parametrized by a path α from 0 to ∞. Let γ be an arbitrary loop in U_1, we need to show γ becomes trivial in X. Let $R > 0$ be sufficient large such that γ is contained in the ball of radius R centered at the origin in $\mathbb{R}^2$. Consider the homotopy

$$F(t, s) = \gamma(t) - \alpha(s), \quad \gamma_s := F(-, s).$$

We have $\gamma_0 = \gamma$. Assume that $\alpha(t_0) > R$, then γ_{t_0} lies inside the ball of radius R centered at $\alpha(t_0)$, which is contractible in X. This implies that γ is trivial in X. The same argument applies to A_2.

Finally, we show that:

$$\text{the complement of } C \text{ in } S^2 \text{ has exactly two connected components.}$$

Let $C = A_1 \cup A_2$ and U_1, U_2 as in the previous step. Let D be a set which contains exactly one point from each connected component of $S^2 \backslash C$. We have a pushout in **Groupoid**

$$\begin{tikzcd}
\Pi_1(U_1 \cap U_2, D) \arrow[r] \arrow[d] & \Pi_1(U_2, D) \arrow[d] \\
\Pi_1(U_1, D) \arrow[r] & \Pi_1(X, D).
\end{tikzcd}$$

Suppose that D contains at least three points, say c, d, e. Since U_1, U_2 are connected, and points in D are disconnected in $U_1 \cap U_2$, the following two compositions

$$c \xrightarrow{\text{in } U_1} d \xlongequal{\text{in } U_1 \cap U_2} d \xrightarrow{\text{in } U_2} c \quad \text{and} \quad c \xrightarrow{\text{in } U_1} e \xlongequal{\text{in } U_1 \cap U_2} e \xrightarrow{\text{in } U_2} c$$

give two free generators in $\pi_1(X, c)$. But $\pi_1(X, c) = \mathbb{Z}$, contradiction. $\square$

7 A CONVENIENT CATEGORY OF SPACES

In homotopy theory, it would be convenient to work with a category of spaces which has all limits, colimits, and enjoys nice properties about mapping space (especially the Exponential Law). The full category $\underline{\mathbf{Top}}$ does not work since the Exponential Law fails. The subcategory of locally compact Hausdorff spaces has the Exponential Law, but does not preserve limits and colimits in general. It turns out that there is some complete and cocomplete category that sits in between locally compact Hausdorff spaces and all topological spaces, and enjoys the Exponential Law. Compactly generated weak Hausdorff spaces give such a category $\underline{\mathbf{CGWH}}$, which we briefly discuss in this section. This will be a convenient category for homotopy theory.

Compactly generated space

Definition 7.1. A subset $Y \subset X$ is called "**compactly closed**" (or "**k-closed**") if $f^{-1}(Y)$ is closed in K for every continuous map $f : K \to X$ with K compact Hausdorff. We define a new topology on X, denoted by kX, where close subsets of kX are compactly closed subsets of X. The identity

$$kX \to X$$

is a continuous map. X is called **compactly generated** if $kX = X$.

Let $\underline{\mathbf{CG}}$ denote the full subcategory of $\underline{\mathbf{Top}}$ consisting of compactly generated spaces.

If a space X is compactly generated, then for any Y, a map $f : X \to Y$ is continuous if and only if the composition $K \to X \to Y$ is continuous for any continuous $K \to X$ with K compact Hausdorff. Note

$$k^2X = kX.$$

Proposition 7.2. *Every locally compact Hausdorff space is compactly generated.*

Proof. Let X be locally compact Hausdorff, and Z be a k-closed subset. We need to show $Z = \bar{Z}$ is closed.

Let $x \in \bar{Z}$. Since X is locally compact Hausdorff, x has a neighborhood U with $K = \bar{U}$ compact Hausdorff. Then $x \in \overline{K \cap Z}$. Since Z is k-closed, $K \cap Z$ is closed in K, hence closed in X. So $x \in Z$. □

Proposition 7.3. *The assignment $X \to kX$ defines a functor $\underline{\mathbf{Top}} \to \underline{\mathbf{CG}}$, which is right adjoint to the embedding $i : \underline{\mathbf{CG}} \subset \underline{\mathbf{Top}}$. In other words, we have an adjoint pair*

$$i : \underline{\mathbf{CG}} \rightleftarrows \underline{\mathbf{Top}} : k$$

Proof. Let $X \in \underline{\mathbf{CG}}, Y \in \underline{\mathbf{Top}}$, we need to show that $f : X \to Y$ is continuous if and only if the same map $f : X \to kY$ is continuous. Assume $f : X \to kY$ is continuous. Then the composition $X \to kY \to Y$ is continuous. Conversely, assume $f : X \to Y$ is continuous. Let $Z \subset Y$ be a k-closed subset. Then for any $g : K \to X$ with K compact Hausdorff,

$$g^{-1}(f^{-1}(Z)) = (f \circ g)^{-1}(Z)$$

is closed in K. It follows that $f^{-1}(Z)$ is k-closed in X, hence closed. So $f : X \to kY$ is continuous. □

Proposition 7.4. *Let $X \in \underline{\mathbf{CG}}$ and $p : X \to Y$ be a quotient map. Then $Y \in \underline{\mathbf{CG}}$.*

Proof. By Proposition 7.3, p factor through $X \to kY$. Since the quotient topology is the finest topology making the quotient map continuous, we find $Y = kY$. □

Theorem 7.5. *The category $\underline{\mathbf{CG}}$ is complete and cocomplete. Colimits in $\underline{\mathbf{CG}}$ inherit the colimits in $\underline{\mathbf{Top}}$. The limits in $\underline{\mathbf{CG}}$ are obtained by applying k to the limits in $\underline{\mathbf{Top}}$.*

Proof. Let $F \in \mathrm{Fun}(I, \underline{\mathbf{CG}})$ and $\hat{F} = i \circ F \in \mathrm{Fun}(I, \underline{\mathbf{Top}})$ where $i : \underline{\mathbf{CG}} \to \underline{\mathbf{Top}}$ is the embedding.

The left adjoint functor $i : \underline{\mathbf{CG}} \to \underline{\mathbf{Top}}$ preserves colimits. Since $F(i) \in \underline{\mathbf{CG}}$, their coproduct $\coprod_{i\in I} F(i)$ in $\underline{\mathbf{Top}}$ (given by the disjoint union) is in $\underline{\mathbf{CG}}$. Since $\mathbf{colim}\,\hat{F}$ is a quotient of $\coprod_{i\in I} F(i)$, it also lies in $\underline{\mathbf{CG}}$ by Proposition 7.4. This implies the statement about $\mathbf{colim}\,F$.

The right adjoint functor $k : \underline{\mathbf{Top}} \to \underline{\mathbf{CG}}$ preserves limits. Therefore

$$\mathbf{lim}\,F = \mathbf{lim}(k \circ \hat{F}) = k\,\mathbf{lim}\,\hat{F}.$$

□

Corollary 7.6. *Let $\{X_i\}_{i\in I}$ be a family of objects in $\underline{\mathbf{CG}}$. Then their product in $\underline{\mathbf{CG}}$ is*

$$k\prod_{i\in I} X_i$$

where $\prod\limits_{i\in I} X_i$ is the topological product of X_i's.

Definition 7.7.

$$\boxed{\text{We will use } \times, \prod \text{ to denote the product in } \underline{\mathbf{CG}} \text{ and } \overset{t}{\times}, \overset{t}{\prod} \text{ to denote the product in } \underline{\mathbf{Top}}.}$$

Proposition 7.8. *Assume X is compactly generated and Y is locally compact Hausdorff, then $X \times Y = X \overset{t}{\times} Y$.*

Definition 7.9. Let $X, Y \in \underline{\mathbf{CG}}$. We define the compactly generated topology on $\mathrm{Hom}_{\underline{\mathbf{Top}}}(X, Y)$ by

$$\mathrm{Map}(X, Y) = kC(X, Y) \in \underline{\mathbf{CG}}.$$

Here $C(X, Y)$ is the compact-open topology generated by

$$\{f \in \mathrm{Hom}_{\underline{\mathbf{Top}}}(X, Y) | f(g(K)) \subset U\}, \quad \text{where } g : K \to X \text{ with } K \text{ compact Hausdorff and } U \subset Y \text{ is open.}$$

Note that the compact-open topology we use here for $\underline{\mathbf{CG}}$ is slightly different from the usual one: we ask for a map from K which is compact Hausdorff. We will also use the exponential notation

$$Y^X := \mathrm{Map}(X, Y).$$

Lemma 7.10. *Let $X, Y \in \underline{\mathbf{CG}}$, K compact Hausdorff, and $f : K \to X$ continuous. Then the evaluation map*

$$ev_K : \mathrm{Map}(X, Y) \overset{t}{\times} K \to Y, \quad (g, k) \to g(f(k))$$

is continuous. In particular, $\mathrm{Map}(X, Y) \times K \to Y$ *is continuous.*

Proof. Let $U \subset Y$ be open, and $(g, k) \in ev_K^{-1}(U)$. Then $g \circ f^{-1}(U)$ is open in K and contains k. Since K is compact Hausdorff, k has a neighborhood V such that $\bar{V} \subset g \circ f^{-1}(U)$. Then

$$\{h | h(f(\bar{V})) \subset U\} \times V$$

is an open neighborhood of (g, k).

□

Proposition 7.11. *Let $X, Y \in \underline{\mathbf{CG}}$. Then the evaluation map* $\mathrm{Map}(X, Y) \times X \to Y$ *is continuous.*

Proof. Let K be compact Hausdorff, and a continuous map $K \to \mathrm{Map}(X,Y) \times X$. We need to show the composition $K \to \mathrm{Map}(X,Y) \times X \to Y$ is continuous. But this is the same as the composition

$$K \to \mathrm{Map}(X,Y) \times K \to Y$$

which is continuous by the previous lemma. □

Proposition 7.12. *Let $X, Y, Z \in \underline{\mathbf{CG}}$ and $f : X \times Y \to Z$ continous. Then the induced map*

$$\hat{f} : X \to \mathrm{Map}(Y,Z), \quad \{x \to f(x,-) | x \in X\}$$

is also continuous.

Proof. We need to show $\hat{f} : X \to C(Y,Z)$ is continuous. Let $h : K \to Y$ be a continuous map with K compact Hausdorff, and $U \subset Z$ open. Let

$$W = \{g : Y \to Z | g(h(K)) \subset U\}.$$

Let $x \in \hat{f}^{-1}(W)$, i.e., $f(x, h(K)) \subset U$. Since f is continuous and K is compact, there exists an open neighborhood V of x such that $f(V, h(K)) \subset U$. Then $V \subset f^{-1}(W)$ as required. □

Theorem 7.13 (Exponential Law)**.** *Let $X, Y, Z \in \underline{\mathbf{CG}}$. Then the natural map*

$$\mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y,Z)), \quad f \to \{x \to f(x,-) | x \in X\}$$

is a homeomorphism.

Proof. We first show that

$$\mathrm{Hom}_{\underline{\mathbf{Top}}}(X \times Y, Z) \to \mathrm{Hom}_{\underline{\mathbf{Top}}}(X, \mathrm{Map}(Y,Z))$$

is a set isomorphism. Note that this map is well-defined by Proposition 7.12, which is obviously injective.

For any continuous $g : X \to \mathrm{Map}(Y,Z)$, we obtain

$$f : X \times Y \stackrel{g \times 1}{\to} \mathrm{Map}(Y,Z) \times Y \to Z$$

which is continuous. This proves the surjectivity and we have established the set isomorphism.

The fact on homeomorphism is a formal consequence. In fact, for any $W \in \underline{\mathbf{CG}}$, we have

$$\begin{aligned}
\mathrm{Hom}_{\underline{\mathbf{Top}}}(W, \mathrm{Map}(X \times Y, Z)) &\cong \mathrm{Hom}_{\underline{\mathbf{Top}}}(W \times X \times Y, Z) \\
&\cong \mathrm{Hom}_{\underline{\mathbf{Top}}}(W \times X, \mathrm{Map}(Y,Z)) \\
&\cong \mathrm{Hom}_{\underline{\mathbf{Top}}}(W, \mathrm{Map}(X, \mathrm{Map}(Y,Z))).
\end{aligned}$$

This says that we have a natural isomorphism between the two functors

$$\mathrm{Hom}_{\underline{\mathbf{Top}}}(-, \mathrm{Map}(X \times Y, Z)) \cong \mathrm{Hom}_{\underline{\mathbf{Top}}}(-, \mathrm{Map}(X, \mathrm{Map}(Y,Z))) : \underline{\mathbf{CG}} \to \underline{\mathbf{Set}}.$$

Then Yoneda Lemma gives rise to the homeomorphism

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y,Z)).$$

□

Proposition 7.14. *Let $X, Y, Z \in \underline{\mathbf{CG}}$. Then the composition*

$$\mathrm{Map}(X,Y) \times \mathrm{Map}(Y,Z) \to \mathrm{Map}(X,Z), \quad (f,g) \to g \circ f$$

is continuous, i.e., a morphism in $\underline{\mathbf{CG}}$.

Proof. This follows from the Exponential Law. By Yoneda Lemma, we need to find a natural transformation

$$\mathrm{Hom}_{\underline{\mathbf{Top}}}(W, \mathrm{Map}(X,Y) \times \mathrm{Map}(Y,Z)) \to \mathrm{Hom}_{\underline{\mathbf{Top}}}(W, \mathrm{Map}(X,Z)), \quad \forall W \in \underline{\mathbf{CG}}.$$

First we observe that

$$\begin{aligned}\mathrm{Hom}_{\underline{\mathbf{Top}}}(W, \mathrm{Map}(X,Y) \times \mathrm{Map}(Y,Z)) &\cong \mathrm{Hom}_{\underline{\mathbf{Top}}}(W, \mathrm{Map}(X,Y)) \times \mathrm{Hom}_{\underline{\mathbf{Top}}}(W, \mathrm{Map}(Y,Z)) \\ &\cong \mathrm{Hom}_{\underline{\mathbf{Top}}}(W \times X, Y) \times \mathrm{Hom}_{\underline{\mathbf{Top}}}(W \times Y, Z).\end{aligned}$$

Now given two maps $f : W \times X \to Y, g : W \times Y \to Z$, we consider the composition

$$W \times X \xrightarrow{\Delta \times 1_X} W \times W \times X \xrightarrow{1 \times f} W \times Y \xrightarrow{g} Z.$$

Here $\Delta : W \to W \times W$ is the diagonal map. This gives naturally the required element of

$$\mathrm{Hom}_{\underline{\mathbf{Top}}}(W \times X, Z) \cong \mathrm{Hom}_{\underline{\mathbf{Top}}}(W, \mathrm{Map}(X,Z)).$$

□

Another nice property of the category $\underline{\mathbf{CG}}$ is that product of quotient maps is a quotient.

Proposition 7.15. *Let $p_i : X_i \to Y_i, i = 1, 2$, be quotients in $\underline{\mathbf{CG}}$. Then $p_1 \times p_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a quotient.*

Proof. We only need to show that if $p : X \to Y$ is a quotient map, then the induced map $q : X \times Z \to Y \times Z$ is a quotient. Here $X, Y, Z \in \underline{\mathbf{CG}}$. Evidently, q is surjective on sets. This is equivalent to show that for any map $f : Y \times Z \to W$, if $q \circ f$ is continuous, that f is continuous. By the Exponential Law,

$$\mathrm{Hom}_{\underline{\mathbf{Top}}}(X \times Z, W) = \mathrm{Hom}_{\underline{\mathbf{Top}}}(X, \mathrm{Map}(Z, W)).$$

So $q \circ f$ is equivalent to a continuous map $X \to \mathrm{Map}(Z, W)$. Since $p : X \to Y$ is a quotient, this shows that f corresponds to a continuous map $Y \to \mathrm{Map}(Z, W)$. Using the Exponential Law again,

$$\mathrm{Hom}_{\underline{\mathbf{Top}}}(Y, \mathrm{Map}(Z, W)) = \mathrm{Hom}_{\underline{\mathbf{Top}}}(Y \times Z, W).$$

This implies the continuity of f. □

Compactly generated weak Hausdorff space

Definition 7.16. A space X is **weak Hausdorff** if for every compact Hausdorff K and every continuous map $f : K \to X$, the image $f(K)$ is closed in X.

Let $\underline{\mathbf{wH}}$ denote the full subcategory of $\underline{\mathbf{Top}}$ consisting of weak Hausdorff spaces. Let $\underline{\mathbf{CGWH}}$ denote the full subcategory of $\underline{\mathbf{Top}}$ consisting of compactly generated weak Hausdorff spaces.

Example 7.17. Hausdorff spaces are weak Hausdorff since compact subsets of Hausdorff spaces are closed. Therefore locally compact Hausdorff spaces are compactly generated weak Hausdorff spaces.

Proposition 7.18. *The functor $k : \underline{\mathbf{wH}} \to \underline{\mathbf{CGWH}}$ is right adjoint to the embedding $i : \underline{\mathbf{CGWH}} \subset \underline{\mathbf{wH}}$. In other words, we have an adjoint pair*

$$i : \underline{\mathbf{CGWH}} \rightleftarrows \underline{\mathbf{wH}} : k$$

Proof. This follows from Proposition 7.3.

□

Lemma 7.19. *Let $X \in \underline{\mathbf{wH}}$, K compact Hausdorff, and $f : K \to X$ continuous. Then $f(K)$ is compact Hausdorff.*

Proof. $f(K)$ is compact and closed. Moreover, $f : K \to X$ is a closed map by hypothesis.

Let $x_1, x_2 \in f(K)$ be two points. Since $X \in \underline{\mathbf{wH}}$, x_1, x_2 are closed, hence $f^{-1}(x_1), f^{-1}(x_2)$ are disjoint closed. Since K is compact Hausdorff, there exists disjoint open subsets U_1, U_2 of K such that $f^{-1}(x_i) \subset U_i$. Then $f(K) - f(K - U_i)$ give disjoint open neighborhoods of x_i.

□

Remark 7.20. For a weak Hausdorff X, this lemma says that $Z \subset X$ is k-closed if and only if $Z \cap K$ is closed in K for any compact Hausdorff subspace $K \subset X$.

Proposition 7.21. *Let $X \in \underline{\mathbf{CG}}$. Then X is weak Hausdorff if and only if the diagonal subspace $\Delta_X = \{(x,x) | x \in X\}$ is closed in $X \times X$. Here $X \times X$ is the product in the category $\underline{\mathbf{CG}}$.*

Proof. Assume $X \in \underline{\mathbf{CGWH}}$. We need to show that Δ_X is k-closed in $X \times X$. Let

$$f = (f_1, f_2) : K \to X \times X, \quad f_i : K \to X$$

where K is compact Hausdorff. Let

$$L = f_1(K) \cap f_2(K)$$

which is compact Hausdorff by Lemma 7.19. Consider the diagonal Δ_L in $L \times L$, which lies in the image

$$L \to X \times X.$$

Since L is compact Hausdorff, Δ_L is a compact Hausdorff subspace of $X \times X$, hence closed in $X \times X$. It follows that $f^{-1}(\Delta_X) = f^{-1}(\Delta_L)$ is closed.

Conversely, assume $X \in \underline{\mathbf{CG}}$ and Δ_X is closed in $X \times X$. Let $f : K \to X$ be a continuous map with K compact Hausdorff. We need to show $f(K)$ is k-closed in X. Let $g : L \to X$ be any continuous map with L compact Hausdorff. Consider

$$(f, g) : K \times L \to X \times X.$$

Then

$$g^{-1}(f(K)) = (f, g)^{-1}(\Delta_X)$$

which is closed. This shows that $f(K)$ is k-closed in X, hence closed in X. □

Remark 7.22. Recall that $X \in \underline{\mathbf{Top}}$ is Hausdorff if and only if Δ_X is closed in $X \times X$. This proposition says that $\underline{\mathbf{CGWH}}$ relative to $\underline{\mathbf{CG}}$ is the analogue of Hausdorff spaces relative to $\underline{\mathbf{Top}}$.

Corollary 7.23. *Let $\{X_i\}_{i \in I}$ be a family of objects in $\underline{\mathbf{CGWH}}$. Then their product $\prod\limits_{i \in I} X_i$ in $\underline{\mathbf{CG}}$ also lies in $\underline{\mathbf{CGWH}}$.*

Proof. Let $X = \prod\limits_{i \in I} X_i$ with $\pi_i : X \to X_i$. We need to show that the diagonal Δ_X is closed in $X \times X$. Let

$$\pi_i \times \pi_i : X \times X \to X_i \times X_i, \quad D_i = (\pi_i \times \pi_i)^{-1}(\Delta_{X_i}).$$

Since Δ_i is closed in $X_i \times X_i$, it follows that $\Delta_X = \cap_{i \in I} D_i$ is closed in $X \times X$. □

Proposition 7.24. *Let $X \in \underline{\mathbf{CG}}$, and $E \subset X \times X$ be an equivalence relation on X. Then the quotient space X/E by the equivalence relation E lies in $\underline{\mathbf{CGWH}}$ if and only if E is closed in $X \times X$.*

Proof. By Proposition 7.4, $X/E \in \underline{\mathbf{CG}}$. We need to check the weak Hausdorff property.

Let $q : X \to Y = X/E$ denote the quotient map. By Proposition 7.15, the product

$$q \times q : X \times X \to Y \times Y$$

is also a quotient map. So Δ_Y is closed in $Y \times Y$ if and only if $(q \times q)^{-1}(\Delta_Y) = E$ is closed in $X \times X$. □

Given $X \in \underline{\mathbf{CG}}$, let E_X denote the smallest closed equivalence relation on X. E_X is constructed as the intersection of all closed equivalence relations on X. Then the quotient X/E_X by the equivalence relation E_X is an object in $\underline{\mathbf{CGWH}}$. This construction is functorial, so defines a functor

$$h : \underline{\mathbf{CG}} \to \underline{\mathbf{CGWH}}\,.$$

Proposition 7.25. *The functor $h : \underline{\mathbf{CG}} \to \underline{\mathbf{CGWH}}$ is left adjoint to the inclusion $j : \underline{\mathbf{CGWH}} \to \underline{\mathbf{CG}}$. That is, we have an adjoint pair*

$$h : \underline{\mathbf{CG}} \rightleftarrows \underline{\mathbf{CGWH}} : j$$

Moreover, h preserves the subcategory $\underline{\mathbf{CGWH}}$, i.e, $h \circ j$ is the identity functor.

Proof. Let $X \in \underline{\mathbf{CG}}, Y \in \underline{\mathbf{CGWH}}$, and $f : X \to Y$ continuous. We need to show that f factors through $X/E_X \to Y$. Consider

$$f \times f : X \times X \to Y \times Y.$$

Since Δ_Y is closed in $Y \times Y$, $(f \times f)^{-1}(\Delta_Y)$ defines a closed equivalence relation on X. Therefore $E_X \subset (f \times f)^{-1}(\Delta_Y)$. It follows that f factors through $X \to X/E_X \to Y$.

□

Theorem 7.26. *The category $\underline{\mathbf{CGWH}}$ is complete and cocomplete. Limits in $\underline{\mathbf{CGWH}}$ inherit the limits in $\underline{\mathbf{CG}}$. The colimits in $\underline{\mathbf{CGWH}}$ are obtained by applying h to the colimits in $\underline{\mathbf{CG}}$.*

Proof. Let $F \in \mathrm{Fun}(I, \underline{\mathbf{CGWH}})$, then we need to show

$$\mathbf{colim}\, F = h\left(\mathbf{colim}(j \circ F)\right), \quad j(\mathbf{lim}\, F) = \mathbf{lim}(j \circ F).$$

The statement about colimit follows from the fact that $h \circ j$ is the identity functor and h perserves colimits. For the limit, let

$$X = \prod_{i \in I} F(i), \quad Y = \prod_{i \xrightarrow{f} j} F(j)$$

be the products in $\underline{\mathbf{CG}}$, which also lie in $\underline{\mathbf{CGWH}}$ by Lemma 7.23. Consider two maps $g_1, g_2 : X \to Y$ where

$$g_1(\{x_i\}) = \{x_j\}_{i \xrightarrow{f} j}, \quad g_2(\{x_i\}) = \{f(x_i)\}_{i \xrightarrow{f} j}.$$

Then

$$\mathbf{lim}(j \circ F) = \{x \in X | g_1(x) = g_2(x)\} = (g_1 \times g_2)^{-1}(\Delta_Y)$$

is a closed subspace of X, hence also lies in $\underline{\mathbf{CGWH}}$. It can be checked that this is the limit of F. □

Remark 7.27. The proof of the limit part of this theorem does not rely on h. It shows that $j : \underline{\mathbf{CGWH}} \to \underline{\mathbf{CG}}$ preserves all limits. The Adjoint Functor Theorem implies an abstract existence of h.

Proposition 7.28. *Let $X, Y \in \underline{\mathbf{CGWH}}$. Then* $\mathrm{Map}(X, Y) \in \underline{\mathbf{CGWH}}$.

Proof. We need to show that the diagonal $\Delta_{\mathrm{Map}(X,Y)}$ in $\mathrm{Map}(X, Y) \times \mathrm{Map}(X, Y)$ is closed. Let

$$ev_x : \mathrm{Map}(X, Y) \to Y, \quad f \to f(x), \quad \text{for } x \in X,$$

which is continuous. Then

$$\Delta_{\mathrm{Map}(X,Y)} = \cap_{x \in X}(ev_x \times ev_x)^{-1}(\Delta_Y)$$

is closed since Δ_Y is closed in $Y \times Y$. □

Theorem 7.29. *Let $X, Y, Z \in \underline{\mathbf{CGWH}}$. Then*

1°. *the evaluation map* $\mathrm{Map}(X,Y) \times X \to Y$ *is continuous;*
2°. *the composition map* $\mathrm{Map}(X,Y) \times \mathrm{Map}(Y,Z) \to \mathrm{Map}(X,Z)$ *is continuous;*
3°. *the Exponential Law holds: we have a homeomorphism*

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y,Z)).$$

Proof. The theorem follows from Proposition 7.28, Proposition 7.11, Theorem 7.13, Proposition 7.14.

□

Therefore **CGWH** is a full complete and cocomplete subcategory of **Top** that enjoys the Exponential Law.

We give a brief discussion on "subspace topology" in **CG** to end this section.

Let $X \in$ **CG** and A be a subset of X. The subspace topology on A may not be compactly generated. We equip A with a compactly generated topology by applying k to the usual subspace topology. This will be called the subspace topology in the category **CG**. When we write $A \subset X$, A is understood as s subspace of X with this compatly generated topology. It is clear that if $X \in$ **CGWH**, then $A \in$ **CGWH**. It can be checked that if A is the intersection of an open and a closed subset of X, then the usual subspace topology on A is already compactly generated, so these two notions of subspace coincide in this case.

This new notion of subspace satisfies the standard characteristic property in **CG**: given $Y \in$ **CG**, a map $Y \to A$ is continuous if and only if it is continuous viewed as a map $Y \to X$.

Definition 7.30. In the category **CG**, a map $i: A \to X$ in **CG** is called an **inclusion** if $A \to i(A)$ is a homeomorphism, where $i(A)$ is the image of A with the compactly generated subspace topology from X.

Proposition 7.31. *Let* $X \xrightarrow{i} Y \xrightarrow{r} X$ *be maps in* **CGWH** *such that* $r \circ i = 1_X$*. Then* i *is a closed inclusion and* r *is a quotient map.*

Proof. It is clear that i is an inclusion and r is a quotient. We show $i(X)$ is a closed inclusion. Consider

$$(i \circ r, 1_Y): Y \to Y \times Y.$$

Let $\Delta_Y \subset Y \times Y$ be the diagonal which is closed. Then $i(X) = (i \circ r, 1_Y)^{-1}(\Delta_Y)$ is also closed. □

Proposition 7.32. *Let* $X, Y, Z \in$ **CGWH** *and* $i: X \to Y$ *is an inclusion. Then* $i \times 1_Z : X \times Z \to Y \times Z$ *is also an inclusion. If* i *is closed, then so is* $i \times 1_Z$*.*

We will often need the notion of a pair. Given $X, Y \in$ **CGWH**, and subspaces $A \subset X, B \subset Y$, we let

$$\mathrm{Map}((X,A),(Y,B)) = \{f \in \mathrm{Map}(X,Y) | f(A) \subset B\}$$

be the subspace of $\mathrm{Map}(X,Y)$ that maps A to B. It fits into the following pull-back diagram

$$\begin{tikzcd}
\mathrm{Map}((X,A),(Y,B)) \arrow[r] \arrow[d] & \mathrm{Map}(X,Y) \arrow[d] \\
\mathrm{Map}(A,B) \arrow[r] & \mathrm{Map}(A,Y).
\end{tikzcd}$$

In our later discussion on homotopy theory, we will mainly work with **CGWH**. In particular, a space there always means an object in **CGWH**. All the limits and colimits are in **CGWH**. For example, given $X, Y \in$ **CGWH**, their product $X \times Y$ always means the categorical product in **CGWH**. Subspace refers to the compacted generated subspace topology.

To simplify notations, we will write

$$\underline{\mathscr{T}} = \underline{\mathbf{CGWH}}, \quad \underline{\mathrm{h}\mathscr{T}}$$

for the category **CGWH**, the quotient category of $\underline{\mathscr{T}}$ by homotopy classes of maps.

We will also need the category of pointed spaces.

Definition 7.33. We define the category $\underline{\mathscr{T}_\star}$ of pointed spaces where

- an object (X, x_0) is a space $X \in \underline{\mathscr{T}}$ with a based point $x_0 \in X$
- morphisms are based continuous maps that map based point to based point

$$\mathrm{Hom}_{\underline{\mathscr{T}_\star}}((X, x_0), (Y, y_0)) = \mathrm{Map}((X, x_0), (Y, y_0)).$$

We will write

$$\mathrm{Map}_\star(X, Y) = \mathrm{Map}((X, x_0), (Y, y_0))$$

when base points are not explicitly mentioned. $\mathrm{Map}_\star(X, Y)$ is viewed as an object in $\underline{\mathscr{T}_\star}$, whose base point is the constant map from X to the base point of Y.

The following theorem follows from the analogue for $\underline{\mathscr{T}}$ described above.

Theorem 7.34. *The category $\underline{\mathscr{T}_\star}$ is complete and cocomplete. Let $X, Y, Z \in \underline{\mathscr{T}_\star}$. Then*

1°. *the evaluation map* $\mathrm{Map}_\star(X, Y) \wedge X \to Y$ *is continuous;*
2°. *the composition map* $\mathrm{Map}_\star(X, Y) \wedge \mathrm{Map}_\star(Y, Z) \to \mathrm{Map}_\star(X, Z)$ *is continuous;*
3°. *the Exponential Law holds: we have a homeomorphism*

$$\mathrm{Map}_\star(X \wedge Y, Z) \cong \mathrm{Map}_\star(X, \mathrm{Map}_\star(Y, Z)).$$

Here $\wedge$ is the smash product

$$X \wedge Y = \frac{X \times Y}{X \times \{y_0\} \cup \{x_0\} \times Y}.$$

8 GROUP OBJECT AND LOOP SPACE

Definition 8.1. Let $X, Y \in \underline{\mathscr{T}_\star}$ be two pointed spaces. A based homotopy between two based maps $f_0, f_1 : X \to Y$ is a homotopy between f_0, f_1 relative to the base points. We denote $[X, Y]_0$ to be based homotopy classes of based maps. We define the category $\underline{\mathrm{h}\mathscr{T}_\star}$ by the quotient of $\underline{\mathscr{T}_\star}$ where

$$\mathrm{Hom}_{\underline{\mathrm{h}\mathscr{T}_\star}}(X, Y) = [X, Y]_0.$$

Definition 8.2. Given $(X, x_0) \in \underline{\mathscr{T}_\star}$, we define the **based loop space** $\Omega_{x_0} X$ or simply ΩX by

$$\Omega X = \mathrm{Map}_\star(S^1, X).$$

In the unpointed case, we define the **free loop space**

$$\mathcal{L}X = \mathrm{Map}(S^1, X).$$

Our goal in this section is to explore some basic algebraic structures of based loop spaces.

Theorem 8.3. *The based loop space Ω defines functors*

$$\Omega : \underline{\mathscr{T}_\star} \mapsto \underline{\mathscr{T}_\star}, \quad \Omega : \underline{\mathrm{h}\mathscr{T}_\star} \mapsto \underline{\mathrm{h}\mathscr{T}_\star}.$$

Proof. Let us first consider $\Omega : \underline{\mathscr{T}_\star} \mapsto \underline{\mathscr{T}_\star}$. This amounts to show that given $f : X \to Y$, the induced map

$$f_* : \mathrm{Map}_\star(S^1, X) \to \mathrm{Map}_\star(S^1, Y), \quad \gamma \to f \circ \gamma$$

is continuous. This follows from Proposition 7.14 since this map is the same as

$$\mathrm{Map}_\star(S^1, X) \times \{f\} \to \mathrm{Map}_\star(S^1, Y).$$

Now we consider $\Omega : \underline{\mathrm{h}\mathscr{T}_\star} \mapsto \underline{\mathrm{h}\mathscr{T}_\star}$. We need to show that if we have a homotopy $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & Y \end{tikzcd}$ realized by $F : X \times I \to Y$, then the induced maps $f_*, g_* : \mathrm{Map}_\star(S^1, X) \to \mathrm{Map}_\star(S^1, Y)$ are also homotopic. The required homotopy is given by

$$\Omega F : \Omega X \times I \to \Omega Y, \quad (\gamma, t) \to F(-, t) \circ \gamma.$$

To see the continuity of ΩF, we first use Exponential Law to express F equivalently as a continuous map $\tilde{F} : I \to \mathrm{Map}_\star(X, Y)$. Then ΩF is given by the composition

$$\mathrm{Map}_\star(S^1, X) \times I \xrightarrow{1 \times \tilde{F}} \mathrm{Map}_\star(S^1, X) \times \mathrm{Map}_\star(X, Y) \to \mathrm{Map}_\star(S^1, Y),$$

which is continuous by Proposition 7.14. $\square$

Definition 8.4. Let $\mathcal{C}$ be a category with finite product and terminal object $\star$. A **group object** in $\mathcal{C}$ is an object G in $\mathcal{C}$ together with morphisms

$$\mu : G \times G \to G, \quad \eta : G \to G, \quad \epsilon : \star \to G$$

such that the following diagrams commute

1°. associativity:

$$\begin{tikzcd} G \times G \times G \arrow[r, "1 \times \mu"] \arrow[d, "\mu \times 1"'] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G \end{tikzcd}.$$

2°. unit:

$$\begin{tikzcd} G \times \star \arrow[r, "1\times\epsilon"] \arrow[dr] & G \times G \arrow[d, "\mu"] & \star \times G \arrow[l, "\epsilon\times 1"'] \arrow[dl] \\ & G & \end{tikzcd}$$

3°. inverse

$$\begin{tikzcd} G \arrow[r, "1\times\eta"] \arrow[d] & G \times G \arrow[d, "\mu"] & G \arrow[l, "\eta\times 1"'] \arrow[d] \\ \star \arrow[r, "\epsilon"] & G & \star \arrow[l, "\epsilon"'] \end{tikzcd}$$

μ is called the multiplication, η is called the inverse, ϵ is called the unit.

Example 8.5. Here are some classical examples.

- Group objects in $\underline{\mathbf{Set}}$ are groups.
- Group objects in $\underline{\mathbf{Top}}$ are topological groups.
- Group objects in $\underline{\mathrm{h}\mathbf{Top}}$ are called H-groups.

Proposition 8.6. *Let $\mathcal{C}$ be a category with finite products and a terminal object. Let G be a group object. Then*

$$\mathrm{Hom}(-, G) : \mathcal{C} \to \underline{\mathbf{Group}}$$

defines a contravariant functor from $\mathcal{C}$ to $\underline{\mathbf{Group}}$.

Proof. For any $X \in \mathcal{C}$, we define the group structure on $\mathrm{Hom}(X, G)$ as followings:

- Multiplication: $f \cdot g = \mu(f, g)$ as

$$\begin{array}{ccccc} \mathrm{Hom}(X,G) & \times & \mathrm{Hom}(X,G) & \longrightarrow & \mathrm{Hom}(X,G) \\ X \xrightarrow{f} G & & X \xrightarrow{g} G & \mapsto & X \xrightarrow{(f,g)} G \times G \xrightarrow{\mu} G, \end{array}$$

- Inverse: $f^{-1} = \eta(f)$ as

$$\begin{array}{ccc} \mathrm{Hom}(X,G) & \longrightarrow & \mathrm{Hom}(X,G) \\ X \xrightarrow{f} G & \mapsto & X \xrightarrow{f} G \xrightarrow{\eta} G, \end{array}$$

- Identity is the image of the morphism $\mathrm{Hom}(X, \star) \to \mathrm{Hom}(X, G)$. □

Remark 8.7. The converse is also true, by Yoneda Lemma.

In the category $\underline{\mathscr{T}_\star}$, product exists and is given by

$$(X, x_0) \times (Y, y_0) = (X \times Y, x_0 \times y_0).$$

It admits a zero (both initial and terminal) object $\star$, which is a single point space.

Lemma 8.8. *The quotient functor $\underline{\mathscr{T}_\star} \to \underline{\mathrm{h}\mathscr{T}_\star}$ preserves finite product.*

Proof. Exercise. □

Theorem 8.9. *Let $X \in \underline{\mathscr{T}_\star}$. Then ΩX is a group object in $\underline{\mathrm{h}\mathscr{T}_\star}$.*

Proof. The multiplication is the $\star$ composition of paths as in Definition 2.5

$$\Omega X \times \Omega X \to \Omega X.$$

The inverse is the usual reverse of paths. The constant path 1_{x_0} is the zero object. The associativity follows from Proposition 2.7. We leave the details to the readers. $\square$

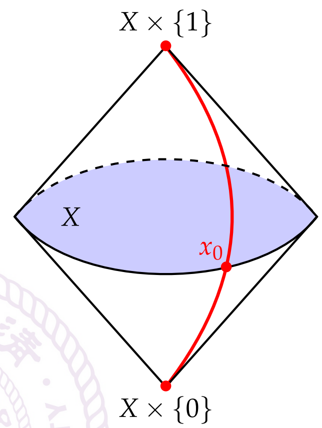
By Proposition 8.6, an immediate consequence is:

Corollary 8.10. *Any $X \in \underline{\mathrm{h}\mathscr{T}}$ defines a functor*

$$[-, \Omega X]_0 \colon \underline{\mathrm{h}\mathscr{T}_\star} \to \underline{\mathbf{Group}}\,.$$

Definition 8.11. Let $(X, x_0) \in \underline{\mathscr{T}_\star}$. We define its **suspension** ΣX by the quotient of $X \times I$:

$$\Sigma X = X \times I \big/ (X \times \partial I \cup x_0 \times I)$$



The suspension is the same as the smash product with S^1

$$\Sigma X = S^1 \wedge X.$$

It defines functors

$$\Sigma : \underline{\mathscr{T}_\star} \to \underline{\mathscr{T}_\star}, \quad \underline{\mathrm{h}\mathscr{T}_\star} \to \underline{\mathrm{h}\mathscr{T}_\star}\,.$$

Example 8.12. $\Sigma S^n \cong S^{n+1}$ are homeomorphic for any $n \geq 0$.

Theorem 8.13. (Σ, Ω) *defines adjoint pairs*

$$\Sigma : \underline{\mathscr{T}_\star} \rightleftarrows \underline{\mathscr{T}_\star} : \Omega \qquad \Sigma : \underline{\mathrm{h}\mathscr{T}_\star} \rightleftarrows \underline{\mathrm{h}\mathscr{T}_\star} : \Omega$$

Proof. This follows from Theorem 7.34. $\square$

Definition 8.14. Let $(X, x_0) \in \underline{\mathscr{T}_\star}$. We define the n-th **homotopy group**

$$\boxed{\pi_n(X, x_0) = [S^n, X]_0.}$$

Sometimes we simply denote it by $\pi_n(X)$.

In particular, we have

- π_0 is the path connected component.
- π_1 is the fundamental group.

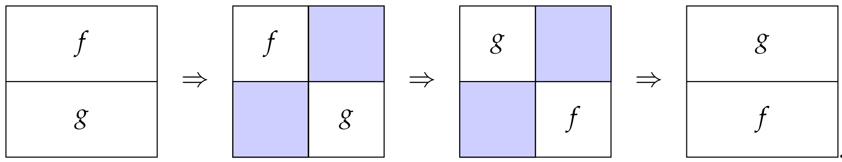
- For $n \geq 1$, we know that

$$\pi_n(X) = [\Sigma S^{n-1}, X]_0 = [S^{n-1}, \Omega X]$$

  which is a group since ΩX is a group object.

Proposition 8.15. *$\pi_n(X)$ is abelian if $n \geq 2$.*

Proof. This statement can be also illustrated as follows: □



.

The following statements are the analogue of what we did in Section 2.

Proposition 8.16. *Let X be path connected. There is a natural functor*

$$T_n : \Pi_1(X) \to \underline{\mathbf{Group}}$$

which sends x_0 to $\pi_n(X, x_0)$. In particular, there is a natural action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ and all $\pi_n(X, x_0)$'s are isomorphic for different choices of x_0.

Proposition 8.17. *Let $f : X \to Y$ be a' homotopy equivalence. Then*

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

is a group isomorphism.

9 FIBER HOMOTOPY AND HOMOTOPY FIBER

Path space

Definition 9.1. Given a space $X \in \underline{\mathscr{T}}$, and $x \in X$, we define

- free path space: $PX = \mathrm{Map}(I, X)$ and
- based path space: $P_x X = \mathrm{Map}((I, 0), (X, x))$.

We denote the two maps

$$\begin{tikzcd} & PX \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\ X & & X \end{tikzcd},$$

where $p_0(\gamma) = \gamma(0)$ is the start point and $p_1(\gamma) = \gamma(1)$ is the end point of the path γ. It induces

$$p = (p_0, p_1) : PX \to X \times X.$$

Theorem 9.2. *Let $X \in \underline{\mathscr{T}}$. Then*

1°. *$p : PX \to X \times X$ is a fibration.*
2°. *The map $p_0 : PX \to X$ is a fibration whose fiber at x_0 is $P_{x_0} X$.*
3°. *The map $p_1 : P_{x_0} X \to X$ is a fibration whose fiber at x_0 is $\Omega_{x_0} X$.*
4°. *$p_0 : PX \to X$ is a homotopy equivalence. $P_{x_0} X$ is contractible.*

Proof. (1) We need to prove the HLP of the diagram

$$\begin{tikzcd} Y \times \{0\} \arrow[r] \arrow[d, hook] & PX \arrow[d, "p"] \\ Y \times I \arrow[r] \arrow[ur, dashed, "?"] & X \times X \end{tikzcd}.$$

By the Exponential Law, this is equivalent to the extension problem

$$\begin{tikzcd} Y \times \{0\} \times I \cup Y \times I \times \partial I \arrow[r] \arrow[d] & X \\ Y \times I \times I \arrow[ur, dashed, "?"'] & \end{tikzcd}.$$

This follows by observing that $\{0\} \times I \cup I \times \partial I$ is a deformation retract of $I \times I$.

(2) follows from the composition of two fibrations

$$\begin{tikzcd} P_X \arrow[r] \arrow[dr] & X \times X \arrow[d] \\ & X \end{tikzcd}.$$

(3) follows from the pull-back diagram and the fact that fibrations are preserved under pull-back

$$\begin{tikzcd}
P_{x_0}X \arrow[r] \arrow[d] & PX \arrow[d] \\
X \arrow[r, "x_0 \times \mathrm{id}"] & X \times X.
\end{tikzcd}$$

(4) follows from the retracting path trick, which we have seen before in Section 2. ☐

Definition 9.3. Let $f : X \to Y$. We define the **mapping path space** P_f by the pull-back diagram

$$\begin{tikzcd}
P_f \arrow[r] \arrow[d] & PY \arrow[d, "p_1"] \\
X \arrow[r, "f"] & Y
\end{tikzcd} .$$

An element of P_f is a pair (x, γ) where γ is a path in Y that ends at $f(x)$. Let

$$\iota : X \hookrightarrow P_f, \quad x \mapsto (x, 1_{f(x)})$$

represent the constant path map and $p : P_f \to Y$ be the start point of the path. We have

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[dr, "f"'] & P_f \arrow[d, "p"] \\
& Y
\end{tikzcd}$$

Theorem 9.4. *$\iota : X \to P_f$ is a strong deformation retract (hence homotopy equivalence) and $p : P_f \to Y$ is a fibration. In particular, any map $f : X \to Y$ is a composition of a homotopy equivalence with a fibration.*

Proof. The first statement follows from the retracting path trick. We prove p is a fibration.

Consider the pull-back diagram

$$\begin{tikzcd}
P_f \arrow[r] \arrow[d] & PY \arrow[d] \\
Y \times X \arrow[r, "\mathrm{id} \times f"] & Y \times Y.
\end{tikzcd}$$

This implies $P_f \to Y \times X$ is a fibration. Since $Y \times X \to Y$ is also a fibration, so is the composition

$$p : P_f \to Y \times X \to Y. \quad \square$$

This theorem says that in $\underline{\mathrm{h}\mathscr{T}}$, every map is equivalent to a fibration.

Fiber homotopy

Definition 9.5. Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two fibrations. A fiber map from p_1 to p_2 is a map $f : E_1 \to E_2$ such that $p_1 = p_2 \circ f$:

$$\begin{tikzcd}
E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\
& B &
\end{tikzcd} .$$

Two fiber maps $f_0, f_1 : p_1 \to p_2$ are said to be **fiber homotopic**

$$f_0 \simeq_B f_1$$

if there exists a homotopy $F : E_1 \times I \to E_2$ from f_0 to f_1 such that $F(-,t)$ is a fiber map for each $t \in I$. $f : p_1 \to p_2$ is a **fiber homotopic equivalence** if there exists $g : p_2 \to p_1$ such that both $f \circ g$ and $g \circ f$ are fiber homotopic to identity maps.

Proposition 9.6. *Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two fibrations and $f : E_1 \to E_2$ be a fiber map. Assume $f : E_1 \to E_2$ is a homotopy equivalence, then f is a fiber homotopy equivalence. In particular, $f : p_1^{-1}(b) \to p_2^{-1}(b)$ is a homotopy equivalence for any $b \in B$.*

$$\begin{tikzcd} E_1 \arrow[rr, bend left, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[ll, dotted, bend left, "g"] \arrow[dl, "p_2"] \\ & B & \end{tikzcd}.$$

Proof. We only need to prove that for any fiber map $f : E_1 \to E_2$ which is a homotopy equivalence, there is a fiber map $g : E_2 \to E_1$ such that $g \circ f \simeq_B 1$. In fact, such a g is also a homotopy equivalence and we can find $h : E_1 \to E_2$ such that $h \circ g \simeq_B 1$. Then $f \simeq_B h \circ g \circ f \simeq_B h$, which implies $f \circ g \simeq_B 1$ as well.

Let $g : E_2 \to E_1$ represent the inverse of the homotopy class $[f]$ in $\underline{\mathrm{h}\mathscr{T}}$.

We first show that we can choose g to be a fiber map, i.e., $p_1 \circ g = p_2$ in the following diagram

$$\begin{tikzcd} & E_1 \arrow[d, "p_1"] \\ E_2 \arrow[ur, "g"] \arrow[r, "p_1 \circ g"'] & B \end{tikzcd}.$$

Otherwise, we observe that $p_1 \circ g = p_2 \circ f \circ g \simeq p_2$. We can use the fibration p_1 to lift the homotopy $p_1 \circ g \simeq p_2$ to a homotopy $g \simeq g'$. Then g' is a fiber map, and we can replace g by g'.

$$\begin{tikzcd} & E_1 \arrow[dd, "p_1"] \\ E_2 \arrow[ur, bend left=40, "g"{name=G}] \arrow[ur, bend right=10, "g'"'{name=GP}] \arrow[dr, bend left=10, "p_1 \circ g"{name=PG}] \arrow[dr, bend right=40, "p_2"'{name=P}] \arrow[from=G, to=GP, Rightarrow, shorten=2pt] \arrow[from=PG, to=P, Rightarrow, shorten=2pt] & \\ & B \end{tikzcd}.$$

Now we assume $g : E_2 \to E_1$ is a fiber map. The problem can be further reduced to the following

"Claim": Let $p : E \to B$ be a fibration and $f : E \to E$ is a fiber map that is homotopic to 1_E, then there is a fiber map $h : E \to E$ such that $h \circ f \simeq_B 1$.

In fact, let $f : E_1 \to E_2$ as in the proposition, $g : E_2 \to E_1$ be a fiber map such that $g \circ f \simeq 1$ as chosen above. The "Claim" implies that we can find a fiber map $h : E_1 \to E_1$ such that $h \circ g \circ f \simeq_B 1$. Then the fiber map $\tilde{g} = h \circ g$ has the required property that $\tilde{g} \circ f \simeq_B 1$.

Now we prove the "Claim". Let F be a homotopy from f to 1_E and $G = p \circ F$. Since p is a fibration, we can construct a homotopy H that starts from 1_E and lifts G. Here is the picture

$$
\begin{tikzcd}
E \arrow[r, bend left, "f"] \arrow[r, bend right, "1_E"'] \arrow[r, phantom, "\Downarrow F"] \arrow[d, equal] & E \arrow[r, "h"] \arrow[d, "p"] & E \arrow[dl, "p"] \\
E \arrow[r, bend left, "p"] \arrow[r, bend right, "p"'] \arrow[r, phantom, "\Downarrow G"] & B &
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, bend left, "1_E"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow H"] \arrow[d, equal] & E \arrow[d, "p"] \\
E \arrow[r, bend left, "p"] \arrow[r, bend right, "p"'] \arrow[r, phantom, "\Downarrow G"] & B
\end{tikzcd}.
$$

Combining these two homotopies we find a homotopy $\tilde{F}$ from $h \circ f$ to 1_E that lifts the following homotopy

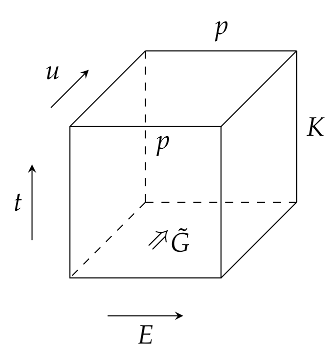
$$\tilde{G} : E \times I \to B, \quad \tilde{G}(-,t) = \begin{cases} G(-.2t) & 0 \le t \le 1/2 \\ G(-,2-2t) & 1/2 \le t \le 1 \end{cases}.$$

Here is the picture

$$
\begin{tikzcd}
E \arrow[r, bend left, "h\circ f"] \arrow[r, bend right, "1_E"'] \arrow[r, phantom, "\Downarrow \tilde{F}"] \arrow[d, equal] & E \arrow[d, "p"] \\
E \arrow[r, bend left, "p"] \arrow[r, bend right, "p"'] \arrow[r, phantom, "\Downarrow \tilde{G}"] & B
\end{tikzcd}.
$$

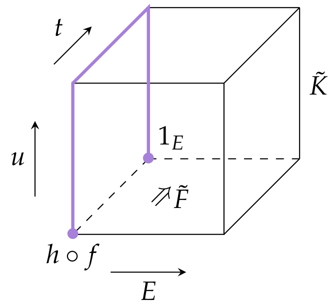
We can construct a map $K : E \times I \times I \to B$ that gives a homotopy between $\tilde{G} : E \times I \to B$ and the projection $E \times I \to E \xrightarrow{p} B$ (by pushing the two copies of G in $\tilde{G}$) :

$$
\begin{aligned}
K(-,u,0) &= \tilde{G}(-,u), \\
K(-,u,1) &= p(-), \\
K(-,0,t) &= p(-), \\
K(-,1,t) &= p(-), \quad \forall u,t \in I.
\end{aligned}
$$



Since p is a fibration, we can find a lift $\tilde{K} : E \times I \times I \to E$ of K such that

$$\tilde{K}(-,u,0) = \tilde{F}(-,u).$$



Then we have the following fiber homotopy

$$h \circ f = \tilde{K}(-,0,0) \simeq_B \tilde{K}(-,0,1) \simeq_B \tilde{K}(-,1,1) \simeq_B \tilde{K}(-,1,0) = 1_E.$$ □

Homotopy fiber

Definition 9.7. Let $f : X \to Y$, we define its **homotopy fiber** over $y \in Y$ to be the fiber of $P_f \to Y$ over y.

Proposition 9.8. *If Y is path connected, then all homotopy fibers of $f : X \to Y$ are homotopic equivalent.*

Proof. Let $y_1, y_2 \in Y$, and F_1, F_2 be the homotopy fiber over y_1, y_2. Then

$$F_i = \{(x,\gamma) | \gamma : I \to Y, \gamma(0) = y_i, \gamma(1) = f(x)\}$$

and composition with a path in Y from y_1 to y_2 gives a homotopy equivalence between F_1, F_2. □

In this case we will usually write the following diagram

$$\begin{tikzcd} F \arrow[r] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

where F denotes the homotopy fiber.

Proposition 9.9. *If $f : X \to Y$ is a fibration, then its homotopy fiber at y is homotopy equivalent to $f^{-1}(y)$.*

Proof. We have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & P_f \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where ι is a homotopy equivalence. Then ι is a fiber homotopy equivalence by Proposition 9.6. □

Corollary 9.10. *Let $f : X \to Y$ be a fibration and Y path connected. Then all fibers of f are homotopy equivalent.*

Proof. Given any two points y_1, y_2 in Y, their fibers $f^{-1}(y_1), f^{-1}(y_2)$ are homotopy equivalent to the corresponding homotopy fibers. The corollary follows since all homotopy fibers are homotopy equilvalent. □

Recall the following theorem which gives a criterion for fibration that is very useful in practice.

Theorem 9.11. *Let $p : E \to B$ with B paracompact Hausdorff. Assume there exists an open cover $\{U_\alpha\}$ of B such that $p^{-1}(U_\alpha) \to U_\alpha$ is a fibration. Then p is a fibration.*

Corollary 9.12. *Let $p : E \to B$ be a fiber bundle with B paracompact Hausdorff. Then p is a fibration.*

10 EXACT PUPPE SEQUENCE

Definition 10.1. A sequence of maps of sets with base points (i.e. in the category $\underline{\mathbf{Set}}_*$)

$$(A, a_0) \xrightarrow{f} (B, b_0) \xrightarrow{g} (C, c_0)$$

is said to be **exact** at B if $\mathrm{im}(f) = \ker(g)$, where $\mathrm{im}(f) = f(A)$ and $\ker(g) = g^{-1}(c_0)$. A sequence

$$\cdots \to A_{n+1} \to A_n \to A_{n-1} \to \cdots$$

is called an exact sequence if it is exact at every A_i.

Example 10.2. Let $H \triangleleft G$ be a normal subgroup of G. Then there is a short exact sequence

$$1 \to H \to G \to G/H \to 1$$

in $\underline{\mathbf{Group}}$. Here we view $\underline{\mathbf{Group}}$ as a subcategory of $\underline{\mathbf{Set}}_*$, where a group is based at its identity element.

Definition 10.3. A sequence of maps in $\underline{\mathrm{h}\mathscr{T}_\star}$

$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots$$

is called exact if for any $Y \in \underline{\mathrm{h}\mathscr{T}_\star}$, the following sequence of pointed sets is exact

$$\cdots \to [Y, X_{n+1}]_0 \to [Y, X_n]_0 \to [Y, X_{n-1}]_0 \to \cdots$$

The goal of this section is to study the relationship between homotopy groups via exact sequence.

Definition 10.4. Let $f : (X, x_0) \to (Y, y_0)$ be a map in $\underline{\mathscr{T}_\star}$. We define its homotopy fiber F_f in $\underline{\mathscr{T}_\star}$ by the pull-back diagram

$$\begin{tikzcd}
F_f \arrow[r] \arrow[d, "\pi"'] & P_{y_0}Y \arrow[d, "p_1"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

$$F_f = \{(x, \gamma) \in X \times PY | \gamma(0) = y_0, \gamma(1) = f(x)\}.$$

Recall that $p_1 \colon P_{y_0}Y \to Y$ is a fibration, thus

Lemma 10.5. *$\pi \colon F_f \to X$ is a fibration.*

Note that F_f is precisely the fiber of $P_f \to Y$ over y_0:

$$\begin{tikzcd}
F_f \arrow[r] \arrow[d, "\pi"'] & P_f \arrow[d, "p_0"] \\
y_0 \arrow[r, hook] & Y
\end{tikzcd}.$$

So this is the same as our definition before. We will emphasize on the role of based point in this section.

The following lemma is the same as Proposition 9.9. We restate here for convenience.

Lemma 10.6. *If $f \colon X \to Y$ is a fibration, then $f^{-1}(y_0)$ is homotopy equivalent to its homotopy fiber F_f.*

For arbitrary map $f : X \to Y$, we still have a canonical map

$$j : f^{-1}(y_0) \to F_f,$$

which may not be a homotopy equivalence. The homotopy fiber can be viewed as a good replacement of fiber in homotopy category that behaves nicely for fibrations.

Lemma 10.7. *The sequence*

$$F_f \xrightarrow{\pi} X \xrightarrow{f} Y$$

is exact at X *in* $\underline{\mathrm{h}\mathcal{T}_\star}$.

Proof. Let y_0 be the base point of Y. We first observe that $f \circ \pi$ factors through $P_{y_0}Y$ which is contractible. Therefore $f \circ \pi$ is null homotopy. Let $Z \in \underline{\mathrm{h}\mathcal{T}_\star}$. Consider

$$[Z, F_f]_0 \xrightarrow{\pi_*} [Z, X]_0 \xrightarrow{f_*} [Z, Y]_0.$$

Since $f \circ \pi$ is null homotopic, we have $\operatorname{im} \pi_* \subset \ker f_*$.

Let $g : Z \to X$ such that $[g]_0 \in \ker f_*$. Let G be a based homotopy of $f \circ g$ to the trivial map:

$$G\colon Z \times I \to Y.$$

Since $G\,|_{Z\times\{0\}} = y_0$, it can be regarded as a map (via the Exponential Law)

$$G\colon Z \to P_{y_0}Y$$

that fits into the following diagram

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, "G"] \arrow[ddr, "g"'] & & \\
& F_f \arrow[r] \arrow[d, "\pi"] & P_{y_0}Y \arrow[d, "p_1"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}.$$

Therefore the pair (G, g) factors through F_f. This implies $[g]_0 \in \operatorname{im} \pi_*$. So $\ker f_* \subset \operatorname{im} \pi_*$. $\square$

Notice that the fiber of F_f over x_0 is precisely ΩY

$$\begin{tikzcd}
\Omega Y \arrow[r] \arrow[d] & F_f \arrow[d, "\pi"] \\
x_0 \arrow[r, hook] & X
\end{tikzcd}.$$

We find the following sequence of pointed maps

$$\Omega X \xrightarrow{\Omega f} \Omega Y \to F_f \xrightarrow{\pi} X \xrightarrow{f} Y.$$

Lemma 10.8. *The sequence* $\Omega X \xrightarrow{\Omega f} \Omega Y \to F_f \xrightarrow{\pi} X \xrightarrow{f} Y$ *is exact in* $\underline{\mathrm{h}\mathcal{T}_\star}$.

Proof. We construct the commutative diagram (†) in $\underline{h\mathcal{T}_\star}$ with all vertical arrows homotopy equivalences

(†)
$$\begin{tikzcd}
\Omega X \arrow[r, "\Omega f"] \arrow[d, "\tilde{j}'"] & \Omega Y \arrow[r] \arrow[d, "j"] & F_f \arrow[r, "\pi"] \arrow[d] & X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
F_{\pi'} \arrow[r, "\pi''"] & F_\pi \arrow[r, "\pi'"] & F_f \arrow[r, "\pi"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

F_π is the homotopy fiber of $\pi : F_f \to X$, given by the pull-back

$$\begin{tikzcd}
F_\pi \arrow[r] \arrow[d] & P_{x_0}X \arrow[d, "p_1"] \\
F_f \arrow[r, "\pi"] & X
\end{tikzcd}$$

or explicitly

$$F_\pi = \{([\gamma], [\beta]) \in P_{y_0}Y \times P_{x_0}X | f(\beta(1)) = \gamma(1)\}.$$

Since $F_f \xrightarrow{\pi} X$ is a fibration with fiber ΩY, the map $j : \Omega Y \to F_\pi$ is the natural map of fiber into homotopy fiber which is a homotopy equivalence by Lemma 10.6. By construction, the second square in (†) commutes.

Explicitly, the map $j : \Omega Y \to F_\pi$ sends a loop $[\beta]$ based at y_0 to the pair

$$j([\beta]) = ([1_{x_0}], [\beta]).$$

Similarly, the fiber of the fibration $F_\pi \to F_f$ is ΩX. We find the natural map

$$j' : \Omega X \to F_{\pi'}$$

from fiber into homotopy fiber, which is a homotopy equivalence. Let

$$(-)^{-1} : \Omega X \to \Omega X, \quad \gamma \to \gamma^{-1}$$

be the inverse of the loop. We define

$$\tilde{j}' = j' \circ (-)^{-1} : \Omega X \to F_{\pi'}.$$

which is again a homotopy equivalence. Let us form the commutative diagram

$$\begin{tikzcd}
\Omega X \arrow[d, "\tilde{j}'"'] \arrow[dr, "k"] & \\
F_{\pi'} \arrow[r, "\pi''"'] & F_\pi
\end{tikzcd}$$

which defines the map $k : \Omega X \to F_\pi$. Consider the diagram

$$\begin{tikzcd}
\Omega X \arrow[r, "\Omega f"] \arrow[dr, "k"'] & \Omega Y \arrow[d, "j"] \\
& F_\pi .
\end{tikzcd}$$

This diagram is NOT commutative in $\underline{\mathscr{T}_\star}$. However, $j \circ \Omega f$ is homotopic to k, so this diagram commutes in $\underline{\mathrm{h}\mathscr{T}_\star}$. To see this, let us explicitly write

$$k([\gamma]) = ([\gamma^{-1}], [1_{y_0}]), \quad (j \circ \Omega f)(\gamma) = ([1_{x_0}], [f(\gamma)]).$$

They are homotopic via

$$F([\gamma], t) = \left([(\gamma\mid_{[t,1]})^{-1}], f\left[\gamma\mid_{[0,t]}\right] \right).$$

Therefore the first square in (†) commutes in $\underline{\mathrm{h}\mathscr{T}_\star}$. The lemma follows. ☐

Lemma 10.9. *Let $X_1 \to X_2 \to X_3$ be exact in $\underline{\mathrm{h}\mathscr{T}_\star}$, then so is $\Omega X_1 \to \Omega X_2 \to \Omega X_3$.*

Proof. For any Y, apply $[Y, -]_0$ to the exact sequence $X_1 \to X_2 \to X_3$ and use the fact that Ω is right adjoint to the suspension Σ, i.e. $[\Sigma Y, X_i]_0 = [Y, \Omega X_i]_0$, we obtain an exact sequence. This implies the lemma. ☐

Theorem 10.10 (Exact Puppe Sequence)**.** *Let $f : X \to Y$ in $\underline{\mathscr{T}_\star}$. Then the following sequence in exact in $\underline{\mathrm{h}\mathscr{T}_\star}$*

$$\cdots \to \Omega^2 Y \to \Omega F_f \to \Omega X \to \Omega Y \to F_f \to X \to Y.$$

Proof. The theorem follows from Lemma 10.8 and Lemma 10.9.

☐

Theorem 10.11. *Let $p : E \to B$ be a map in $\underline{\mathscr{T}_\star}$. Assume p is a fibration whose fiber over the base point is F. Then we have an exact sequence of homotopy groups*

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \to \pi_0(E) \to \pi_0(B).$$

Proof. Since p is a fibration, F is homotopy equivalent to F_p. Observe that

$$[S^0, \Omega^n X]_0 = [\Sigma^n S_0, X]_0 = [S^n, X] = \pi_n(X).$$

The theorem follows by applying $[S^0, -]_0$ to the Puppe Sequence associated to $p : E \to B$. ☐

This theorem give a very effective method to compute homotopy groups via fibrations.

Example 10.12. Consider the universal cover $\exp : \mathbb{R}^1 \to S^1$. The associated long exact sequence implies

$$\pi_n(S^1) = 0, \quad \forall n > 1.$$

Proposition 10.13. *If $i < n$, then $\pi_i(S^n) = 0$.*

Sketch of proof. Let $f : S^i \to S^n$. We need the following fact: any continuous map from a compact smooth manifold X to S^n can be uniformly approximated by a smooth map. Furthermore, two smooth maps are continuously homotopic, then they are smoothly homotopic. This follows by performing perturbation locally (in small neighbourhoods at each point) while compactness implies that the perturbation can be performed globally.

Thus, we can assume that f is homotopic to a smooth map f'. Then f' is not surjective (for dimension reason). Thus, $f' \colon S^i \to (S^n - \{\mathrm{pt}\}) \simeq \mathbb{R}^n \simeq \{\mathrm{pt}\}$ is null homotopic. ☐

Example 10.14. Consider the Hopf fibration $S^3 \to S^2$ with fiber S^1. The associated long exact sequence of homotopy groups implies

$$\pi_2(S^2) \cong \mathbb{Z} \quad \text{and} \quad \pi_n(S^3) \cong \pi_n(S^2) \ \text{ for } n \geq 3.$$

11 COFIBRATION

Cofibration

Definition 11.1. A map $i : A \to X$ is said to have the **homotopy extension property** (HEP) with respect to Y if for any map $f : X \to Y$ and any homotopy $F : A \times I \to Y$ where $F(-,0) = f \circ i$, there exists a homotopy $\bar{F} : X \times I \to Y$ such that

$$\bar{F}(i(a), t) = F(a,t), \quad F(x,0) = f(x), \quad \forall a \in A, x \in X, t \in I.$$

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d] & X \arrow[d] \arrow[ddr, "f"] & \\
A \times I \arrow[r] \arrow[drr, "F"'] & X \times I \arrow[dr, dashed, "\bar{F}"] & \\
& & Y .
\end{tikzcd}$$

Definition 11.2. A map $i : A \to X$ is called a **cofibration** if it has HEP for any spaces.

The notion of cofibration is dual to that of the fibration: fibration is defined by the HLP of the diagram

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists\tilde{F}"] & B .
\end{tikzcd}$$

If we reverse the arrows and observe that $Y \times I$ is dual to the path space Y^I via the adjointness of $(-) \times I$ and $(-)^I$, we arrive at HEP (using Exponential Law)

$$\begin{tikzcd}
Y & X \arrow[l, "f"'] \arrow[dl, dashed, "\exists\tilde{F}"'] \\
Y^I \arrow[u, "p_0"] & A . \arrow[l, "F"] \arrow[u, "i"']
\end{tikzcd}$$

Definition 11.3. Let $f : A \to X$. We define its **mapping cylinder** M_f by the push-out

$$\begin{tikzcd}
A \times \{0\} \arrow[r, hook] \arrow[d, "f"'] & A \times I \arrow[d] \arrow[ddr, "f\times 1"] & \\
X \times \{0\} \arrow[r] \arrow[drr, hook] & M_f \arrow[dr, "j"] & \\
& & X \times I .
\end{tikzcd}$$

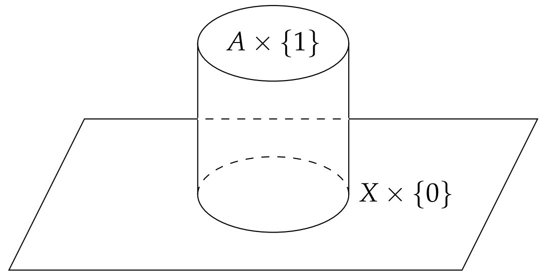


FIGURE 16. The mapping cylinder M_f

There is a natural map $j\colon M_f \to X \times I$ induced by the inclusion $X \times \{0\} \to X \times I$ and $f \times 1\colon A \times I \to X \times I$. The mapping cylinder topology (i.e. the push-out topology) of M_f says that a map $g : M_f \to Z$ is continuous if and only if g is continuous when it restricted to $X \times \{0\}$ and to $A \times I$.

Lemma 11.4. *The HEP of $i : A \to X$ is equivalent to the property of filling the commutative diagram*

$$\begin{tikzcd} M_i \arrow[r] \arrow[d] & Y \\ X \times I \arrow[ur, dashed, "\exists ?"'] & \end{tikzcd}.$$

Proposition 11.5. *Let $i : A \to X$ and $j : M_i \to X \times I$ be defined as above. Then i is a cofibration if and only there exists $r : X \times I \to M_i$ such that $r \circ j = 1_{M_i}$.*

Proof. If i is a cofibration, then take $Y = M_i$ in the lemma above and we obtain the required map r. On the other hand, if r exists, then any $f\colon M_i \to Y$ lifts to $f \circ r$. □

Proposition 11.6. *Let $i : A \to X$ be a cofibration. Then i is a homeomorphism to its image (i.e. embedding). If we work in $\underline{\mathscr{T}}$, so A, X are compactly generated weak Hausdorff. Then i has closed image (i.e. closed inclusion).*

Proof. Consider the following commutative diagram obtained from the previous proposition

$$\begin{tikzcd} M_i \arrow[r, "1_{M_i}"] \arrow[d, "j"'] & M_i \\ X \times I \arrow[ur, dashed, "r"'] & \end{tikzcd}.$$

This implies that M_i is homeomorphic to its image $j(M_i)$. Consider the next commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & M_i \arrow[d, "j"] \\ X \arrow[r, "1_X \times \{1\}"'] & X \times I \end{tikzcd}.$$

Since $A \to M_i, M_i \to X \times I, X \to X \times I$ are all embeddings, so is $i : A \to X$.

Assume now that $A, X \in \underline{\mathscr{T}}$ are compactly generated weak Hausdorff. By Proposition 7.31, $j : M_i \to X \times I$ is a closed inclusion. Since $A \to M_i, X \to X \times I$ are also closed inclusions, so is $i : A \to X$.

□

Remark 11.7. A cofibration is not closed in general. An example is $X = \{a, b\}$ having two points with the trivial topology and $A = \{a\}$ is one of the point.

Definition 11.8. Let A be a subspace of X. We say (X, A) is **cofibered** if the inclusion $A \subset X$ is a cofibration.

Proposition 11.9. *Let A be a closed subspace of X. Then the inclusion map $i : A \subset X$ is a cofibration if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.*

Proof. If i is closed, then M_i is homeomorphic to the subspace $X \times \{0\} \cup A \times I$ of $X \times I$. □

Remark 11.10. If $A \subset X$ is not closed, then the mapping cylinder topology for M_i and the subspace topology for $X \times \{0\} \cup A \times I$ may not be the same. For example, we take choose

$$X = [0, 1], \quad A = \{1, 1/2, 1/3, \cdots, 1/n, \cdots\}.$$

Consider the subspace $Z = \{(1, 1), (1/2, 1/2), \cdots, (1/n, 1/n), \cdots\} \subset A \times I$. Then Z is closed in $A \times I$ and $Z \cap (X \times \{0\}) = \emptyset$. So Z is closed in the mapping cylinder, but not closed in $X \times \{0\} \cup A \times I$.

Example 11.11. The inclusion $S^{n-1} \hookrightarrow D^n$ is a cofibration, cf. Figure 17.

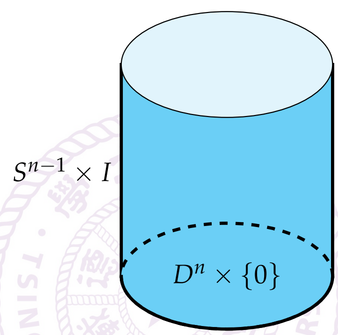


FIGURE 17. $D^n \times \{0\} \cup S^{n-1} \times I$ is a retract of $D^n \times I$

Proposition 11.12. *Let $f : A \to X$ be any map. Then the closed inclusion*

$$i_1 : A \to M_f, \quad a \to (a, 1)$$

is a cofibration.

Proof. Figure 11 shows $M_f \times \{0\} \cup A \times I$ is a retract of $M_f \times I$. So i_1 is a cofibration.

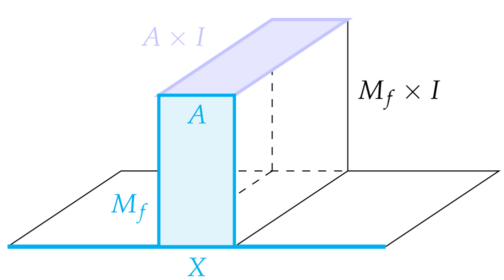


FIGURE 18. Retract of $M_f \times I$

□

Example 11.13. The inclusion $A \to A \times I, a \to a \times \{0\}$, is a cofibration. In fact, we can view it as

$$A \to M_{1_A}$$

where $1_A : A \to A$ is the identity map.

Definition 11.14. Let A be a subspace of X. A is called a **neighborhood deformation retract (NDR)** if there exists a continuous map $u : X \to I$ with $A = u^{-1}(0)$ and a homotopy $H : X \times I \to X$ such that

$$\begin{cases} H(x,0) = x & \forall x \in X \\ H(a,t) = a & \text{if } (a,t) \in A \times I \\ H(x,1) \in A & \text{if } u(x) < 1\,. \end{cases}$$

Note that if A is a NDR of X, then A is a strong deformation retract of the open subset $u^{-1}([0,1))$ of X.

Theorem 11.15. *Let A be a closed subspace of X. Then the following conditions are equivalent*

1°. *(X, A) is a cofibered pair.*
2°. *A is a NDR of X.*
3°. *$X \times \{0\} \cup A \times I$ is a retract of $X \times I$.*
4°. *$X \times \{0\} \cup A \times I$ is a strong deformation retract of $X \times I$.*

Proof. We have seen the equivalence between (1) and (3).

(3) $\Longrightarrow$ (2). Let r be a retraction map

$$r : X \times I \to X \times \{0\} \cup A \times I.$$

Let $\pi_X : X \times I \to X, \pi_I : X \times I \to I$ be the projections. We obtain the data for NDR by

$$u : X \to I, \quad u(x) = \sup_{t \in I} |t - \pi_I \circ r(x,t)|$$

and

$$H : X \times I \to X, \quad H(x,t) = \pi_X \circ r(x,t).$$

(2) $\Longrightarrow$ (3). Given the data (u, H) for NDR. We define a retraction $r : X \times I \to X \times \{0\} \cup A \times I$ by

$$\begin{cases} r(x,t) = (x,0) & \text{if} \quad u(x) = 1 \\ r(x,t) = (H(x, 2(1-u(x))t), 0) & \text{if} \quad 1/2 \le u(x) < 1 \\ r(x,t) = (H(x, t/(2u(x))), 0) & \text{if} \quad 0 < u(x) \le 1/2, \quad 0 \le t \le 2u(x) \\ r(x,t) = (H(x,1), t - 2u(x)) & \text{if} \quad 0 < u(x) \le 1/2, \quad 2u(x) \le t \le 1 \\ r(x,t) = (x,t) & \text{if} \quad u(x) = 0\,. \end{cases}$$

(4) $\Longrightarrow$ (3). Obvious.

(3) $\Longrightarrow$ (4). Let $r : X \times I \to X \times \{0\} \cup A \times I$ be a retraction map. Then the following homotopy

$$F : X \times I \times I \to X \times \{0\} \cup A \times I \ \text{ rel } X \times \{0\} \cup A \times I$$
$$F(x,t,s) = (\pi_X \circ r(x, (1-s)t), (1-s)\pi_I \circ r(x,t) + st)$$

gives the required strong deformation retract. $\square$

Basic properties

Proposition 11.16. *Let $i : A \to X$ be a cofibration, $f : A \to B$ is a map. Consider the push-out*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r] & Y . \end{tikzcd}$$

Then $j : B \to Y$ is also a cofibration. In other words, the push-out of a cofibration is a cofibration.

Proof. The proof is dual to Proposition 5.15.

□

Proposition 11.17. *Let $i : X \to Y$ and $j : Y \to Z$ be cofibrations. Then $j \circ i : X \to Z$ is also a cofibration.*

Proof. Exercise. □

Proposition 11.18. *If $i : A \to X$ is a cofibration and A is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence.*

Proof. Exercise. □

The next proposition is very useful in constructing homotopies.

Proposition 11.19. *Let $A \subset X$ and $B \subset Y$ be closed inclusions which are both cofibrations. Then the inclusion*

$$X \times B \cup A \times Y \subset X \times Y$$

is also a cofibration. As a consequence, $A \times B \to X \times Y$ is a cofibration.

Proof. Let $u : X \to I, H : X \times I \to X$ be the data of NDR for $A \hookrightarrow X$, and $v : Y \to I, K : Y \times I \to Y$ be the data of NDR for $B \hookrightarrow Y$. Consider the following maps

$$\varphi : X \times Y \to I, \quad \varphi(x, y) = \min\{u(x), v(y)\}$$

and

$$\Sigma : X \times Y \times I \to X \times Y, \quad \Sigma(x,y,t) = \begin{cases} (x,y) & \text{if } u(x) = v(y) = 0 \\ \left(H(x,t), K(y, t\frac{u(x)}{v(y)})\right) & \text{if } u(x) \leq v(y) \neq 0 \\ \left(H(x, t\frac{v(y)}{u(x)}), K(y,t)\right) & \text{if } 0 \neq u(x) \geq v(y) \end{cases}$$

Then (φ, Σ) defines a data of NDR for $X \times B \cup A \times Y \subset X \times Y$, so a cofibration. As a special case, if $B = \emptyset$, then $A \times Y \to X \times Y$ is a cofibration.

Now consider the following push-out diagram

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] & A \times Y \arrow[d] \\ X \times B \arrow[r] & X \times B \cup A \times Y \end{tikzcd}$$

So all arrows in this diagram are cofibrations. It follows that $A \times B \to X \times Y$ is a cofibration.

□

Let $f : A \to X$ be a map. Consider the diagram of mapping cylinder

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, "f"'] & A \times I \arrow[d] \\ X \arrow[r] & M_f \,. \end{tikzcd}$$

There is a natural commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "i_1"'] \arrow[dr, "f"] & \\ M_f \arrow[rr, "r"] & & X \,. \end{tikzcd}$$

Here $i_1(a) = (a, 1), r(a, t) = f(a), r(x, 0) = x$.

It is easy to see that r is a homotopy equivalence. We have the following dual statement of Theorem 9.4.

Theorem 11.20. *The map $r : M_f \to X$ is a homotopy equivalence, and $i_1 : A \to M_f$ is a cofibration. In particular, any map $f : A \to X$ is a composition of a cofibration with a homotopy equivalence.*

Definition 11.21. Let $i : A \to X, j : A \to Y$ be cofibrations. A map $f : X \to Y$ is called a **cofiber map** if the following diagram is commutative

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f"] & & Y \,. \end{tikzcd}$$

A **cofiber homotopy** between two cofiber maps $f, g : X \to Y$ is a homotopy of cofiber maps between f and g. Cofiber homotopy equivalence is defined similarly.

The following result is the cofibration analogue of Proposition 9.6.

Proposition 11.22. *Let $i : A \to X, j : A \to Y$ be cofibrations. Let $f : X \to Y$ be a cofiber map. Assume f is a homotopy equivalence. Then f is a cofiber homotopy equivalence.*

Cofiber exact sequence

Now we work with the category $\underline{\mathscr{T}_\star}$ and $\underline{\mathrm{h}\mathscr{T}_\star}$. All maps and testing diagrams are required to be based.

Definition 11.23. A based space (X, x_0) is called well-pointed, if the inclusion of the base point $x_0 \in X$ is a cofibration in the unbased sense.

Definition 11.24. Let $(X, x_0) \in \underline{\mathscr{T}_\star}$. We define its (reduced) **cone** by

$$C_\star X = X \wedge I = X \times I / \left(X \times \{0\} \cup x_0 \times I \right).$$

Proposition 11.25. *If X is well-pointed, then the embedding $i_1 : X \to C_\star X$ where $i_1(x) = (x, 1)$ is a cofibration.*

Definition 11.26. Let $f : (X, x_0) \to (Y, y_0) \in \underline{\mathscr{T}_\star}$. We define its (reduced) **mapping cylinder** by

$$M_{\star f} = M_f / \{x_0 \times I\}.$$

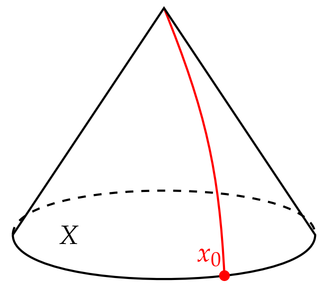


FIGURE 19. The reduced cone $C_\star X$

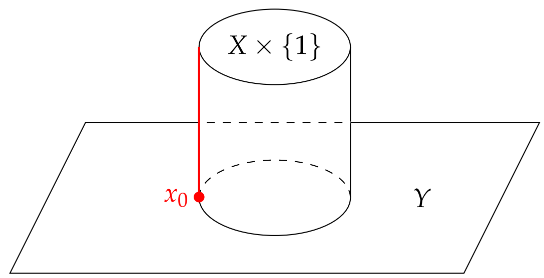


FIGURE 20. The reduced mapping cylinder M_f

If (X, x_0) is well-pointed, then the quotient $M_f \to M_{\star f}$ is a homotopy equivalence.

Given $f : X \to Y$ in $\underline{\mathscr{T}_\star}$, we define its (reduced) **homotopy cofiber** $C_{\star f}$ by the push-out

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "f"'] & C_\star X \arrow[d] \\ Y \arrow[r, "j"] & C_{\star f} \end{tikzcd}$$

If X is well-pointed, then $j : Y \to C_{\star f}$ is also a cofibration. Note that the quotient of $C_{\star f}$ by Y is precisely ΣX. We can extend the above maps by

$$X \longrightarrow Y \longrightarrow C_{\star f} \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma C_{\star f} \longrightarrow \Sigma^2 X \longrightarrow \cdots .$$

Definition 11.27. A sequence of maps in $\underline{\mathrm{h}\mathscr{T}_\star}$

$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots$$

is called co-exact if for any $Y \in \underline{\mathrm{h}\mathscr{T}_\star}$, the following sequence of pointed sets is exact

$$\cdots \to [X_{n-1}, Y]_0 \to [X_n, Y]_0 \to [X_{n+1}, Y]_0 \to \cdots$$

Theorem 11.28 (Co-exact Puppe Sequence). *Let $f : X \to Y$ in $\underline{\mathscr{T}_\star}$ between well-pointed spaces. The following sequence is co-exact in $\underline{\mathrm{h}\mathscr{T}_\star}$*

$$X \longrightarrow Y \longrightarrow C_{\star f} \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma C_{\star f} \longrightarrow \Sigma^2 X \longrightarrow \cdots .$$

Lemma 11.29. *Let $f : A \to X$ be a cofibration between well-pointed spaces. Then the natural embedding*

$$C_\star(A) \to C_{\star f}$$

is a cofibration.

Proof. This follows from the push-out diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C_\star(A) \arrow[d] \\ X \arrow[r, "j"] & C_{\star f} \end{tikzcd}$$

□

Theorem 11.30. *Let $f : A \to X$ be a cofibration between well-pointed spaces. Then the natural map*

$$\bar{r} : C_{\star f} \to X/A$$

is a homotopy equivalence. In other words, the cofiber is homotopy equivalent to the homotopy cofiber.

Proof. Since $C_\star(A) \to C_{\star f}$ is a cofibration and $C_\star(A)$ is contractible, Proposition 11.18 implies

$$C_{\star f} \to C_{\star f}/C_\star(A) = X/A$$

is a homotopy equivalence. □

Theorem 11.31. *Let $i : A \to X$ be a cofibration between well-pointed spaces. The following sequence*

$$A \longrightarrow X \longrightarrow X/A \longrightarrow \Sigma A \longrightarrow \Sigma X \longrightarrow \Sigma(X/A) \longrightarrow \Sigma^2 A \longrightarrow \cdots$$

is co-exact in $\underline{\mathrm{h}\mathscr{T}_\star}$.

12 CW COMPLEX

CW complex

Recall that $S^{n-1} \hookrightarrow D^n$ is a cofibration satisfying HEP, where, D^n is the n-**disk** and $S^{n-1} = \partial D^n$ is its boundary, the $(n-1)$-sphere. Let

$$e^n = (D^n)^\circ = D^n - \partial D^n$$

denote the interior of D^n, the open disk known as the n-**cell**.

The category of CW-complex consists of topological spaces that can be built from n-cells (like lego, and thus behaves nicely just like $S^{n-1} \hookrightarrow D^n$). Moreover, it is large enough to cover most interesting examples.

Definition 12.1. A **cell decomposition** of a space X is a family

$$\mathcal{E} = \{e^n_\alpha | \alpha \in J_n\}$$

of subspaces of X such that each e^n_α is a n-cell and we have a disjoint union of sets

$$X = \coprod e^n_\alpha.$$

The n-**skeleton** of X is the subspace

$$X^n = \coprod_{\alpha \in J_m, m \leq n} e^m_\alpha.$$

Definition 12.2. A **CW complex** is a pair $(X, \mathcal{E})$ of a Hausdorff space X with a cell decomposition such that

1°. Characteristic map: for each n-cell e^n_α, there is a characteristic map

$$\Phi_{e^n_\alpha} : D^n \to X$$

such that the restriction of $\Phi_{e^n_\alpha}$ to $(D^n)^\circ$ is a homeomorphism to e^n_α and $\Phi_{e^n_\alpha}(S^{n-1}) \subset X^{n-1}$.

2°. **C=Closure finiteness**: for any cell $e \in \mathcal{E}$ the closure $\bar{e}$ intersects only a finite number of cells in $\mathcal{E}$.

3°. **W=Weak topology**: a subset $A \subset X$ is closed if and only if $A \cap \bar{e}$ is closed in $\bar{e}$ for each $e \in \mathcal{E}$.

We say X is an n-dim CW complex if the maximal dimension of cells in $\mathcal{E}$ is n (n could be ∞).

Note that the Hausdorff property of X implies that $\bar{e} = \Phi_e(D^n)$ for each cell $e \in \mathcal{E}$. The surjective map $\Phi_e : D^n \to \bar{e}$ is a quotient since D^n is compact and $\bar{e}$ is Hausdorff. Let us denote the full characteristic maps

$$\Phi : \coprod_{e \in \mathcal{E}} D^n \xrightarrow{\coprod \Phi_e} X.$$

Then the weak topology implies that Φ is a quotient map. This implies the following proposition.

Proposition 12.3. *Let $(X, \mathcal{E})$ be a CW complex. Then $f : X \to Y$ is continuous if and only if*

$$f \circ \Phi_e : D^n \to Y$$

is continuous for each $e \in \mathcal{E}$.

Proposition 12.4. *Let $(X, \mathcal{E})$ be a CW complex. Then any compact subspace of X meets only finitely many cells.*

Proof. Assume K is a compact subspace of X which meets infinitely many cells. Let $x_i \in K \cap e_i, i = 1, 2, \cdots$, where e_i's are different cells. Consider the subset

$$Z_m = \{x_m, x_{m+1}, \cdots\}, \quad m \geq 1.$$

By the closure finiteness, Z_m intersects each closure $\bar{e}$ by finite points, hence closed in $\bar{e}$ by the Hausdorff property. By the weak topology, Z_m is a closed subset of X, hence closed in K. Observe

$$\bigcap_{m \geq 1} Z_m = \emptyset$$

but any finite intersection of Z_m's is non-empty. This contradicts the compactness of K. □

Proposition 12.5. *Let $(X, \mathcal{E})$ be a CW complex and X^n be the n-skeleton. Then X is the colimit (i.e. direct limit) of the telescope diagram*

$$X^1 \to X^2 \to \cdots \to X^n \to \cdots$$

Proof. This is because $f : X \to Y$ is continuous if and only if $f : X^n \to Y$ is continuous for each n.

□

Proposition 12.6. *Let $(X, \mathcal{E})$ be a CW complex. Then X is compactly generated weak Hausdorff.*

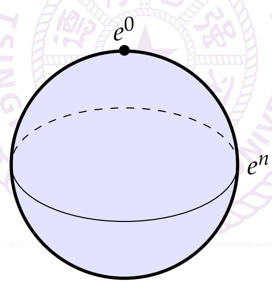
Proof. X is Hausdorff, hence also weak Hausdorff. We check X is compactly generated.

Assume $Z \subset X$ is k-closed. Since the closure of each cell $\bar{e}$ is compact Hausdorff, $Z \cap \bar{e}$ is closed in $\bar{e}$. The weak topology implies that Z is closed in X.

□

Example 12.7. Here are some classical examples.

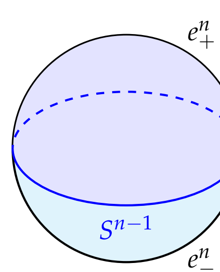
- The n-sphere S^n as a 0-cell and an n-cell:



  In this case, we have

$$S^n = e^0 \cup e^n$$

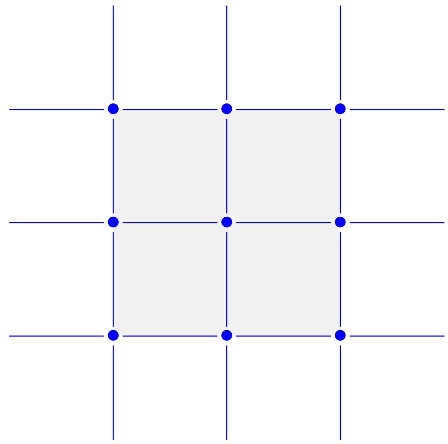
- The n-sphere S^n with two n-cells and a $(n-1)$-sphere:



  Thus we have

$$\begin{aligned} S^n &= e^n_+ \cup e^n_- \cup S^{n-1} \\ &= (e^n_+ \cup e^n_-) \cup \left(e^{n-1}_+ \cup e^{n-1}_-\right) \cup \cdots \cup \left(e^0_+ \cup e^0_-\right) \end{aligned}$$

- Gride/cube decomposition of $\mathbb{R}^n$ into n-cubes $I^n \simeq D^n$.



- $\mathbb{CP}^n$: $(\mathbb{C}^{n+1} - \{0\})/\sim$ and we have

$$\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \cdots \mathbb{CP}^{n-1} \subset \mathbb{CP}^n \subset \cdots \subset \mathbb{CP}^\infty.$$

  Moreover,

$$\begin{aligned} \mathbb{CP}^n - \mathbb{CP}^{n-1} &= \{[z_0, \ldots, z_n] \mid z_n \neq 0\} \\ &\simeq \mathbb{C}^n \simeq e^{2n}. \end{aligned}$$

  Thus $\mathbb{CP}^n$ has one cell in every even dimension from 0 to $2n$ with characteristic map

$$\begin{aligned} \Phi_{2n} \colon D^{2n} &\longrightarrow \mathbb{CP}^n \\ (z_0, \ldots, z_n) &\mapsto \left[z_0, \ldots, z_{n-1}, \sqrt{1 - \Sigma_{i-0}^{n-1} |z_i|^2}\right] \end{aligned}$$

Definition 12.8. A subcomplex $(X', \mathcal{E}')$ of the CW complex $(X, \mathcal{E})$ is a closed subspace $X' \subset X$ with a cell decomposition $\mathcal{E}' \subset \mathcal{E}$. We will just write $X' \subset X$ when the cell decomposition is clear. We will also write $X' = |\mathcal{E}'|$. Equivalently, a subcomplex is described by a subset $\mathcal{E}' \subset \mathcal{E}$ such that

$$e_1 \in \mathcal{E}', e_2 \in \mathcal{E}, \bar{e}_1 \cap e_2 \neq \emptyset \Longrightarrow e_2 \in \mathcal{E}'.$$

Example 12.9. The n-skeleton X^n is a subcomplex of X of dimension $\leq n$.

Attaching cells

Definition 12.10. Given $f : S^{n-1} \to X$. Consider the push-out

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, hook] \\ D^n \arrow[r, "\Phi_f"] & D^n \coprod_f X \end{tikzcd}$$

We say $D^n \coprod_f X$ is obtained by attaching an n-cell to X.

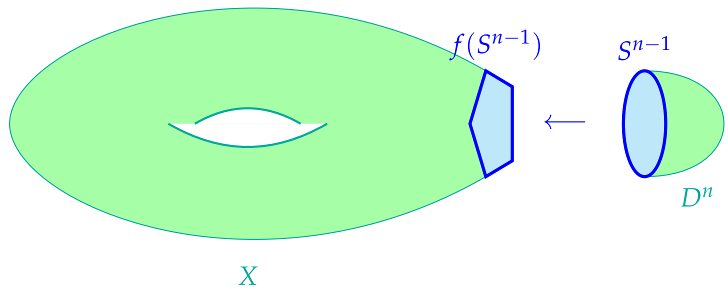


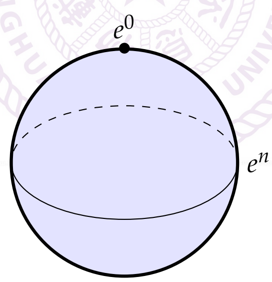
FIGURE 21. Attaching a cell

Φ_f is called the characteristic map of the attached n-cell. More generally, if we have a set of maps $f_\alpha : S^{n-1} \to X$, then the push-out

$$\begin{tikzcd}
\coprod_\alpha S^{n-1} \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, hook] \\
\coprod_\alpha D^n \arrow[r, "\Phi_f"] & (\coprod D^n) \coprod_f X
\end{tikzcd} \qquad f = \coprod f_\alpha$$

is called attaching n-cells to X.

Example 12.11. The n-sphere S^n can be obtained by attaching an n-cell to a point.



Proposition 12.12. *Let $(X, \mathcal{E})$ be a CW complex, and $\mathcal{E} = \coprod \mathcal{E}^n$ where $\mathcal{E}^n$ is the set of n-cells. Then the diagram*

$$\begin{tikzcd}
\coprod_{e \in \mathcal{E}^n} S^{n-1} \arrow[r, "\partial\Phi^n"] \arrow[d, hook] & X^{n-1} \arrow[d] \\
\coprod_{e \in \mathcal{E}^n} D^n \arrow[r, "\Phi^n"] & X^n
\end{tikzcd} \qquad \Phi^n = \coprod_{e \in \mathcal{E}^n} \Phi_e$$

is a push-out. In particular, X^n is obtained from X^{n-1} by attaching n-cells in X.

Proof. This follows from the fact that X^{n-1} is a closed subspace of X^n and the weak topology. $\square$

The converse is also true. The next theorem can be viewed as an alternate definition of CW complex.

Theorem 12.13. *Suppose we have a sequence of spaces*

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^n \subset X^{n+1} \subset \cdots$$

where X^n is obtained from X^{n-1} by attaching n-cells. Let $X = \cup_{n\geq 0} X^n$ be the union with the weak topology: $A \subset X$ is closed if and only if $A \cap X^n$ is closed in X^n for each n. Then X is a CW complex.

The theorem follows directly from the next lemma.

Lemma 12.14. *Let X be a $(n-1)$-dim CW complex and Y is obtained from X by attaching n-cells. Then Y is a n-dim CW complex.*

Proof. We need to check the following properties of Y.

H: The Hausdroff property of Y. Take $x, y \in Y$. If x lies in an n-cell, then it is easy to separate x from y. Otherwise, let $x, y \in X$ and take their open neighbourhoods U, V in X that separate them. Consider attaching the n-cells via the push-out:

$$\begin{tikzcd} \coprod_\alpha S^{n-1} \arrow[r, "\coprod_\alpha g_\alpha"] \arrow[d, hook] & X \arrow[d, hook] \\ \coprod_\alpha D^n \arrow[r, "\coprod_\alpha \Phi_\alpha"] & Y \end{tikzcd}$$

Then $g_\alpha^{-1}(U), g_\alpha^{-1}(V)$ are open in S^{n-1}. Take their open neighbourhoods U_α, V_α in D^n, i.e.

$$U_\alpha \cap S^{n-1} = g_\alpha^{-1}(U), \qquad V_\alpha \cap S^{n-1} = g_\alpha^{-1}(V)$$

such that $U_\alpha \cap V_\alpha = \emptyset$. Then $U \cup (\bigcup_\alpha U_\alpha)$ and $V \cup (\bigcup_\alpha V_\alpha)$ are separated neighbourhoods of x, y.

C: Closure finiteness follows from the fact that S^{n-1} is compact.

W: Weak topology follows from the push-out construction.

□

Definition 12.15. Let A be a subspace of X. A CW decomposition of (X, A) consists of a sequence

$$A = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X$$

such that X^n is obtained from X^{n-1} by attaching n-cells and X carries the weak topology with respect to the subspaces X^n. The pair (X, A) is called a **relative CW complex**.

We say (X, A) has relative dimension n if the maximal dimension of cells attached is n (n could be ∞).

Note that for a relative CW complex (X, A), A itself may not be a CW complex.

Proposition 12.16. *Let (X, A) be a relative CW complex. Then $A \subset X$ is a cofibration.*

Proof. $S^{n-1} \hookrightarrow D^n$ is a cofibration, and cofibration is preserved under push-out, so each

$$X^{n-1} \to X^n$$

is a cofibration. The proposition follows since composition of cofibrations is a cofibration. □

Corollary 12.17. *Let X be a CW complex and X' be a CW subcomplex. Then $X' \to X$ is a cofibration.*

Proof. (X, X') is a relative CW complex. □

Product of CW complexes

Let $(X,\mathcal{E}),(Y,\tilde{\mathcal{E}})$ be two CW complexes. We can define a cellular structure on $X\times Y$ with n-skeleton

$$(X\times Y)^n=\{e^k_\alpha\times\tilde{e}^l_\beta|0\le k+l\le n,\quad e^k_\alpha\in\mathcal{E},\tilde{e}^l_\beta\in\tilde{\mathcal{E}}\}$$

and characteristic maps

$$\Phi^{k,l}_{\alpha,\beta}=(\Phi^l_\alpha,\Phi^l_\beta):D^{k+l}_{\alpha,\beta}\to X\times Y.$$

Here we use the fact that $D^{k+l}_{\alpha,\beta}\equiv D^k_\alpha\times D^l_\beta$ topologically.

Example 12.18. Cellular decomposition for $S^1\times S^1$.

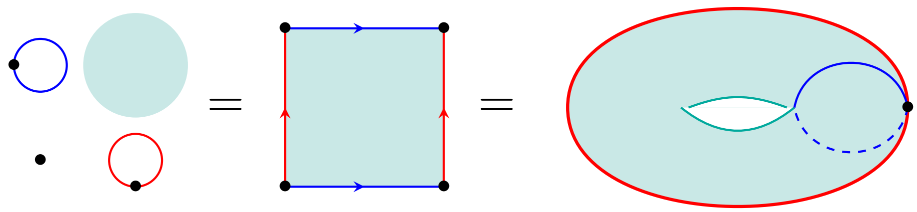


FIGURE 22. Cellular decomposition for $S^1\times S^1$

This natural cellular structure is closure finite. However, the product topology on $X\times Y$ may not be the same as the weak topology, so the topological product may not be a CW complex. Observe that X,Y are compactly generated weak Hausdorff, and we can take their categorical product in the category $\underline{\mathscr{T}}$. Then this compactly generated product will have the weak topology, and becomes a CW complex.

By Proposition 7.8, we have the following useful criterion.

Theorem 12.19. *Let $X.Y$ be CW complexes and Y be locally compact. Then the topological product $X\times Y$ is a CW complex.*

Example 12.20. If X is a CW complex, then $X\times I$ is a CW complex.

Definition 12.21. A CW complex X is called locally finite if each point in X has an open neighborhood that intersects only finite many cells.

It is easy to see that locally finite CW complexes are locally compact Hausdorff.

Corollary 12.22. *Let $X.Y$ be CW complexes and Y be locally finite. Then the topological product $X\times Y$ is a CW complex.*

13 WHITEHEAD THEOREM AND CW APPROXIMATION

Relative homotopy group

Definition 13.1. We define the category $\underline{\mathbf{TopP}}$ of topological pairs where an object (X, A) is a topological space X with a subspace A, and morphisms $(X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ such that $f(A) \subset B$. A homotopy between two maps $f_1, f_2 : (X, A) \to (Y, B)$ is a homotopy $F : X \times I \to Y$ between f_0, f_1 such that $F|_{X \times t}(A) \subset B$ for any $t \in I$.

The quotient category of $\underline{\mathbf{TopP}}$ by homotopy of maps is denoted by $\underline{\mathrm{h}\mathbf{TopP}}$. The pointed versions are defined similarly and denoted by $\underline{\mathbf{TopP}_\star}$ and $\underline{\mathrm{h}\mathbf{TopP}_\star}$. Morphisms in $\underline{\mathrm{h}\mathbf{TopP}}$ and $\underline{\mathrm{h}\mathbf{TopP}_\star}$ are denoted by

$$[(X, A), (Y, B)], \qquad [(X, A), (Y, B)]_0.$$

When we work with the convenient category $\underline{\mathscr{T}}$, we have similar notions of $\underline{\mathscr{T}\mathbf{P}}$ for a pair of spaces, $\underline{\mathrm{h}\mathscr{T}\mathbf{P}}$ for the quotient homotopy category, and $\underline{\mathscr{T}\mathbf{P}_\star}, \underline{\mathrm{h}\mathscr{T}\mathbf{P}_\star}$ for the pointed cases.

Theorem 13.2. *Let $f : (X, A) \to (Y, B)$ in $\underline{\mathrm{h}\mathscr{T}\mathbf{P}_\star}$. Let $\bar{f} = f|_A$. Then the sequence*

$$(X, A) \to (Y, B) \to (C_f, C_{\bar{f}}) \to \Sigma(X, A) \to \Sigma(Y, B) \to \Sigma(C_f, C_{\bar{f}}) \to \Sigma^2(X, A) \to \cdots$$

is co-exact in $\underline{\mathrm{h}\mathscr{T}\mathbf{P}_\star}$.

This generalizes the co-exact Puppe sequence to the pair case. See [Spanier] for a proof.

Definition 13.3. Let $(X, A) \in \underline{\mathscr{T}\mathbf{P}_\star}$. We define the **relative homotopy group** $\pi_n(X, A)$ by

$$\pi_n(X, A) = [(D^n, S^{n-1}), (X, A)]_0.$$

We will also write $\pi_n(X, A; x_0)$ when we want to specify the base point.

Note that

$$(D^n, S^{n-1}) \simeq \Sigma^{n-1}(D^1, S^0), \quad n \geq 2.$$

Therefore $\pi_n(X, A)$ is a group for $n \geq 2$ due to the adjunct pair (Σ, Ω).

Lemma 13.4. *$f : (D^n, S^{n-1}) \to (X, A)$ is zero in $\pi_n(X, A)$ if and only if f is homotopic rel S^{n-1} to a map whose image lies in A.*

Proof. Assume $[f]_0 = 0$ in $\pi_n(X, A)$. Then we can find a homotopy

$$F : D^n \times I \to X \quad \text{such that} \quad F(-, 0) = x_0, \quad F(S^{n-1}, t) \subset A, \quad F(-, 1) = f(-).$$

Let us view the restriction of F to $S^{n-1} \times I \cup D^n \times \{0\}$ as defining a map (via a natural homeomorphism)

$$g : (D^n, S^{n-1}) \to (X, A).$$

Then F can be viewed as defining a homotopy $g \simeq f$ rel S^{n-1} as required.

Conversely, assume there exists $g : (D^n, S^{n-1}) \to (X, A)$ such that $g \simeq f$ rel S^{n-1}. Let

$$F : D^n \times I \to D^n$$

be a homotopy from the identity to the trivial map. Then the homotopy

$$F \circ g : D^n \times I \to X$$

shows that $[g]_0 = 0$, hence $[f]_0 = 0$ as well.

□

This lemma can be illustrated by the following diagram

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, hook] & A \arrow[d, hook] \\ D^n \arrow[ur, dashed, "g"] \arrow[r, bend left] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow"] & X \end{tikzcd}$$

Here g maps D^n to A and $g \simeq f$ rel S^{n-1}.

Theorem 13.5. *Let $A \subset X$ in $\underline{\mathscr{T}_\star}$. Then there is a long exact sequence*

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \cdots \to \pi_0(X)$$

Here the boundary map ∂ sends $\varphi \in [(D^n, S^{n-1}), (X, A)]_0$ to its restriction to S^{n-1}.

Proof. Consider

$$f : (S^0, \{0\}) \to (S^0, S^0).$$

Let $\bar{f} = f|_{\{0\}} : \{0\} \to S^0$. It is easy to see that

$$(C_f, C_{\bar{f}}) \simeq (D^1, S^0).$$

Since $\Sigma^n(S^0) = S^n, \Sigma(D^n, S^{n-1}) = (D^{n+1}, S^n)$, the co-exact Puppe sequence

$$(S^0, \{0\}) \to (S^0, S^0) \to (D^1, S^0) \to (S^1, \{0\}) \to (S^1, S^1) \to (D^2, S^1) \to (S^2, \{0\}) \to \cdots$$

implies the exact sequence

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \cdots \to \pi_0(X)$$

□

Definition 13.6. A pair (X, A) is called **n-connected** ($n \geq 0$) if $\pi_0(A) \to \pi_0(X)$ is surjective and

$$\pi_k(X, A; x_0) = 0 \quad \forall 1 \leq k \leq n, x_0 \in A.$$

From the long exact sequence

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \cdots \to \pi_0(X)$$

we see that (X, A) is n-connected if and only if for any $x_0 \in A$

$$\begin{cases} \pi_r(A, x_0) \to \pi_r(X, x_0) \text{ is bijective for } r < n \\ \pi_n(A, x_0) \to \pi_n(X, x_0) \text{ is surjective} \end{cases}$$

Definition 13.7. A map $f : X \to Y$ is called an **n-equivalence** ($n \geq 0$) if for any $x_0 \in X$

$$\begin{cases} f_* : \pi_r(X, x_0) \to \pi_r(Y, f(x_0)) \text{ is bijective for } r < n \\ f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0)) \text{ is surjective} \end{cases}$$

f is called a **weak homotopy equivalence** or **∞-equivalence** if f is n-equivalence for any $n \geq 0$.

Example 13.8. For any $n \geq 0$, the pair (D^{n+1}, S^n) is n-connected.

Whitehead Theorem

Lemma 13.9. *Let X be obtained from A by attaching n-cells. Let (Y, B) be a pair such that*

$$\begin{cases} \pi_n(Y, B; b) = 0, \forall b \in B & \text{if} \quad n \geq 1 \\ \pi_0(B) \to \pi_0(Y) \text{ is surjective} & \text{if} \quad n = 0. \end{cases}$$

Then any map from $(X, A) \to (Y, B)$ is homotopic rel A to a map from X to B.

Proof. This follows from the universal property of push-out and Lemma 13.4.

$$\begin{tikzcd}
\coprod S^{n-1} \arrow[r] \arrow[d, hook] & A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\
\coprod D^n \arrow[r] \arrow[urr, dashed] & X \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y
\end{tikzcd}$$

□

Theorem 13.10. *Let (X, A) be a relative CW complex with relative dimension $\leq n$. Let (Y, B) be n-connected ($0 \leq n \leq \infty$). Then any map from (X, A) to (Y, B) is homotopic relative to A to a map from X to B.*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\
X \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y
\end{tikzcd}$$

Proof. Apply the previous Lemma to

$$A \subset X^0 \subset X^1 \subset \cdots \subset X^n = X$$

and observe that all embeddings are cofibrations. □

Proposition 13.11. *Let $f : X \to Y$ be a weak homotopy equivalence, P be a CW complex. Then*

$$f_* : [P, X] \to [P, Y]$$

is a bijection.

Proof. We can assume f is an embedding and (Y, X) is ∞-connected. Otherwise replace Y by M_f.

Surjectivity is illustrated by the diagram (applying Theorem 13.10 to the pair $(P, \varnothing)$)

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\
P \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y
\end{tikzcd}$$

Injectivity is illustrated by the diagram (observing $P \times I, P \times \partial I$ are CW complexes)

$$\begin{tikzcd}
P \times \partial I \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\
P \times I \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y
\end{tikzcd}$$

□

Theorem 13.12 (Whitehead Theorem)**.** *A map between CW complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

Proof. Let $f : X \to Y$ be a weak homotopy equivalence between two CW complexes. Apply Proposition 13.11 to $P = X$ and $P = Y$, we find bijections

$$f_* : [X, X] \to [X, Y], \quad f_* : [Y, X] \to [Y, Y].$$

Let $g \in [Y, X]$ such that $f_*[g] = 1_Y$. Then $f \circ g \simeq 1_Y$. On the other hand,

$$f_*[g \circ f] = [f \circ g \circ f] \simeq [f \circ 1] = [f] = f_*[1_X].$$

We conclude $[g \circ f] = 1_X$. Therefore f is a homotopy equivalence. The reverse direction is obvious. □

Remark 13.13. This is basically the combination of Proposition 13.11 and Yoneda Lemma.

Cellular Approximation

Definition 13.14. Let (X, Y) be CW complexes. A map $f : X \to Y$ is called **cellular** if $f(X^n) \subset Y^n$ for any n. We define the category **CW** whose objects are CW complexes and morphisms are cellular maps.

Definition 13.15. A **cellular homotopy** between two cellular maps $X \to Y$ of CW complexes is a homotopy $X \times I \to Y$ that is itself a cellular map. Here I is naturally a CW complex. We define the quotient category **hCW** of **CW** whose morphisms are cellular homotopy class of cellular maps.

Lemma 13.16. *Let X be obtained from A by attaching n-cells $(n \geq 1)$, then (X, A) is $(n-1)$-connected.*

Proof. Let $r < n$. Consider a diagram

$$\begin{tikzcd} S^{r-1} \arrow[r] \arrow[d, hook] & A \arrow[d, hook] \\ D^r \arrow[r, "f"] & X \end{tikzcd}$$

Since D^r is compact, $f(D^r)$ meets only finitely many attached n-cells on X, say $e_1, \cdots, e_m$. Let p_i be the center of e_i. Let $e_i^* = e_i - \{p_i\}$. $Y = X - \{p_1, \cdots, p_m\}$. We subdivide D^r into small disks $D^r = \cup_\alpha D^r_\alpha$ such that $f(D^r_\alpha) \subset Y$ or $f(D^r_\alpha) \subset e_i$. For each D^r_α such that $f(D^r_\alpha) \subset e_i$ but not in Y, we use the fact that $(e_i, e_i^*) \simeq (D^n, S^{n-1})$ is $(n-1)$-connected to find a homotopy rel ∂D^r_α to adjust mapping D^r_α into e_i^*. It glues together to obtain

$$\begin{tikzcd} S^{r-1} \arrow[r] \arrow[d, hook] & Y \arrow[d, hook] \\ D^r \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & X \end{tikzcd}$$

Then we can further find a homotopy

$$\begin{tikzcd} S^{r-1} \arrow[r] \arrow[d, hook] & A \arrow[d, hook] \\ D^r \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y \end{tikzcd}$$

□

Corollary 13.17. *Let (X, A) be a relative CW complex, then for any $n \geq 0$, the pair (X, X^n) is n-connected.*

Theorem 13.18. *Let $f : (X, A) \to (\tilde{X}, \tilde{A})$ between relative CW complexes which is cellular on a subcomplex (Y, B) of (X, A). Then f is homotopic rel Y to a cellular map $g : (X, A) \to (\tilde{X}, \tilde{A})$.*

Proof. Assume we have constructed $f_{n-1} : (X, A) \to (\tilde{X}, \tilde{A})$ which is homotopic to f rel Y and cellular on the $(n-1)$-skeleton X^{n-1}. Let X^n be obtained from X^{n-1} by attaching n-cells. Consider

$$\begin{tikzcd} X^{n-1} \arrow[r] \arrow[d, hook] & \tilde{X}^n \arrow[d, hook] \\ X^n \arrow[r, "f_{n-1}"] & \tilde{X} \end{tikzcd}$$

Since X^n is obtained from X^{n-1} by attaching n-cells and $(\tilde{X}, \tilde{X}^n)$ is n-connected,

$$\begin{tikzcd} X^{n-1} \arrow[r] \arrow[d, hook] & \tilde{X}^n \arrow[d, hook] \\ X^n \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right, "f_{n-1}"'] \arrow[r, phantom, "\Downarrow"] & \tilde{X} \end{tikzcd}$$

we can find a homotopy rel X^{n-1} from $f_{n-1}|_{X^n} : X^n \to \tilde{X}$ to a map $X^n \to \tilde{X}^n$. Since f is cellular on Y, we can choose this homotopy rel Y by adjusting only those n-cells not in Y. This homotopy extends to a homotopy rel $X^{n-1} \cup Y$ from f_{n-1} to a map $f_n : X \to \tilde{X}$ since $X^n \subset X$ is a cofibration. Then f_∞ works. $\square$

Theorem 13.19 (Cellular Approximation Theorem)**.** *Any map between relative CW complexes is homotopic to a cellular map. If two cellular maps between relative CW complexes are homotopic, then they are cellular homotopic.*

Proof. Apply the previous Theorem to $(X, \emptyset)$ and $(X \times I, X \times \partial I)$. $\square$

This theorem says that <u>**hCW**</u> is a full subcategory of <u>h**Top**</u>.

CW Approximation

Definition 13.20. A CW approximation of a topological space Y is a CW complex X with a weak homotopy equivalence $f : X \to Y$.

Theorem 13.21. *Any space has a CW approximation.*

Proof. We may assume Y is path connected. We construct a CW approximation X of Y by induction on the skeleton X^n. Assume we have constructed $f_n : X^n \to Y$ which is an n-equivalence. We attach an $(n+1)$-cell to every generator of $\ker(\pi_n(X^n) \to \pi_n(Y))$ to obtain $\tilde{X}^{n+1}$. We can extend f_n to a map $\tilde{f}_{n+1} : \tilde{X}^{n+1} \to Y$

$$\begin{tikzcd} \coprod S^n \arrow[r] \arrow[d, hook] & X^n \arrow[d] \arrow[ddr, "f_n"] & \\ \coprod D^{n+1} \arrow[r] \arrow[drr, dashed] & \tilde{X}^{n+1} \arrow[dr, dashed, "\tilde{f}_{n+1}"] & \\ & & Y \end{tikzcd}$$

Since $(\tilde{X}^{n+1}, X^n)$ is also n-connected, $\tilde{f}_{n+1}$ is an n-equivalence. By construction and the surjectivity of $\pi_n(X^n) \to \pi_n(\tilde{X}^{n+1})$, $\tilde{f}_{n+1}$ defines also an isomorphism for $\pi_n(\tilde{X}^{n+1}) \to \pi_n(Y)$.

Now for every generator S^{n+1}_α of $\mathrm{coker}(\pi_{n+1}(\tilde{X}^{n+1}) \to \pi_{n+1}(Y))$, we take a wedge sum to obtain

$$X^{n+1} = \tilde{X}^{n+1} \vee (\vee_\alpha S^{n+1}).$$

Then the induced map $f_{n+1} : X^{n+1} \to Y$ extends f_n to an $(n+1)$-equivalence. Inductively we obtain a weak homotopy equivalence $f_\infty : X = X^\infty \to Y$. □

Theorem 13.22. *Let $f : X \to Y$. Let $\Gamma X \to X$, and $\Gamma Y \to Y$ be CW approximations. Then there exists a unique map in $[\Gamma X, \Gamma Y]$ making the following diagram commutative in* <u>h**Top**</u>

$$\begin{tikzcd}
\Gamma X \arrow[r, "\Gamma f"] \arrow[d] & \Gamma Y \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Proof. Weak homotopy equivalence of $\Gamma Y \to Y$ implies the bijection $[\Gamma_X, \Gamma_Y] \to [\Gamma_X, Y]$.

□

Definition 13.23. Two spaces X_1, X_2 are said to have the same **weak homotopy type** if there exist a space Y and weak homotopy equivalences $f_i : Y \to X_i, i = 1, 2$.

Theorem 13.24. *Weak homotopy type is an equivalence relation.*

Proof. Exercise.

□

14 EILENBERG-MACLANE SPACE

$\pi_n(S^n)$ and Degree

We have seen that $\pi_k(S^n) = 1$ for $k < n$. In this subsection we compute

$$\pi_n(S^n) = [S^n, S^n]_0 \cong \mathbb{Z}.$$

Given $f : S^n \to S^n$, its class $[f] \in \mathbb{Z}$ under the above isomorphism is called the **degree** of f.

Theorem 14.1 (Homotopy Excision Theorem). *Let $(A, C), (B, C)$ be relative CW complex. Let X be the push-out*

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

If (A, C) is m-connected and (B, C) is n-connected, then

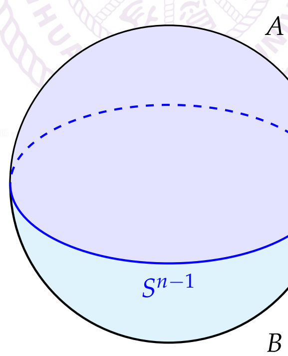
$$\pi_i(A, C) \to \pi_i(X, B)$$

is an isomorphism for $i < m + n$, and a surjection for $i = m + n$.

Corollary 14.2 (Freudenthal Suspension Theorem). *The suspension map*

$$\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$

is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$.



Proof. Apply Homotopy Excision to $X = S^{n+1}, C = S^n$, A the upper half disk, B the lower half disk. □

Freudenthal Suspension Theorem holds similarly replacing S^n by general $(n - 1)$-connected space.

Proposition 14.3. $\pi_n(S^n) \cong \mathbb{Z}$ *for* $n \geq 1$.

Proof. Freudenthal Suspension Theorem reduces to show $\pi_2(S^2) \cong \mathbb{Z}$. This follows from the Hopf fibration

$$S^1 \to S^3 \to S^2.$$

□

Eilenberg-MacLane Space

Definition 14.4. An **Eilenberg-MacLane Space** of type (G, n) is a CW complex X such that $\pi_n(X) \cong G$ and $\pi_k(X) = 0$ for $k \neq n$. Here G is abelian if $n > 1$.

As we will show next, Eilenberg-MacLane Space of any type (G, n) exists and is unique up to homotopy. It will be denoted by $K(G, n)$. The importance of $K(G, n)$ is that it is the representing space for cohomology functor with coefficients in G

$$H^n(X; G) \cong [X, K(G, n)] \quad \text{for any CW complex } X.$$

Theorem 14.5. *Eilenberg-MacLane Spaces exist.*

Proof. We prove the case for $n \geq 2$. There exists an exact sequence

$$0 \to F_1 \to F_2 \to G \to 0$$

where F_1, F_2 are free abelian groups. Let B_i be a basis of F_i. Let

$$A = \bigvee_{i \in B_1} S^n, \quad B = \bigvee_{j \in B_2} S^n.$$

A, B are $(n-1)$-connected and $\pi_n(A) = F_1, \pi_n(B) = F_2$. Using the degree map, we can construct

$$f : A \to B$$

such that $\pi_n(A) \to \pi_n(B)$ realizes the map $F_1 \to F_2$. Let X be obtained from B by attaching $(n+1)$-cells via f. Then X is $(n-1)$-connected and $\pi_n(X) = G$. Now we proceed as in the proof of Theorem 13.21 to attach cells of dimension $\geq (n+2)$ to kill all higher homotopy groups of X to get $K(G, n)$. □

Theorem 14.6. *Let X be an $(n-1)$-connected CW complex. Let Y be an Eilenberg-MacLane Space of type (G, n). Then the map*

$$\phi : [X, Y] \to \mathrm{Hom}(\pi_n(X), \pi_n(Y)), \quad f \to f_*$$

is a bijection. In particular, any two Eilenberg-MacLane Spaces of type (G, n) are homotopy equivalent.

Proof. Let us first do two simplifications. First, as in the proof of Theorem 13.21, we can find a CW complex Z and a weak homotopy equivalence $g : Z \to X$ such that the n-skeleton of Z is

$$Z^n = \bigvee_{j \in J} S^n.$$

By Whitehead Theorem, g is also a homotopy equivalence. So we can assume the n-skeleton of X is

$$X^n = \bigvee_{j \in J} S^n.$$

Secondly, let X^{n+1} be the $(n+1)$-skeleton of X. Then $\pi_n(X) = \pi_n(X^{n+1})$. Let $f : X \to Y$. Since X is obtained from X^{n+1} by attaching cells of dimension $\geq n+2$ and $\pi_k(Y) = 0$ for all $k > n$, any map $X^{n+1} \to Y$ can be extended to $X \to Y$. So the natural map

$$[X, Y] \to [X^{n+1}, Y]$$

is a surjection. Now assume $f : X \to Y$ such that its restriction to X^{n+1} is null-homotopic. Since $X^{n+1} \subset X$ is a cofibration, f is homotopic to a map which shrinks the whole X^{n+1} to a point. Since $\pi_k(Y) = 0$ for all $k > n$, f is further null-homotopic. This implies that

$$[X, Y] \to [X^{n+1}, Y]$$

is a bijection. So we can also assume $X = X^{n+1}$ has dimension at most $n+1$.

Assume X is obtained from X^n by attaching $(n+1)$-cells via the map

$$\chi : \bigvee_{i\in I} S^n \to \bigvee_{j\in J} S^n.$$

- Injectivity of ϕ. Assume $f : X \to Y$ such that $\phi(f) = 0$. Then the restriction of f to
$$X^n = \bigvee_{j\in J} S^n \to Y$$
is null-homotopic. Since $X^n \hookrightarrow X$ is a cofibration, f is homotopic to a map which shrinks X^n to a point, so can be viewed as a map
$$\bigvee_{i\in I} S^{n+1} \to Y.$$
Since $\pi_{n+1}(Y) = 0$, this map is also null-homotopic. This shows $[f] = 0$.
- Surjectivity of ϕ. Let $g : \pi_n(X) \to \pi_n(Y)$ be a group homomorphism. Since
$$j : \pi_n(X^n) \to \pi_n(X)$$
is surjective and $\pi_n(X^n)$ is free, we can find a map
$$f_n : X^n \to Y$$
such that $f_{n*} : \pi_n(X^n) \to \pi_n(Y)$ coincides with $g \circ j$. By construction, $f_n \circ \chi$ is null-homotopic, so we can extend f_n to a map $f : X \to Y$ which gives the required group homomorphism.

Now assume we have two Eilenberg-MacLane Spaces Y_1, Y_2 of type (G, n). We have the identification

$$[Y_1, Y_2] = \mathrm{Hom}(\pi_n(Y_1), \pi_n(Y_2)).$$

Then a group isomorphism $\pi_n(Y_1) \to \pi_n(Y_2)$ gives a homotopy equivalence $Y_1 \to Y_2$. □

Remark 14.7. A classical result of Milnor says the loop space of a CW complex is homotopy equivalent to a CW complex. Since for any X, we have $\pi_k(\Omega X) = \pi_{k+1}(X)$. Therefore

$$\Omega K(G, n) \simeq K(G, n-1).$$

Example 14.8. $S^1 = K(\mathbb{Z}, 1)$ and $\bigvee_{i=1}^m S^1 = K(Z^m, 1)$.

Example 14.9. Consider the fibration

$$S^1 \to S^{2n+1} \to \mathbb{CP}^n.$$

We have natural embeddings

$$\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \cdots \mathbb{CP}^{n-1} \subset \mathbb{CP}^n \subset \cdots \subset \mathbb{CP}^\infty$$

and

$$S^1 \subset S^3 \subset \cdots S^{2n-1} \subset S^{2n+1} \subset \cdots \subset S^\infty.$$

Here $\mathbb{CP}^\infty$ and S^∞ are the corresponding colimits. This gives rise to the fibration

$$S^1 \to S^\infty \to \mathbb{CP}^\infty.$$

Observe that $\pi_k(S^\infty) = 0$ for any k. In fact, for any map $f : S^k \to S^\infty$, since S^k is compact,

$$f(S^k) \subset S^n, \quad \text{for some} \quad n > k.$$

Since $\pi_k(S^n) = 0$, f is homotopic to the trivial map in S^n, hence also in S^∞. Using the long exact sequence of homotopy groups associated to the fibration $S^1 \to S^\infty \to \mathbb{CP}^\infty$, we find

$$\mathbb{CP}^\infty = K(\mathbb{Z}, 2).$$

Example 14.10. A **knot** is an embedding $K : S^1 \hookrightarrow S^3$. Let $G = \pi_1(S^3 - K)$. Then

$$S^3 - K = K(G, 1).$$

Postnikov Tower

Postnikov tower for a space is a decomposition dual to a cell decomposition. In the Postnikov tower description of a space, the building blocks of the space are Eilenberg-MacLane spaces.

Definition 14.11. A **Postnikov tower** of a path-connected space X is a tower diagram of spaces

$$\cdots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \ .$$

with a sequence of compatible maps $f_n : X \to X_n$ satisfying

1°. $f_n : X \to X_n$ induces an isomorphism $\pi_k(X) \to \pi_k(X_n)$ for any $k \leq n$
2°. $\pi_k(X_n) = 0$ for $k > n$
3°. each $X_n \to X_{n-1}$ is a fibration with fiber $K(\pi_n(X), n)$.

$$\begin{tikzcd}
& X_n \arrow[d] & K(\pi_n(X), n) \arrow[l] \\
X \arrow[ur, "f_n"] \arrow[r, "f_{n-1}"'] & X_{n-1} &
\end{tikzcd}$$

X_n is called a **n-th Postnikov approximation** of X.

Note that if X is $(n-1)$-connected, then $X_n = K(\pi_n(X), n)$. In general, a Postnikov tower can be viewed as an approximation of a space by twisted product of Eilenberg-MacLane spaces.

Theorem 14.12. *Postnikov Tower exists for any connected CW complex.*

Proof. Let X be a connected CW complex. Let us construct Y_n which is obtained from X by successively attaching cells of dimensions $n+2, n+3, \cdots$ to kill homotopy groups $\pi_k(X)$ for $k > n$. Then we have a CW subcomplex $X \subset Y_n$ such that

$$\begin{cases} \pi_k(X) \to \pi_k(Y_n) \text{ is an isomorphism} & \text{if} \quad k \leq n \\ \pi_k(Y_n) = 0 & \text{if} \quad k > n. \end{cases}$$

Since $\pi_k(Y_{n-1}) = 0$ for $k \geq n$, we can extend the map $X \to Y_{n-1}$ to a map $Y_n \to Y_{n-1}$ making the following diagram commutative

$$\begin{tikzcd}
& X \arrow[dl] \arrow[dr] & \\
Y_n \arrow[rr] & & Y_{n-1}
\end{tikzcd}$$

In this way we find a tower diagram

$$\begin{tikzcd}
& & X \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\
\cdots \arrow[r] & Y_{n+1} \arrow[r] & Y_n \arrow[r] & \cdots \arrow[r] & Y_2 \arrow[r] & Y_1
\end{tikzcd}.$$

Now we can replace $Y_2 \to Y_1$ by a fibration, and then similarly adjust $Y_3, Y_4, \cdots$ successively to end up with

$$\begin{tikzcd}
& & X \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\
\cdots \arrow[r] \arrow[d] & Y_{n+1} \arrow[r] \arrow[d] & Y_n \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\
\cdots \arrow[r] & X_{n+1} \arrow[r] & X_n \arrow[r] & \cdots \arrow[r] & X_2 \arrow[r] & X_1 = Y_1
\end{tikzcd}.$$

such that each $X_n \to X_{n-1}$ is a fibration with fiber F_n. Since X_n is homotopy equivalent to Y_n, we have

$$\begin{cases} \pi_k(X_n) = \pi_k(X) & \text{if} \quad k \leq n \\ \pi_k(X_n) = 0 & \text{if} \quad k > n. \end{cases}$$

Then the long exact sequence of homotopy groups associated to the fibration $F_n \to X_n \to X_{n-1}$ implies

$$F_n \simeq K(\pi_n(X), n).$$

□

Whitehead Tower

Whitehead Tower is a sequence of fibrations that generalize the universal covering of a space.

Theorem 14.13 (Whitehead Tower)**.** *Let X be a connected CW complex. There is a sequence of maps*

$$\cdots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 = X$$

where each map $X_n \to X_{n-1}$ is a fibration with fiber $K(\pi_n(X), n-1)$. Each X_n satisfies

$$\begin{cases} \pi_k(X_n) \to \pi_k(X) \text{ is an isomorphism} & \text{if} \quad k > n \\ \pi_k(X_n) = 0 & \text{if} \quad k \leq n. \end{cases}$$

Proof. Let $Y_1 \simeq K(\pi_1(X), 1)$ be obtained from X by successively attaching cells to kill $\pi_k(X)$ for $k > 1$. Let $j_1 : X \subset Y_1$ and $X_1 = F_{j_1}$ be the homotopy fiber. Then we have a fibration

$$\begin{tikzcd}
\Omega Y_1 \arrow[r] & X_1 \arrow[d] \\
& X
\end{tikzcd}$$

Note that $\Omega Y_1 \simeq K(\pi_1(X), 0)$ and $\pi_1(X_1) = 0$. So X_1 can be viewed as the universal cover of X up to homotopy equivalence.

Similarly, assume we have constructed the Whitehead Tower up to X_n. Let $Y_n \simeq K(\pi_n(X), n)$ be obtained from X_n by killing homotopy groups $\pi_k(X)$ for $k > n$. Let $j_n : X_n \subset Y_n$. Then we define $X_{n+1} = F_{j_n}$ to be the homotopy fiber. Repeating this process, we obtain the Whitehead Tower. □

15 SINGULAR HOMOLOGY

Chain complex

Definition 15.1. Let R be a commutative ring. A **chain complex** over R is a sequence of R-module maps

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

such that $\partial_n \circ \partial_{n+1} = 0\ \forall n$. When R is not specified, we mean chain complex of abelian groups (i.e. $R = \mathbb{Z}$).

Sometimes we just write the map by ∂ and the chain complex by $(C_\bullet, \partial)$. Then $\partial_n = \partial|_{C_n}$ and $\partial^2 = 0$.

Definition 15.2. A **chain map** $f : C_\bullet \to C'_\bullet$ between two chain complexes over R is a sequence of R-module maps $f_n : C_n \to C'_n$ such that the following diagram is commutative

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"'] & C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

This can be simply expressed as

$$\boxed{f \circ \partial = \partial' \circ f}$$

Definition 15.3. We define the category $\underline{\mathrm{Ch}_\bullet(\mathrm{R})}$ whose objects are chain complexes over R and morphisms are chain maps. We simply write $\underline{\mathbf{Ch}_\bullet}$ when $R = \mathbb{Z}$.

Definition 15.4. Given a chain complex $(C_\bullet, \partial)$, we define its n**-cycles** Z_n and n**-boundaries** B_n by

$$Z_n = \mathrm{Ker}(\partial : C_n \to C_{n-1}), \quad B_n = \mathrm{Im}(\partial : C_{n+1} \to C_n).$$

The equation $\partial^2 = 0$ implies $B_n \subset Z_n$. We define the n**-th homology group** by

$$\boxed{\mathrm{H}_n(C_\bullet, \partial) := \frac{Z_n}{B_n} = \frac{\ker(\partial_n)}{\mathrm{im}(\partial_{n+1})}}.$$

A chain complex $C_\bullet$ is called **acyclic** or **exact** if $\mathrm{H}_n(C_\bullet) = 0$ for any n.

Proposition 15.5. *The n-th homology group defines a functor*

$$\mathrm{H}_n : \underline{\mathbf{Ch}_\bullet} \to \underline{\mathbf{Ab}}.$$

Proof. We only need to check any $f\colon C_\bullet \to C'_\bullet$ induces a group homomorphism

$$f_*\colon\ \mathrm{H}_n(C_\bullet) \to \mathrm{H}_n(C'_\bullet).$$

This is because

- if $\alpha \in Z_n(C_\bullet)$, then $f(\alpha) \in Z_n(C'_\bullet)$;
- if $\alpha \in B_n(C_\bullet)$, then $f(\alpha) \in B_n(C'_\bullet)$.

□

Definition 15.6. A chain map $f : C_\bullet \to D_\bullet$ is called a **quasi-isomorphism** if

$$f_* : \mathrm{H}_n(C_\bullet) \to \mathrm{H}_n(D_\bullet)$$

is an isomorphism for all n.

Definition 15.7. A **chain homotopy** $f \overset{s}{\simeq} g$ between two chain maps $f, g : C_\bullet \to C'_\bullet$ is a sequence of homomorphisms $s_n : C_n \to C'_{n+1}$ such that $f_n - g_n = s_{n-1} \circ \partial_n + \partial'_{n+1} \circ s_n$, or simply

$$\boxed{f - g = s \circ \partial + \partial' \circ s}.$$

Two complexes $C_\bullet, C'_\bullet$ are called **chain homotopy equivalent** if there exist chain maps $f : C_\bullet \to C'_\bullet$ and $h : C'_\bullet \to C_\bullet$ such that $f \circ g \simeq 1$ and $g \circ f \simeq 1$.

Proposition 15.8. *Chain homotopy defines an equivalence relation on chain maps and compatible with compositions.*

In other words, chain homotopy defines an equivalence relation on $\underline{\mathbf{Ch}_\bullet}$. We define the quotient category

$$\underline{\mathrm{h}\mathbf{Ch}_\bullet} = \underline{\mathbf{Ch}_\bullet} / \simeq .$$

Chain homotopy equivalence becomes an isomorphism in $\underline{\mathrm{h}\mathbf{Ch}_\bullet}$.

Proposition 15.9. *Let f, g be chain homotopic chain maps. Then they induce identical map on homology groups*

$$\mathrm{H}_n(f) = \mathrm{H}_n(g) : \mathrm{H}_n(C_\bullet) \to \mathrm{H}_n(C'_\bullet).$$

In other words, the functor H_n factors through

$$\mathrm{H}_n : \underline{\mathbf{Ch}_\bullet} \to \underline{\mathrm{h}\mathbf{Ch}_\bullet} \to \underline{\mathbf{Ab}}\,.$$

Proof. Let $f - g = s \circ \partial + \partial' \circ s$. Consider

$$f_* - g_* \colon \mathrm{H}_n(C_\bullet) \to \mathrm{H}_n(C'_\bullet).$$

Let $\alpha \in C_n$ be a representative of a class $[\alpha]$ in $\mathrm{H}_n(C_\bullet)$. Since $\partial\alpha = 0$, we have

$$(f - g)(\alpha) = (s \circ \partial + \partial' \circ s)(\alpha) = \partial' \circ (s(\alpha)) \in B_n(C'_\bullet).$$

So $[f(\alpha)] = [g(\alpha)]$. Hence $f_* = g_*$ on homologies. ∎

Singular homology

Definition 15.10. We define the **standard n-simplex**

$$\Delta^n = \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1, t_i \geq 0\}$$

We let $\{v_0, \cdots, v_n\}$ denote its vertices. Here $v_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ where 1 sits at the i-th position.

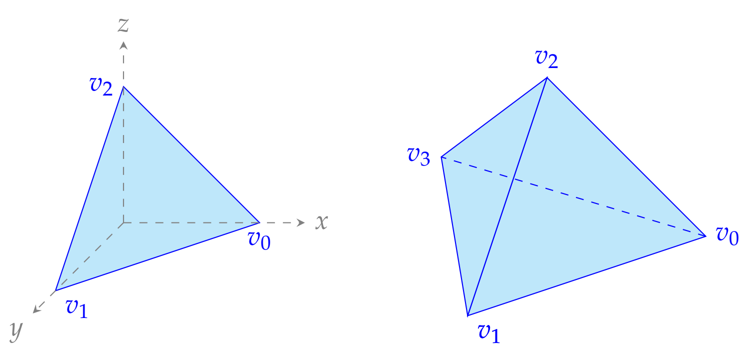


FIGURE 23. Standard 2-simplex Δ^2 and 3-simplex Δ^3

Definition 15.11. Let X be a topological space. A **singular n-simplex** in X is a continuous map $\sigma : \Delta^n \to X$. For each $n \geq 0$, we define $S_n(X)$ to be the free abelian group generated by all singular n-simplexes in X

$$S_n(X) = \bigoplus_{\sigma \in \mathrm{Hom}(\Delta^n, X)} \mathbb{Z}\sigma.$$

An element of $S_n(X)$ is called a **singular n-chain** in X.

A singular n-chain is given by a finite formal sum

$$\gamma = \sum_{\sigma \in \mathrm{Hom}(\Delta^n, X)} m_\sigma \sigma,$$

for $m_\sigma \in \mathbb{Z}$ and only finitely many m_σ's are nonzero. The abelian group structure is:

$$-\gamma := \sum_\sigma (-m_\sigma)\sigma$$

and

$$(\sum_\sigma m_\sigma \sigma) + (\sum_\sigma m'_\sigma \sigma) = \sum_\sigma (m_\sigma + m'_\sigma)\sigma.$$

Definition 15.12. Given a singular n-simplex $\sigma : \Delta^n \to X$ and $0 \leq i \leq n$, we define

$$\partial^{(i)}\sigma : \Delta^{n-1} \to X$$

to be the $(n-1)$-simplex by restricting σ to the i-th face of Δ^n whose vertices are given by $\{v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n\}$. We define the **boundary map**

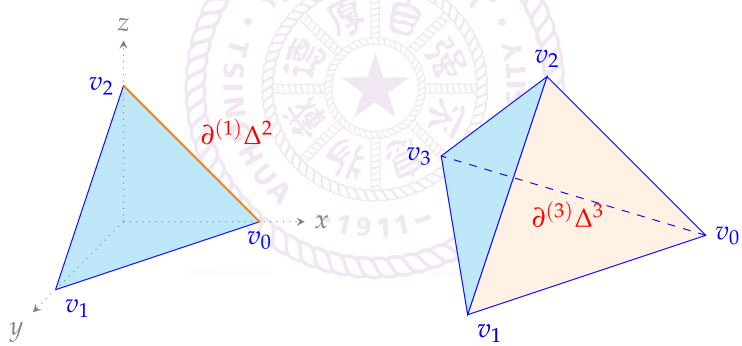


FIGURE 24. Faces of 2-simplex Δ^2 and 3-simplex Δ^3

$$\partial : S_n(X) \to S_{n-1}(X)$$

to be the abelian group homomorphism generated by

$$\boxed{\partial\sigma := \sum_{i=0}^{n} (-1)^i \partial^{(i)}\sigma}.$$

Given a subset $\{v_{i_1}, \cdots, v_{i_k}\}$ of the vertices of Δ^n, we will write

$$\sigma|[v_{i_1}, \cdots, v_{i_k}] \quad \text{or just } [v_{i_1}, \cdots, v_{i_k}] \text{ (when it is clear from the context)}$$

for restricting σ to the face of Δ^n spanned by $\{v_{i_1}, \cdots, v_{i_k}\}$. Then the boundary map can be expressed by

$$\partial[v_0, \cdots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n].$$

Proposition 15.13. *$(S_\bullet(X), \partial)$ defines a chain complex, i.e., $\partial^2 = \partial \circ \partial = 0$.*

Proof.

$$\begin{aligned}\partial \circ \partial [v_0, \cdots, v_n] &= \partial \sum_{i=0}^{n} (-1)^i [v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n] \\ &= \sum_{i<j} (-1)^i (-1)^{j+1} [v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n] + \sum_{j<i} (-1)^i (-1)^j [v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n] \\ &= 0.\end{aligned}$$

□

Example 15.14. Consider a 2-simplex $\sigma : \Delta^2 \to X$. Then

$$\partial \sigma = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

and

$$\partial^2 \sigma = ([v_2] - [v_1]) - ([v_2] - [v_0]) + ([v_1] - [v_0]) = 0.$$

Definition 15.15. For each $n \geq 0$, we define the n-th singular homology group of X by

$$\boxed{\mathrm{H}_n(X) := \mathrm{H}_n(S_\bullet(X), \partial)}.$$

Let $f : X \to Y$ be a continuous map, which gives a chain map

$$S_\bullet(f) : S_\bullet(X) \to S_\bullet(Y).$$

This defines the functor of singular chain complex

$$S_\bullet : \underline{\mathbf{Top}} \to \underline{\mathbf{Ch}_\bullet}.$$

Singular homology group can be viewed as the composition of functors

$$\underline{\mathbf{Top}} \xrightarrow{S_\bullet} \underline{\mathbf{Ch}_\bullet} \xrightarrow{\mathrm{H}_n} \underline{\mathbf{Ab}}.$$

Proposition 15.16. *Let $f, g : X \to Y$ be homotopic maps. Then*

$$S_\bullet(f), S_\bullet(g) : S_\bullet(X) \to S_\bullet(Y)$$

are chain homotopic. In particular, they induce identical map

$$\mathrm{H}_n(f) = \mathrm{H}_n(g) : \mathrm{H}_n(X) \to \mathrm{H}_n(Y).$$

Proof. We only need to prove that for $i_0, i_1 : X \to X \times I$, the induced map

$$S_\bullet(i_0), S_\bullet(i_1) : S_\bullet(X) \to S_\bullet(X \times I)$$

are chain homotopic. Then their composition with the homotopy $X \times I \to Y$ gives the proposition.

Let us define a homotopy

$$s : S_n(X) \to S_{n+1}(X \times I).$$

For $\sigma : \Delta^n \to X$, we define (topologically)

$$s(\sigma) : \Delta^n \times I \xrightarrow{\sigma \times 1} X \times I.$$

Here we treat $\Delta^n \times I$ as a collection of $(n+1)$-simplexes as follows. Let $\{v_0, \cdots, v_n\}$ denote the vertices of Δ^n. The vertices of $\Delta^n \times I$ contain two copies $\{v_0, \cdots, v_n\}$ and $\{w_0, \cdots, w_n\}$. Then

$$\Delta^n \times I = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \cdots v_i, w_i, w_{i+1}, \cdots, w_n]$$

cuts $\Delta^n \times I$ into $(n+1)$-simplexes. Its sum defines $s(\sigma) \in S_{n+1}(X \times I)$.

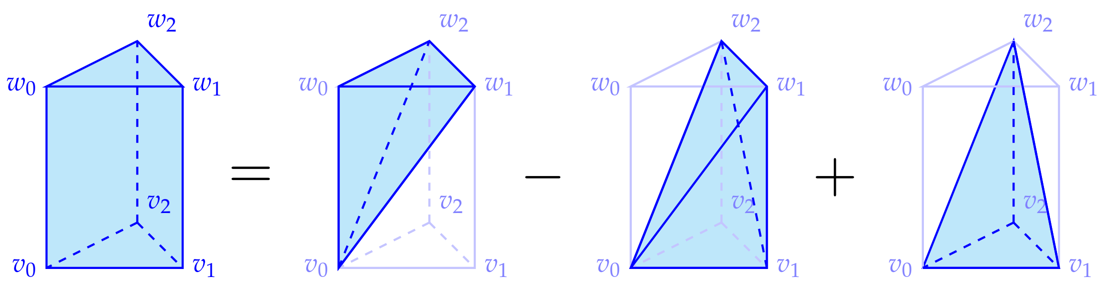


FIGURE 25. Decomposition of $\Delta^n \times I$ for $n = 2$

The following intuitive formula holds

$$\partial(\Delta^n \times I) = \Delta \times \partial I - (\partial \Delta^n) \times I$$

as an equation for singular chains. This leads to the chain homotopy

$$S_\bullet(i_1) - S_\bullet(i_0) = \partial \circ s + s \circ \partial.$$

□

Theorem 15.17. *Singular homologies are homotopy invariants. They factor through*

$$\mathrm{H}_n : \mathrm{h}\underline{\mathbf{Top}} \to \mathrm{h}\underline{\mathbf{Ch_\bullet}} \to \underline{\mathbf{Ab}}\,.$$

Dimension Axiom

Theorem 15.18 (Dimension Axiom)**.** *If X is a contractible, then*

$$\mathrm{H}_n(X) = \begin{cases} 0 & n > 0 \\ \mathbb{Z} & n = 0 \end{cases}$$

Proof. We can assume X is one point. For each $n \geq 0$, there is only one $\sigma_n : \Delta^n \to X$. Therefore

$$S_n(X) = \mathbb{Z}\langle \sigma_n \rangle\,.$$

The boundary operator is

$$\partial \langle \sigma_n \rangle = \sum_{i=0}^{n} (-1)^i \langle \sigma_{n-1} \rangle = \begin{cases} 0 & n = \text{odd} \\ \sigma_{n-1} & n = \text{even.} \end{cases}$$

The singular chain complex of X becomes

$$\cdots \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \to \mathbb{Z} \to 0$$

which implies the theorem. □

Some Algebraic tools

We collect several useful propositions in dealing with chain complexes. The proofs are left to the readers.

Proposition 15.19 (Five Lemma)**.** *Consider the commutative diagram of abelian groups with **exact** rows*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

Then

1°. *If f_2, f_4 are surjective and f_5 is injective, then f_3 is surjective.*
2°. *If f_2, f_4 are injective and f_1 is surjective, then f_3 is injective.*
3°. *If f_1, f_2, f_4, f_5 are isomorphisms, then f_3 is an isomorphism.*

Definition 15.20. Let $f : (C_\bullet, \partial) \to (C'_\bullet, \partial')$ be a chain map. The **mapping cone** of f is the chain complex

$$cone(f)_n = C_{n-1} \oplus C'_n$$

with the differential

$$d : cone(f)_n \to cone(f)_{n-1}, \quad (c_{n-1}, c'_n) \to (-\partial(c_{n-1}), \partial'(c'_n) - f(c_{n-1})).$$

Proposition 15.21. *Let $f : (C_\bullet, \partial) \to (C'_\bullet, \partial')$ be a chain map.*

1°. *There is an exact sequence*

$$0 \to C'_\bullet \to cone(f)_\bullet \to C[-1]_\bullet \to 0$$

Here $C[-1]_\bullet$ is the chain complex with $C[-1]_n := C_{n-1}$ and differential $-\partial$ where ∂ is the differential in C.

2°. *f is a quasi-isomorphism if and only if $cone(f)_\bullet$ is acyclic.*

3°. *Let $j : C'_\bullet \hookrightarrow cone(f)_\bullet$ be the embedding above. Then $cone(j)_\bullet$ is chain homotopic equivalent to $C[-1]_\bullet$.*

In homological algebra, a chain map $f : (C_\bullet, \partial) \to (C'_\bullet, \partial')$ leads to a triangle

$$\begin{tikzcd} C_\bullet \arrow[rr, "f"] & & C'_\bullet \arrow[dl, "j"] \\ & cone(f)_\bullet \arrow[ul, dashed] & \end{tikzcd}$$

Here the dotted arrow is a chain map from $cone(f)_\bullet$ to the shifted one $C[-1]_\bullet$.

The above proposition says $C[-1]_\bullet$ can be identified with $cone(j)$ (up to chain homotopy). So we can rotate the above triangle and still get another triangle

$$\begin{tikzcd} C'_\bullet \arrow[rr, "j"] & & cone(f)_\bullet \arrow[dl] \\ & C_\bullet[-1] \arrow[ul, dashed] & \end{tikzcd}$$

This is closely related to the cofiber exact sequence. $cone(f)_\bullet$ is the analogue of homotopy cofiber of f. $C_\bullet[-1]$ is the analogue of the suspension. Then the above triangle structure can be viewed as

$$C_\bullet \xrightarrow{f} C'_\bullet \to cone(f)_\bullet \to C_\bullet[-1] \xrightarrow{f[-1]} C'_\bullet \to \cdots$$

16 EXACT HOMOLOGY SEQUENCE

Exact homology sequence

Definition 16.1. Chain maps $0 \to C'_\bullet \xrightarrow{i} C_\bullet \xrightarrow{p} C''_\bullet \to 0$ is called a **short exact sequence** if for each n

$$0 \to C'_n \xrightarrow{i} C_n \xrightarrow{p} C''_n \to 0$$

is an exact sequence of abelian groups.

We have the following commuting diagram

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C'_{n+1} \arrow[r, "i"] \arrow[d, "\partial'"] & C_{n+1} \arrow[r, "p"] \arrow[d, "\partial"] & C''_{n+1} \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & C'_n \arrow[r, "i"] \arrow[d, "\partial'"] & C_n \arrow[r, "p"] \arrow[d, "\partial"] & C''_n \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r, "i"] \arrow[d, "\partial'"] & C_{n-1} \arrow[r, "p"] \arrow[d, "\partial"] & C''_{n-1} \arrow[r] \arrow[d, "\partial''"] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

Lemma/Definition 16.2. *Let $0 \to C'_\bullet \xrightarrow{i} C_\bullet \xrightarrow{p} C''_\bullet \to 0$ be a short exact sequence. There is a natural homomorphism*

$$\delta : H_n(C''_\bullet) \to H_{n-1}(C'_\bullet)$$

*called the **connecting map**. It induces a long exact sequence of abelian groups*

$$\cdots \to H_n(C'_\bullet) \xrightarrow{i_*} H_n(C_\bullet) \xrightarrow{p_*} H_n(C''_\bullet) \xrightarrow{\delta} H_{n-1}(C'_\bullet) \xrightarrow{i_*} H_{n-1}(C_\bullet) \xrightarrow{p_*} H_{n-1}(C''_\bullet) \to \cdots$$

The connecting map δ is natural in the sense that a commutative diagram of complexes with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C'_\bullet \arrow[r] \arrow[d] & C_\bullet \arrow[r] \arrow[d] & C''_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D'_\bullet \arrow[r] & D_\bullet \arrow[r] & D''_\bullet \arrow[r] & 0
\end{tikzcd}$$

induces a commutative diagram of abelian groups with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(C''_\bullet) \arrow[r] \arrow[d] & H_n(C_\bullet) \arrow[r] \arrow[d] & H_n(C''_\bullet) \arrow[r, "\delta"] \arrow[d] & H_{n-1}(C'_\bullet) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_n(D''_\bullet) \arrow[r] & H_n(D_\bullet) \arrow[r] & H_n(D''_\bullet) \arrow[r, "\delta"] & H_{n-1}(D'_\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. We first describe the construction of δ. Given a class $[\alpha] \in H_n(C''_\bullet)$, let $\alpha \in C''_n$ be a representative. Since $C_n \to C''_n$ is surjective, we can find $\beta \in C_n$ such that $p(\beta) = \alpha$. Consider $\partial\beta$. Since

$$p(\partial\beta) = \partial(p(\beta)) = \partial\alpha = 0,$$

there exists a unique element in $\gamma \in C'_{n-1}$ such that $i(\gamma) = \partial\beta$. Since

$$i(\partial(\gamma)) = \partial(i(\gamma)) = \partial^2(\beta) = 0$$

and i is injective, we find $\partial(\gamma) = 0$. This is illustrated by chasing the following diagram

$$\begin{tikzcd}
 & \beta \arrow[r] \arrow[d] & \alpha \arrow[d] \\
\gamma \arrow[r] \arrow[d] & \partial\beta \arrow[r] & 0 \\
0 & &
\end{tikzcd}$$

γ defines a class $[\gamma] \in H_{n-1}(C'_\bullet)$. We next show that this class does not depend on the choice of the lifting β and the choice of the representative α.

- Choice of β. Suppose we choose another $\tilde{\beta}$ such that $p(\tilde{\beta}) = \alpha$. Then there exists $x \in C'_n$ such that
$$\tilde{\beta} = \beta + i(x).$$
It follows that $\tilde{\gamma} = \gamma + \partial x$, so $[\tilde{\gamma}] = [\gamma]$.
- Choice of α. Suppose we choose another representative $\tilde{\alpha} = \alpha + \partial x$ of the class $[\alpha]$. We can choose a lifting $\tilde{\beta} = \beta + \partial y$ of $\tilde{\alpha}$ where $p(y) = x$. Since $\partial\tilde{\beta} = \partial\beta$, we have $\tilde{\gamma} = \gamma$.

Therefore we have a well-defined map $\delta : H_n(C''_\bullet) \to H_{n-1}(C'_\bullet)$ by
$$\delta[\alpha] = [\gamma].$$

We next show the exactness of the sequence
$$\cdots \to H_n(C'_\bullet) \xrightarrow{i_*} H_n(C_\bullet) \xrightarrow{p_*} H_n(C''_\bullet) \xrightarrow{\delta} H_{n-1}(C'_\bullet) \xrightarrow{i_*} H_{n-1}(C_\bullet) \xrightarrow{p_*} H_{n-1}(C''_\bullet) \to \cdots$$

- Exactness at $H_n(C_\bullet)$.
  $\mathrm{im}(i_*) \subset \ker(p_*)$ is obvious. If $[\alpha] \in H_n(C_\bullet)$ such that $[p(\alpha)] = 0$, so $p(\alpha) = \partial x$. Let $y \in C_{n+1}$ be a lifting of x so $p(y) = x$. Since $p(\alpha - \partial y) = 0$, $\alpha - \partial y = i(\beta)$ for some $\beta \in C'_n$. Then $\partial\beta = 0$ and
$$i_*([\beta]) = [\alpha] \quad \text{which implies} \quad \ker(p_*) \subset \mathrm{im}(i_*).$$
- Exactness at $H_n(C''_\bullet)$.
  Assume $[\alpha] = p_*[\beta]$, then β is a lift of α and $\partial\beta = 0$. So $\delta[\alpha] = 0$. This shows
$$\mathrm{im}(p_*) \subset \ker(\delta).$$
On the other hand, if $\delta[\alpha] = 0$. We can find a lift β of α such that $\partial\beta = 0$. Then $[\alpha] = p_*[\beta]$. Hence
$$\ker(\delta) \subset \mathrm{im}(p_*).$$
- Exactness at $H_{n-1}(C'_\bullet)$.
  $i_*\delta([\alpha]) = i_*[\gamma] = [\partial\beta] = 0$. This shows
$$\mathrm{im}\,\delta \subset \ker i_*.$$
Assume $[\gamma] \in H_{n-1}(C'_\bullet)$ such that $i_*[\gamma] = 0$. Then $i(\gamma) = \partial\beta$ for some β. Let $\alpha = p(\beta)$. Then
$$\partial(\alpha) = \partial p(\beta) = p(\partial\beta) = pi(\gamma) = 0.$$
So $[\alpha]$ defines a homology class and $\delta[\alpha] = [\gamma]$ by construction. This shows
$$\ker i_* \subset \mathrm{im}\,\delta.$$

The naturality is straight-forward to verify. □

Relative homology

Definition 16.3. Let $A \subset X$ be a subspace. It indues a natural injective chain map $S_\bullet(A) \hookrightarrow S_\bullet(X)$. We define the singular chain complex of X relative to A to be

$$S_n(X, A) := S_n(X)/S_n(A)$$

with the induced differential. Its homology $\mathrm{H}_n(X, A) := \mathrm{H}_n(S_\bullet(X, A))$ is called the **n-th relative homology**.

Theorem 16.4. *For $A \subset X$, there is a long exact sequence of abelian groups*

$$\cdots \to \mathrm{H}_n(A) \to \mathrm{H}_n(X) \to \mathrm{H}_n(X, A) \xrightarrow{\delta} \mathrm{H}_{n-1}(A) \to \cdots$$

Proof. This follows from the short exact sequence of complexes

$$0 \to S_\bullet(A) \to S_\bullet(X) \to S_\bullet(X, A) \to 0.$$

□

Let us define relative n-cycles $Z_n(X, A)$ and relative n-boundaries $B_n(X, A)$ to be

$$Z_n(X, A) = \{\gamma \in S_n(X) : \partial\gamma \in S_{n-1}(A)\}$$
$$B_n(X, A) = B_n(X) + S_n(A) \subset S_n(X).$$

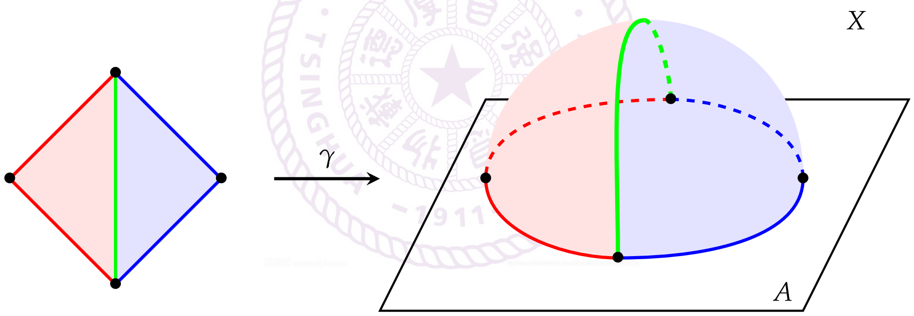


FIGURE 26. A chain γ in $Z_n(X, A)$ with two simplexes. The green face lies outside A but cancelled out from the two simplexes. So $\partial\gamma \subset A$ holds.

Then it is easy to check that $S_n(A) \subset B_n(X, A) \subset Z_n(X, A) \subset S_n(X)$ and

$$\mathrm{H}_n(X, A) = Z_n(X, A)/B_n(X, A).$$

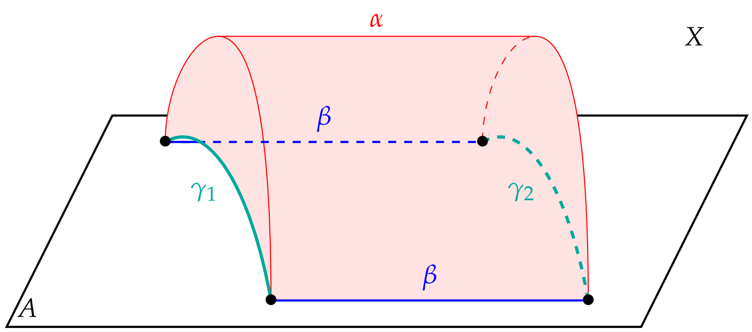
Two relative n-cycles γ_1, γ_2 defines the same class $[\gamma_1] = [\gamma_2]$ in $\mathrm{H}_n(X, A)$ if and only if $\gamma_1 - \gamma_2$ is homologous to a chain in A.

The connecting map

$$\delta : \mathrm{H}_n(X, A) \to \mathrm{H}_{n-1}(A)$$

can be understood as follows: a n-cycle in $\mathrm{H}_n(X, A)$ is represented by an n-chain $\gamma \in S_n(X)$ such that its boundary $\partial(\gamma)$ lies in A. Viewing $\partial(\gamma)$ as an $(n-1)$-cycle in A, then

$$\delta[\gamma] = [\partial(\gamma)].$$



$$\partial\alpha = \beta + \gamma_1 - \gamma_2,\ \beta \in S_\bullet(A)$$

$$\implies [\gamma_1] = [\gamma_2] \quad \in \mathrm{H}_n(X, A)$$

FIGURE 27. Relative n-cycles

Let $f : (X, A) \to (Y, B)$ be a map of pairs. It naturally induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet(A) \arrow[r] \arrow[d] & S_\bullet(X) \arrow[r] \arrow[d] & S_\bullet(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_\bullet(B) \arrow[r] & S_\bullet(Y) \arrow[r] & S_\bullet(Y,B) \arrow[r] & 0
\end{tikzcd}$$

which further induces compatible maps on various homology groups

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_n(A) \arrow[r] \arrow[d] & \mathrm{H}_n(X) \arrow[r] \arrow[d] & \mathrm{H}_n(X,A) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}_n(B) \arrow[r] & \mathrm{H}_n(Y) \arrow[r] & \mathrm{H}_n(Y,B) \arrow[r, "\delta"] & \mathrm{H}_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

This construction can be generalized to the triple $B \subset A \subset X$.

Theorem 16.5. *If $B \subset A \subset X$ are subspaces, then there is a long exact sequence*

$$\cdots \to \mathrm{H}_n(A, B) \to \mathrm{H}_n(X, B) \to \mathrm{H}_n(X, A) \xrightarrow{\delta} \mathrm{H}_{n-1}(A, B) \to \cdots .$$

Proof. This follows from the long exact sequence associated to the exact sequence

$$0 \to \frac{S_\bullet(A)}{S_\bullet(B)} \to \frac{S_\bullet(X)}{S_\bullet(B)} \to \frac{S_\bullet(X)}{S_\bullet(A)} \to 0.$$

□

Theorem 16.6 (Homotopy Axiom for Pairs)**.** *If $f, g : (X, A) \to (Y, B)$ and f is homotopic to g* rel A*. Then*

$$\mathrm{H}_n(f) = \mathrm{H}_n(g) : \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B).$$

Reduced homology

Proposition 16.7. *Let $\{X_\alpha\}$ be path connected components of X, then*

$$\mathrm{H}_n(X) = \bigoplus_\alpha \mathrm{H}_n(X_\alpha).$$

Proof. This is because

$$S_\bullet(X) = \bigoplus_\alpha S_\bullet(X_\alpha).$$

□

Proposition 16.8. *Let X be path connected. Then $\mathrm{H}_0(X) \cong \mathbb{Z}$.*

Proof. $\mathrm{H}_0(X) = S_0(X)/\partial S_1(X)$. Let us define the map

$$\epsilon : S_0(X) \to \mathbb{Z}, \quad \sum_{p\in X} m_p p \to \sum_p m_p.$$

The map ϵ is zero on $\partial S_1(X)$. On the other hand, assume $\epsilon(\sum_{p\in X} m_p p) = 0$, then we can write

$$\sum_{p\in X} m_p p = \sum_i (p_i - q_i)$$

into pairs. Since X is path connected, we can find a path $\gamma_i : I \to X$ such that $\partial\gamma = p_i - q_i$. Therefore $\sum_{p\in X} m_p p = \sum_i \partial\gamma_i \in \partial S_1(X)$. It follows that ϵ induces an isomorphism

$$\epsilon : \mathrm{H}_0(X) \cong \mathbb{Z}$$

□

In general, we have a surjective map

$$\epsilon : \mathrm{H}_0(X) \to \mathbb{Z}, \quad \sum_{p\in X} m_p p \to \sum_p m_p.$$

Definition 16.9. We define the reduced homology group of X by

$$\tilde{\mathrm{H}}_n(X) = \begin{cases} \mathrm{H}_n(X) & n > 0 \\ \ker(\mathrm{H}_0(X) \to \mathbb{Z}) & n = 0 \end{cases}$$

We can think about the reduced homology group as the homology group of the chain complex

$$\cdots \to S_2(X) \to S_1(X) \to S_0(X) \to \mathbb{Z}.$$

The long exact sequence still holds for the reduced case

$$\cdots \to \tilde{\mathrm{H}}_n(A) \to \tilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X, A) \xrightarrow{\delta} \tilde{\mathrm{H}}_{n-1}(A) \to \cdots$$

Example 16.10. If X is contractible, then $\tilde{H}_n(X) = 0$ for all n.

Example 16.11. Let $x_0 \in X$ be a point. Using the long exact sequence for $A = \{x_0\} \subset X$, we find

$$\mathrm{H}_n(X, x_0) = \tilde{\mathrm{H}}_n(X).$$

17 BARYCENTRIC SUBDIVISION AND EXCISION

The fundamental property of homology which makes it computable is **excision**.

Barycentric Subdivision

Definition 17.1. Let Δ^n be the standard n-simplex with vertices $v_0, \cdots, v_n$. We define its **barycenter** to be

$$c(\Delta^n) = \frac{1}{n+1} \sum_{i=0}^{n} v_i \in \Delta^n.$$

Definition 17.2. We define the **barycentric subdivision** $\mathscr{B}\Delta^n$ of a n-simplex Δ^n as follows:

1°. $\mathscr{B}\Delta^0 = \Delta^0$.

2°. Let $F_0, \cdots, F_n$ be the n-simplexes of faces of Δ^{n+1}, c be the barycenter of Δ^{n+1}. Then $\mathscr{B}\Delta^{n+1}$ consists of $(n+1)$-simplexes with ordered vertices $[c, w_0, \cdots, w_n]$ where $[w_0, \cdots, w_n]$ is a n-simplexes in $\mathscr{B}F_0, \cdots, \mathscr{B}F_n$.

Equivalently, a simplex in $\mathscr{B}\Delta^n$ is indexed by a sequence $\{S_0 \subset S_1 \cdots \subset S_n = \Delta^n\}$ where S_i is a face of S_{i+1}. Then its vertices are $[c(S_n), c(S_{n-1}), \cdots, c(S_0)]$. It is seen that Δ^n is the union of simplexes in $\mathscr{B}\Delta^n$.

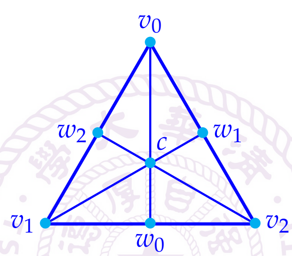


FIGURE 28. $\mathscr{B}\Delta^2 = [c, w_0, v_2] - [c, w_1, v_2] + [c, w_1, v_0] - [c, w_2, v_0] + [c, w_2, v_1] - [c, w_0, v_1]$

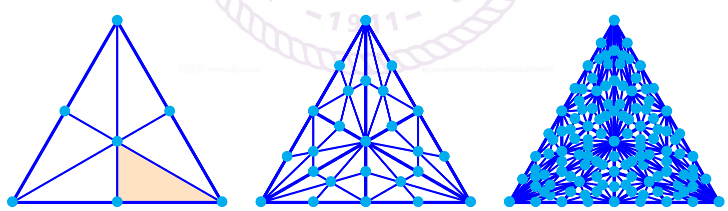


FIGURE 29. Barycentric Subdivision $\mathscr{B}\Delta^2$, two times $\mathscr{B}^2\Delta^2$, and three times $\mathscr{B}^3\Delta^2$

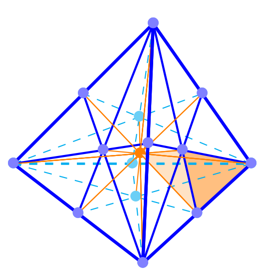


FIGURE 30. Barycentric Subdivision $\mathscr{B}\Delta^3$

Definition 17.3. We define the n-chain of barycentric subdivision $\mathscr{B}_n$ by

$$\mathscr{B}_n = \sum_{\alpha} \pm \sigma_\alpha \in S_n(\Delta^n)$$

where the summation is over all sequence $\alpha = \{S_0 \subset S_1 \cdots \subset S_n = \Delta^n\}$. σ_α is the simplex with ordered vertices $[c(S_n), c(S_{n-1}), \cdots, c(S_0)]$, viewed as a singular n-chain in Δ^n. The sign $\pm$ is about orientation: if the orientation of $[c(S_n), c(S_{n-1}), \cdots, c(S_0)]$ coincides with that of Δ^n, we take $+$; otherwise we take $-$.

Definition 17.4. We define the following composition map

$$S_k(\Delta^m) \times S_n(\Delta^k) \to S_n(\Delta^m), \quad \sigma \times \eta \to \sigma \circ \eta.$$

This is defined on generators via the composition $\Delta^n \to \Delta^k \to \Delta^m$ and extended linearly to singular chains.

Similarly, there is a natural map denoted by

$$S_n(\Delta^m) : S_m(X) \to S_n(X), \quad \eta : \sigma \to \eta^*(\sigma) := \sigma \circ \eta$$

where $\eta^*(\sigma) = \sigma \circ \eta$ is the composition of σ with η. It is easy to see that

$$(\eta_1 \circ \eta_2)^* = \eta_2^* \circ \eta_1^*, \quad \forall \eta_1 \in S_k(\Delta^m), \eta_2 \in S_n(\Delta^k).$$

Example 17.5. Let

$$\partial \Delta^n = \sum_{i=0}^{n} (-1)^i \partial^{(i)} \Delta^n \in S_{n-1}(\Delta^n)$$

be the boundary faces. Then $(\partial \Delta^n) \circ (\partial \Delta^{n-1}) = 0$. The operator

$$\partial_n = (\partial \Delta^n)^* : S_n(X) \to S_{n-1}(x)$$

defines the boundary map in singular chains.

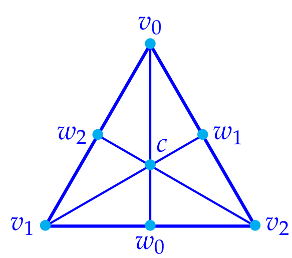
Lemma 17.6. *The barycentric subdivision is compatible with the boundary map*

$$\partial \mathscr{B}_n = \mathscr{B}(\partial \Delta^n)$$

where $\mathscr{B}(\partial \Delta^n)$ is the barycentric subdivision of faces $\partial \Delta^n$ of Δ^n, viewed as an $(n-1)$-chain in Δ^n. Equivalently, we have the following composition relation

$$\mathscr{B}_n \circ (\partial \Delta^n) = (\partial \Delta^n) \circ \mathscr{B}_{n-1}.$$

Proof. The choice of ordering and orientation implies $\partial \mathscr{B}_n = \mathscr{B}(\partial \Delta^n)$. Here is an illustration for $n = 2$.



$\mathscr{B}\Delta^2 = [c, w_0, v_2] - [c, w_1, v_2] + [c, w_1, v_0] - [c, w_2, v_0] + [c, w_2, v_1] - [c, w_0, v_1]$. So

$$\begin{aligned} \partial \mathscr{B}\Delta^2 =& ([c, w_0] - [c, v_2] + [w_0, v_2]) - ([c, w_1] - [c, v_2] + [w_1, v_2]) + ([c, w_1] - [c, v_0] + [w_1, v_0]) \\ & - ([c, w_2] - [c, v_0] + [w_2, v_0]) + ([c, w_2] - [c, v_1] + [w_2, v_1]) - ([c, w_0] - [c, v_1] + [w_0, v_1]) \\ =& ([w_0, v_2] - [w_0, v_1]) - ([w_1, v_2] - [w_1, v_0]) + ([w_2, v_1] - [w_2, v_0]) = \mathscr{B}(\partial \Delta^2). \end{aligned}$$

□

Lemma 17.7. *There exists $T_{n+1} \in S_{n+1}(\Delta^n)$ for all $n \geq 0$ such that*

$$\mathscr{B}_n - 1_{\Delta^n} = T_{n+1} \circ \partial\Delta^{n+1} + \partial\Delta^n \circ T_n.$$

Here $1_{\Delta^n} : \Delta^n \to \Delta^n$ is the identity map, viewed as a n-chain in $S_n(\Delta^n)$.

Proof. We construct T_n inductively. $T_1 = 0$. Suppose we have found T_n. We need to find T_{n+1} such that

$$\partial(T_{n+1}) = \mathscr{B}_n - 1_{\Delta^n} - \partial\Delta^n \circ T_n.$$

Using Lemma 17.6, we have

$$\begin{aligned}\partial\left(\mathscr{B}_n - 1_{\Delta^n} - \partial\Delta^n \circ T_n\right) = &\left(\mathscr{B}_n - 1_{\Delta^n} - \partial\Delta^n \circ T_n\right) \circ \partial\Delta^n \\ =&\partial\Delta^n \circ \left(\mathscr{B}_{n-1} - 1_{\Delta^{n-1}} - T_n \circ \partial\Delta^n\right) \\ =&\partial\Delta^n \circ \partial\Delta^{n-1} \circ T_{n-1} = 0.\end{aligned}$$

Therefore $\mathscr{B}_n - 1_{\Delta^n} - \partial\Delta^n \circ T_n$ is a n-cycle in $S_n(\Delta^n)$. However

$$\mathrm{H}_n(\Delta^n) = 0, \quad \forall n \geq 1$$

since Δ^n is contractible. So $\mathscr{B}_n - 1_{\Delta^n} - \partial\Delta^n \circ T_n$ is a n-boundary and T_{n+1} can be constructed.

□

Definition 17.8. We define the **barycentric subdivision** on singular chain complex by

$$\mathscr{B}^* : S_\bullet(X) \to S_\bullet(X)$$

where $\mathscr{B}^* = \mathscr{B}^*_n$ on $S_n(X)$.

Theorem 17.9. *The barycentric subdivision map $\mathscr{B}^* : S_\bullet(X) \to S_\bullet(X)$ is a chain map. Moreover, it is chain homotopic to the identity map, hence a quasi-isomorphism.*

Proof. Lemma 17.6 implies $\mathscr{B}^*$ is a chain map. It is chain homotopic to the identity map by Lemma 17.7. □

Excision

Theorem 17.10 (Excision). *Let $U \subset A \subset X$ be subspaces such that $\bar{U} \subset A^\circ$ (the interior of A). Then the inclusion $i : (X - U, A - U) \hookrightarrow (X, A)$ induces isomorphisms*

$$i_* : \mathrm{H}_n(X - U, A - U) \cong \mathrm{H}_n(X, A), \quad \forall n.$$

Proof. Let us call $\sigma : \Delta^n \to X$ *small* if

$$\sigma(\Delta^n) \subset A \quad \text{or} \quad \sigma(\Delta^n) \subset X - U.$$

Let $S'_\bullet(X) \subset S_\bullet(X)$ denote the subcomplex generated by small simplexes. The condition $\bar{U} \subset A^\circ$ implies that for any simplex $\sigma : \Delta^n \to X$, there exists a big enough k such that

$$(\mathscr{B}^*)^k(\sigma) \in S'(X).$$

Let $S'_\bullet(X, A)$ be defined by the exact sequence

$$0 \to S_\bullet(A) \to S'(X) \to S'(X, A) \to 0.$$

It is easy to see that

$$S'_\bullet(X, A) \cong S_\bullet(X - U, A - U).$$

There is a natural commutative diagram of chain maps

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet(A) \arrow[r] \arrow[d] & S'_\bullet(X) \arrow[r] \arrow[d] & S'_\bullet(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_\bullet(A) \arrow[r] & S_\bullet(X) \arrow[r] & S_\bullet(X,A) \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma, it is enough to show

$$S'_\bullet(X) \to S_\bullet(X)$$

is a quasi-isomorphism.

• injectivity of $\mathrm{H}(S'_\bullet(X)) \to \mathrm{H}(S_\bullet(X))$.

Let α be a cycle in $S'_\bullet(X)$ and $\alpha = \partial\beta$ for $\beta \in S_\bullet(X)$. Take k big enough that $(\mathscr{B}^*)^k(\beta) \in S'(X)$. Then

$$(\mathscr{B}^*)^k(\alpha) = \partial(\mathscr{B}^*)^k(\beta).$$

Hence the class of $(\mathscr{B}^*)^k(\alpha)$ in $\mathrm{H}(S'_\bullet(X))$ is zero, so is α which is homologous to $(\mathscr{B}^*)^k(\alpha)$.

• surjectivity of $\mathrm{H}(S'_\bullet(X)) \to \mathrm{H}(S_\bullet(X))$.

Let α be a cycle in $S_\bullet(X)$. Take k big enough that $(\mathscr{B}^*)^k(\alpha) \in S'_\bullet(X)$. Then $(\mathscr{B}^*)^k(\alpha)$ is a small cycle which is homologous to α. □

Theorem 17.11. *Let X_1, X_2 be subspaces of X and $X = X_1^\circ \cup X_2^\circ$. Then*

$$\mathrm{H}_\bullet(X_1, X_1 \cap X_2) \to \mathrm{H}_\bullet(X, X_2)$$

is an isomorphism for all n.

Proof. Apply Excision to $U = X - X_1$, $A = X_2$. □

Theorem 17.12 (Mayer-Vietoris). *Let X_1, X_2 be subspaces of X and $X = X_1^\circ \cup X_2^\circ$. Then there is an exact sequence*

$$\cdots \to \mathrm{H}_n(X_1 \cap X_2) \xrightarrow{(i_{1*}, i_{2*})} \mathrm{H}_n(X_1) \oplus \mathrm{H}_n(X_2) \xrightarrow{j_{1*} - j_{2*}} \mathrm{H}_n(X) \xrightarrow{\delta} \mathrm{H}_{n-1}(X_1 \cap X_2) \to \cdots$$

It is also true for the reduced homology.

Proof. Let $S_\bullet(X_1) + S_\bullet(X_2) \subset S_\bullet(X)$ be the subspace spanned by $S_\bullet(X_1)$ and $S_\bullet(X_2)$.

We have a short exact sequence

$$0 \to S_\bullet(X_1 \cap X_2) \xrightarrow{(i_1, i_2)} S_\bullet(X_1) \oplus S_\bullet(X_2) \xrightarrow{j_1 - j_2} S_\bullet(X_1) + S_\bullet(X_2) \to 0.$$

Similar to the proof of Excision via barycentric subdivision, the embedding $S_\bullet(X_1) + S_\bullet(X_2) \subset S_\bullet(X)$ is a quasi-isomorphism. Mayer-Vietoris sequence follows. □

Theorem 17.13. *Let $A \subset X$ be a closed subspace. Assume A is a strong deformation retract of a neighborhood in X. Then the map $(X, A) \to (X/A, A/A)$ induces an isomorphism*

$$\mathrm{H}_\bullet(X, A) = \tilde{\mathrm{H}}_\bullet(X/A).$$

Proof. Let U be an open neighborhood of A that deformation retracts to A. Then $\mathrm{H}_\bullet(A) \cong \mathrm{H}_\bullet(U)$, hence

$$\mathrm{H}_\bullet(X, A) \cong \mathrm{H}_\bullet(X, U)$$

by Five Lemma. Since A is closed and U is open, we can apply Excision to find

$$\mathrm{H}_\bullet(X, A) \cong \mathrm{H}_\bullet(X, U) \cong \mathrm{H}_\bullet(X - A, U - A).$$

The same consideration applied to $(X/A, A/A)$ and U/A gives

$$\mathrm{H}_\bullet(X/A, A/A) \cong \mathrm{H}_\bullet(X/A - A/A, U/A - A/A) = \mathrm{H}_\bullet(X - A, U - A).$$

□

This Theorem in particular applies to cofibrations.

Corollary 17.14. *Let $A \subset X$ be a closed cofibration. Then* $\mathrm{H}_\bullet(X, A) = \tilde{\mathrm{H}}_\bullet(X/A)$.

Suspension

Let (X, x_0) be a well-pointed space. Recall that its reduced cone $C_\star X$ and reduced suspension ΣX are

$$C_\star X = X \wedge I = \frac{X \times I}{(X \times \{0\} \cup x_0 \times I)}, \quad \Sigma X = X \wedge S^1 = \frac{X \times I}{(X \times \{0\} \cup X \times \{1\} \cup x_0 \times I)}.$$

Since (X, x_0) is a well-pointed, we have homotopy equivalences

$$C_\star X \simeq \frac{X \times I}{X \times \{0\}}, \quad \Sigma X \simeq \frac{X \times I}{(X \times \{0\} \cup X \times \{1\})}.$$

Theorem 17.15. *Let (X, x_0) be a well-pointed space. Then* $\tilde{\mathrm{H}}_n(\Sigma X) = \tilde{\mathrm{H}}_{n-1}(X)$.

Proof. Let

$$Z = \frac{X \times I}{X \times \{0\}}, \quad Y = \frac{X \times I}{(X \times \{0\} \cup X \times \{1\})} = Z/X.$$

Since Z is contractible, the homology exact sequence associated to the pair $X \subset Z$ implies

$$\tilde{\mathrm{H}}_n(Z, X) = \tilde{\mathrm{H}}_{n-1}(X).$$

It follows that

$$\tilde{\mathrm{H}}_n(\Sigma X) = \tilde{\mathrm{H}}_n(Y) = \tilde{\mathrm{H}}_n(Z/X) = \tilde{\mathrm{H}}_n(Z, X) = \tilde{\mathrm{H}}_{n-1}(X).$$

□

Proposition 17.16. *The reduced homology of the sphere S^n is given by*

$$\tilde{\mathrm{H}}_k(S^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$$

In particular, spheres of different dimensions are not homotopy equivalent.

Proof. This follows from the previous theorem and $S^n = \Sigma^n S^0$ where $S^0 = \{\pm 1\}$ consists of two points.

□

Applications of Homology of spheres

Proposition 17.17. *If $m \neq n$, then $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.*

Proof. Assume $f : \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism. Then f induces a homeomorphism

$$\mathbb{R}^m - \{p\} \to \mathbb{R}^n - \{f(p)\}$$

hence a homotopy equivalence between S^{m-1} and S^{n-1}. Contradiction. □

Definition 17.18. A continous map $f : S^n \to S^n$ ($n \geq 0$) has **degree** d, denoted by $\deg(f) = d$, if

$$f_* : \tilde{\mathrm{H}}_n(S^n) = \mathbb{Z} \to \tilde{\mathrm{H}}_n(S^n) = \mathbb{Z}$$

is multiplication by d.

We give a geometric interpretation of the degree of $f : S^n \to S^n$. Let $V \subset S^n$ be a small open ball such that $f^{-1}(V) \to V$ is a disjoint union of open balls

$$f^{-1}(V) = U_1 \cup \cdots \cup U_d.$$

Let $f_i : \bar{U}_i / \partial \bar{U}_i \cong S^n \to \bar{V}/\partial \bar{V} \cong S^n$. We have the commutative diagram

$$\begin{tikzcd}
\mathrm{H}_n(S^n) \arrow[r] \arrow[d, "f_*"] & \mathrm{H}_n(S^n/(S^n - \cup_i U_i)) \cong \oplus_i \mathrm{H}_n(S^n) \arrow[d, "\oplus_i (f_i)_*"] \\
\mathrm{H}_n(S^n) \arrow[r] & \mathrm{H}_n(S^n/(S^n - V)) \cong \mathrm{H}_n(S^n)
\end{tikzcd}$$

It is easy to see that first row is $k \to (k, k, \cdots, k)$ and the second row is $k \to k$. It follows that

$$\deg(f) = \sum_{i=1}^{d} \deg(f_i).$$

Note that when $f^{-1}(V) \to V$ is a covering map, then $f : U_i \to V$ is a homeomorphism. We have $\deg(f_i) = \pm 1$ and $\deg(f)$ is given by a counting with signs.

Example 17.19. Identify $S^2 = \mathbb{C} \cup \{\infty\}$. Consider the map $f : S^2 \to S^2, z \to z^k$. Then $\deg(f) = k$.

Lemma 17.20. *Let $f, g : S^n \to S^n$ be continuous maps.*

1°. $\deg(f \circ g) = \deg(f)\deg(g)$.
2°. *If $f \simeq g$ are homotopic, then* $\deg(f) = \deg(g)$
3°. *If f is a homotopy equivalence, then* $\deg(f) = \pm 1$.

Proof. All the statements follow from the fact that H_n defines a functor $\mathrm{H}_n : \underline{\mathrm{h}\mathbf{Top}} \to \underline{\mathbf{Group}}$.

□

Proposition 17.21. *Let $r : S^n \to S^n, (x_0, \cdots, x_n) \to (-x_0, x_1, \cdots, x_n)$ be the reflection. Then*

$$\deg(r) = -1.$$

Proof. We prove by induction on n. This is true for $n = 0$. Assume the case for $n - 1$.

Consider the pair (D^n, S^{n-1}). We find an isomorphism $\tilde{\mathrm{H}}_n(S^n) \to \tilde{\mathrm{H}}_{n-1}(S^{n-1})$ by

$$\tilde{\mathrm{H}}_n(S^n) = \tilde{\mathrm{H}}_n(D^n/S^{n-1}) = \tilde{\mathrm{H}}_n(D^n, S^{n-1}) \xrightarrow{\delta} \tilde{\mathrm{H}}_{n-1}(S^{n-1}).$$

This isomorphism is compatible with the reflection and leads to the commutative diagram

$$\begin{tikzcd} \tilde{\mathrm{H}}_n(S^n) \arrow[r, "\delta"] \arrow[d, "r_*"] & \tilde{\mathrm{H}}_{n-1}(S^{n-1}) \arrow[d, "r_*"] \\ \tilde{\mathrm{H}}_n(S^n) \arrow[r, "\delta"] & \tilde{\mathrm{H}}_{n-1}(S^{n-1}) \end{tikzcd}$$

This proves the case for n.

□

Corollary 17.22. *Let $\sigma : S^n \to S^n, (x_0, \cdots, x_n) \to (-x_0, \cdots, -x_n)$ be the antipodal map. Then*

$$\deg(\sigma) = (-1)^{n+1}.$$

Proof. σ is a composition of $n+1$ reflections. □

Proposition 17.23. *If $f : S^n \to S^n$ has no fixed points. Then f is homotopic to the antipodal map.*

Proof. Let σ be the antipodal map. Then the map

$$F : S^n \times I \to S^n, \quad F(x,t) = \frac{(1-t)\sigma(x) + tf(x)}{\|(1-t)\sigma(x) + tf(x)\|}$$

gives the required homotopy. □

Theorem 17.24 (Hairy Ball Theorem)**.** *S^n has a nowhere vanishing tangent vector field if and only if n is odd.*

Proof. If n is odd, we construct

$$v(x_0, \cdots, x_n) = (-x_1, x_0, -x_3, x_2, \cdots).$$

Conversely, assume v is no-where vanishing vector field. Let

$$f : S^n \to S^n, \quad x \to \frac{v(x)}{|v(x)|}.$$

The map

$$F : S^n \times I \to S^n, \quad F(x,t) = \cos(\pi t)x + \sin(\pi t)f(x)$$

defines a homotopy between the identity map 1 and the antipodal map σ. It follows that

$$\deg(\sigma) = 1 \Longrightarrow n = \text{odd}.$$

□

Theorem 17.25 (Brower's Fixed Point Theorem)**.** *Any continuous map $f : D^n \to D^n$ has a fixed point.*

Proof. Assume f has no fixed point. Define

$$r : D^n \to S^{n-1}$$

where $r(p)$ is the intersection of ∂D^n with the ray starting from $f(p)$ pointing toward p. Then r defines a retract of $S^{n-1} \hookrightarrow D^n$. This implies $\mathrm{H}_\bullet(D^n) = \mathrm{H}_\bullet(S^{n-1}) \oplus \mathrm{H}_\bullet(D^n, S^{n-1})$. Contradiction.

□

18 CELLULAR HOMOLOGY

Cellular homology

Lemma 18.1. *Let $\{(X_i, x_i)\}_{i\in I}$ be well-pointed spaces. Then*

$$\tilde{\mathrm{H}}_n(\bigvee_{i\in I} X_i) = \bigoplus_{i\in I} \tilde{\mathrm{H}}_n(X_i).$$

Proof. Let

$$Y = \coprod_{i\in I} X_i, \quad A = \coprod_{i\in I} \{x_i\}$$

$A \subset Y$ is a cofibration, therefore

$$\tilde{\mathrm{H}}_n(\bigvee_{i\in I} X_i) = \tilde{\mathrm{H}}_n(Y/A) = \mathrm{H}_n(Y, A) = \bigoplus_{i\in I} \mathrm{H}_n(X_i, x_i) = \bigoplus_{i\in I} \tilde{\mathrm{H}}_n(X_i).$$

□

Definition 18.2. Let (X, A) be a relative CW complex with skeletons: $A = X^{-1} \subset X^0 \subset \cdots \subset X^n \subset \cdots$. We define the relative **cellular chain complex** $(C_\bullet^{cell}(X, A), \partial)$

$$\cdots \to C_n^{cell}(X, A) \xrightarrow{\partial} C_{n-1}^{cell}(X, A) \xrightarrow{\partial} \cdots \to C_0^{cell}(X, A) \to 0$$

where

$$C_n^{cell}(X, A) := \mathrm{H}_n(X^n, X^{n-1})$$

and the boundary map ∂ is defined by the commutative diagram

$$\begin{tikzcd}
\mathrm{H}_n(X^n, X^{n-1}) \arrow[rr, "\partial"] \arrow[dr, "\delta"'] & & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) \\
& \mathrm{H}_{n-1}(X^{n-1}, A) \arrow[ur, "j"'] &
\end{tikzcd}$$

Here δ is the connecting map of relative homology for $A \subset X^{n-1} \subset X^n$ and j is the natural map.

Assume X^n is obtained from X^{n-1} by attaching n-cells indexed by J_n

$$\begin{tikzcd}
\coprod_{\alpha\in J_n} S^{n-1} \arrow[r, "f"] \arrow[d, hook] & X^{n-1} \arrow[d, hook] \\
\coprod_{\alpha\in J_n} D^n \arrow[r, "\Phi_f"] & X^n
\end{tikzcd}$$

Since $X^{n-1} \hookrightarrow X^n$ is a cofibration, Lemma 18.1 implies that

$$C_n^{cell}(X, A) \cong \tilde{\mathrm{H}}_n(X^n/X^{n-1}) \cong \bigoplus_{J_n} \tilde{\mathrm{H}}_n(S^n) \cong \bigoplus_{J_n} \mathbb{Z}$$

is the free abelian group generated by each attached $\mathrm{H}_n(D^n, S^{n-1}) = \tilde{\mathrm{H}}_n(S^n)$. Using the diagram

$$\begin{tikzcd}
\mathrm{H}_n(X^n, X^{n-1}) \arrow[rr, "\partial_n"] \arrow[dr, "\delta_n"'] & & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) \arrow[rr, "\partial_{n-1}"] \arrow[dr, "\delta_{n-1}"'] & & \mathrm{H}_{n-2}(X^{n-2}, X^{n-3}) \\
& \mathrm{H}_{n-1}(X^{n-1}, A) \arrow[ur, "j_n"'] & & \mathrm{H}_{n-1}(X^{n-2}, A) \arrow[ur, "j_{n-1}"'] &
\end{tikzcd}$$

and $\delta_{n-1} \circ j_n = 0$, we see that
$$\partial_{n-1} \circ \partial_n = j_{n-1} \circ \delta_{n-1} \circ j_n \circ \delta_n = 0.$$
Therefore $(C_\bullet^{cell}(X, A), \partial)$ indeed defines a chain complex.

Definition 18.3. Let (X, A) be a relative CW complex. We define its **n-th relative cellular homology** by
$$\boxed{\mathrm{H}_n^{cell}(X, A) := \mathrm{H}_n(C_\bullet^{cell}(X, A), \partial)}.$$
When $A = \emptyset$, we simply denote it by $\mathrm{H}_n^{cell}(X)$ called the **n-th cellular homology**.

Lemma 18.4. *Let (X, A) be a relative CW complex. Let $0 \le q < p \le \infty$. Then*
$$\mathrm{H}_n(X^p, X^q) = 0, \quad \text{if} \quad n \le q \quad \text{or} \quad n > p.$$

Proof. Consider the cofibrations
$$X^q \hookrightarrow X^{q+1} \hookrightarrow \cdots \hookrightarrow X^{p-1} \hookrightarrow X^p$$
where each quotient is a wedge of spheres
$$X^{q+1}/X^q = \bigvee S^{q+1}, \quad X^{q+2}/X^{q+1} = \bigvee S^{q+2}, \quad \cdots, \quad X^p/X^{p-1} = \bigvee S^p.$$

Assume $n \le q$ or $n > p$. Then
$$\mathrm{H}_n(X^{q+1}, X^q) = \mathrm{H}_n(X^{q+2}, X^{q+1}) = \cdots = \mathrm{H}_n(X^p, X^{p-1}) = 0.$$
Consider the triple $X^q \hookrightarrow X^{q+1} \hookrightarrow X^{q+2}$. The exact sequence
$$\mathrm{H}_n(X^{q+1}, X^q) \to \mathrm{H}_n(X^{q+2}, X^q) \to \mathrm{H}_n(X^{q+2}, X^{q+1})$$
implies $\mathrm{H}_n(X^{q+2}, X^q) = 0$. The same argument applying to the triple $X^q \hookrightarrow X^{q+2} \hookrightarrow X^{q+3}$ implies $\mathrm{H}_n(X^{q+3}, X^q) = 0$. Repeating this process until arriving at $X^q \hookrightarrow X^{p-1} \hookrightarrow X^p$, we find $\mathrm{H}_n(X^p, X^q) = 0$. $\square$

Theorem 18.5. *Let (X, A) be a relative CW complex. Then the cellular homology coincides with the singular homology*
$$\mathrm{H}_n^{cell}(X, A) \cong \mathrm{H}_n(X, A).$$

Proof. Consider the following commutative diagram
$$\begin{tikzcd}
 & \mathrm{H}_{n+1}(X^{n+1}, X^n) \arrow[d] \arrow[dr, "\partial_{n+1}"] & & \mathrm{H}_n(X^{n-2}, A)(=0) \arrow[d] \\
\mathrm{H}_n(X^{n-1}, A)(=0) \arrow[r] & \mathrm{H}_n(X^n, A) \arrow[r] \arrow[d] & \mathrm{H}_n(X^n, X^{n-1}) \arrow[r] \arrow[dr, "\partial_n"] & \mathrm{H}_{n-1}(X^{n-1}, A) \arrow[d] \\
 & \mathrm{H}_n(X^{n+1}, A) \arrow[d] & & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) \arrow[d] \\
 & \mathrm{H}_n(X^{n+1}, X^n)(=0) & & \mathrm{H}_{n-1}(X^{n-2}, A)(=0)
\end{tikzcd}$$
Diagram chasing implies
$$\mathrm{H}_n(X^{n+1}, A) \cong \mathrm{H}_n^{cell}(X, A).$$
Theorem follows from the exact sequence
$$\mathrm{H}_{n+1}(X, X^{n+1})(=0) \to \mathrm{H}_n(X^{n+1}, A) \to \mathrm{H}_n(X, A) \to \mathrm{H}_n(X, X^{n+1})(=0)$$

□

Let $f : (X, A) \to (Y, B)$ be a cellular map. It induces a map on cellular homology

$$f_* : \mathrm{H}_\bullet^{cell}(X, A) \to \mathrm{H}_\bullet^{cell}(Y, B).$$

Therefore in the category of CW complexes, we can work entirely with cellular homology which is combinatorially easier to compute as we discuss next.

Cellular Boundary Formula

Let us now analyze cellular differential

$$\partial_n : \mathrm{H}_n(X^n, X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}).$$

For each n-cell e_α^n, we have the gluing map

$$f_{e_\alpha^n} : S^{n-1} \to X^{n-1}.$$

This defines a map

$$\bar{f}_{e_\alpha^n} : S^{n-1} \to X^{n-1}/X^{n-2} = \bigvee_{J_{n-1}} S^{n-1}$$

which induces a degree map

$$(\bar{f}_{e_\alpha^n})_* : \tilde{\mathrm{H}}_{n-1}(S^{n-1}) \cong \mathbb{Z} \to \bigoplus_{J_{n-1}} \tilde{\mathrm{H}}_{n-1}(S^{n-1}) \cong \bigoplus_{J_{n-1}} \mathbb{Z}.$$

Collecting all n-cells, this generates the degree map

$$d_n : \bigoplus_{J_n} \mathbb{Z} \to \bigoplus_{J_{n-1}} \mathbb{Z}.$$

Theorem 18.6. *Under the identification $C_n^{cell}(X^n, X^{n-1}) \cong \bigoplus_{J_n} \mathbb{Z}$, cellular differential coincides with the degree map*

$$\partial_n \cong d_n.$$

Proof. This follows from chasing the definition of the connecting map $\partial_n : \mathrm{H}_n(X^n, X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$.

□

Example 18.7. $\mathbb{CP}^n$ has a CW structure with a single $2m$-cell for each $m \le n$. Since there is no odd dim cells, the degree map $d = 0$. We find

$$\mathrm{H}_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k = 0, 2, \cdots, 2n \\ 0 & \text{otherwise} \end{cases}$$

Example 18.8. A closed oriented surface Σ_g of genus g has a CW structure with a 0-cell e_0, 2g 1-cells $\{a_1, b_1, \cdots, a_g, b_g\}$, and a 2-cell e_2.

In the cell complex

$$\mathbb{Z}e_2 \xrightarrow{d_2} \bigoplus_i \mathbb{Z}a_i \oplus \bigoplus_i \mathbb{Z}b_i \xrightarrow{d_1} \mathbb{Z}e_0.$$

the degree map d_2 sends

$$e_2 \to \sum_i (a_i + b_i - a_i - b_i) = 0$$

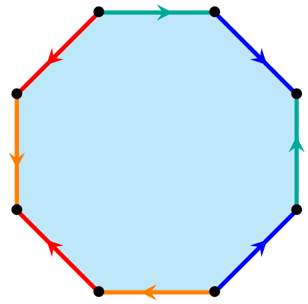


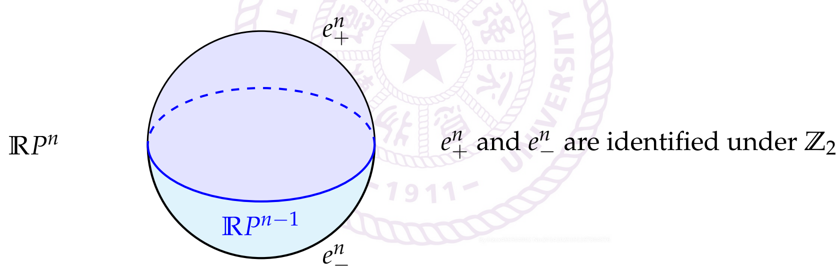
FIGURE 31. The CW structure of a closed oriented surface Σ_g of genus $g = 2$

so $d_2 = 0$. Similarly, d_1 is also 0. We find

$$H_k(\Sigma_g) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^{2g} & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & k > 2. \end{cases}$$

Example 18.9. $\mathbb{R}P^n = S^n/\mathbb{Z}_2$ has a CW structure with a k-cell for each $0 \le k \le n$.

$$\mathbb{R}P^0 \hookrightarrow \mathbb{R}P^1 \hookrightarrow \cdots \hookrightarrow \mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^n.$$



We have the cell oomplex

$$\mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \mathbb{Z} \to \cdots \xrightarrow{d_1} \mathbb{Z}$$

The degree map $d_k : \tilde{H}_{k-1}(S^{k-1}) \to H_{k-1}(S^{k-1})$ is

$$d_k = 1 + \deg(\text{antipodal map}) = 1 + (-1)^k.$$

So the cell complex becomes

$$\cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

It follows that

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 < k < n, k \text{ odd} \\ \mathbb{Z} & k = n = \text{odd} \\ 0 & k = n = \text{even} \\ 0 & k > n \end{cases}$$

Euler characteristic

Definition 18.10. Let X be a finite CW complex of dimension n and denote by c_i the number of i-cells of X. The Euler characteristic of X is defined as:

$$\chi(X) := \sum_i (-1)^i c_i.$$

Recall that any finitely generaed abelian group G is decomposed into a free part and a torsion part

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_k\mathbb{Z}.$$

The integer $r := \mathrm{rk}(G)$ is called the **rank** of G.

Lemma 18.11. *Let $(G_\bullet, \partial)$ be a chain complex of finitely generaed abelian groups such that $G_n = 0$ if $|n| >> 0$. Then*

$$\sum_i (-1)^i \,\mathrm{rk}(G_i) = \sum_i (-1)^i \,\mathrm{rk}(\mathrm{H}_i(G_\bullet)).$$

Proof. We consider the chain complex $(G_\bullet^{\mathbb{Q}}, \partial)$ where

$$G_k^{\mathbb{Q}} = G_k \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}^{\mathrm{rk}(G_k)}.$$

Each $G_k^{\mathbb{Q}}$ is now a vector space over the field $\mathbb{Q}$, and ∂ is a linear map. Moreover

$$\mathrm{H}_i(G_\bullet^{\mathbb{Q}}) = \mathbb{Q}^{\mathrm{rk}(\mathrm{H}_i(G_\bullet))}.$$

The lemma follows from the corresponding statement for linear maps on vector spaces. $\square$

Theorem 18.12. *Let X be a finite CW complex. Then*

$$\chi(X) = \sum_i (-1)^i b_i(X)$$

*where $b_i(X) := rk(\mathrm{H}_i(X))$ is called the i-th **Betti number** of X In particular, $\chi(X)$ is independent of the chosen CW structure on X and only depend on the cellular homotopy class of X.*

Proof. This follows from Lemma 18.11 and the fact that the homologies of celluar complex compute the singular homologies. $\chi(X)$ does not dependent of the chosen CW structure on X since $\mathrm{H}_i(X)$'s do not. $\square$

Example 18.13. $\chi(S^n) = 1 + (-1)^n$.

Example 18.14. Let X be the tetrahedron and Y be the cube. They give two different CW structures on S^2, hence two different counts of the Euler characteristic.

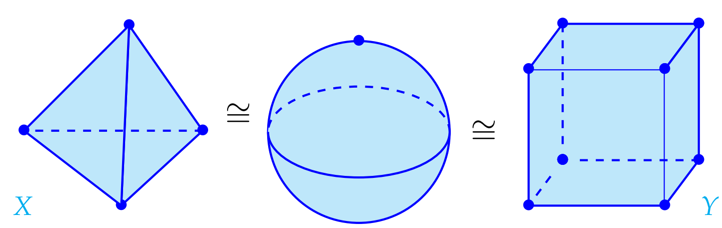


FIGURE 32. The CW structures of sphere S^2

$$\chi(X) = 4 - 6 + 4 = 2, \quad \chi(Y) = 8 - 12 + 6 = 2.$$

19 COHOMOLOGY AND UNIVERSAL COEFFICIENT THEOREM

R refers to a commutative ring in this section.

Cohomology

Definition 19.1. A **cochain complex** over R is a sequence of R-module maps

$$\cdots \to C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n-1} \to \cdots$$

such that $d_n \circ d_{n-1} = 0$. When R is not specified, we mean cochain complex of abelian groups (i.e. $R = \mathbb{Z}$).

Sometimes we just write the map by d and the cochain complex by $(C^\bullet, d)$. Then

$$d_n = d|_{C_n} \quad \text{and} \quad d^2 = 0.$$

Definition 19.2. Given a cochain complex $(C^\bullet, d)$, its n-cocycles Z^n and n-coboundaries B^n are

$$Z^n = \mathrm{Ker}(d : C^n \to C^{n+1}), \quad B^n = \mathrm{Im}(d : C^{n-1} \to C^n).$$

$d^2 = 0$ implies $B^n \subset Z^n$. We define the **n-th cohomology group** by

$$\mathrm{H}^n(C^\bullet, d) := \frac{Z^n}{B^n} = \frac{\ker(d_n)}{\mathrm{im}(d_{n-1})}.$$

A cochain complex $C^\bullet$ is called **acyclic** or **exact** if $\mathrm{H}^n(C^\bullet) = 0$ for all n.

We are interested in the following relation between cochain and chain complex.

Definition 19.3. Let $(C_\bullet, \partial)$ be a chain complex over R, and G be a R-module. We define its dual cochain complex $(C^\bullet, d) = \mathrm{Hom}_R(C_\bullet, G)$ by

$$\cdots \mathrm{Hom}_R(C_{n-1}, G) \to \mathrm{Hom}_R(C_n, G) \to \mathrm{Hom}_R(C_{n+1}, G) \to \cdots$$

Here given $f \in \mathrm{Hom}_R(C_n, G)$, we define $d_n f \in \mathrm{Hom}_R(C_{n+1}, G)$ by

$$d_n f(c) := f(\partial_{n+1}(c)), \quad \forall c \in C_{n+1}.$$

Definition 19.4. Let G be an abelian group and X be a topological space. For $n \geq 0$, we define the group of **singular n-cochains** in X with coefficient in G to be

$$S^n(X; G) := \mathrm{Hom}(S_n(X), G).$$

The dual cochain complex $S^\bullet(X; G) = \mathrm{Hom}(S_\bullet(X), G)$ is called the **singular cochain complex** with coefficient in G. Its cohomology is called the **singular cohomology** and denoted by

$$\mathrm{H}^n(X; G) := \mathrm{H}^n(S^\bullet(X; G)).$$

When $G = \mathbb{Z}$, we simply write it as $\mathrm{H}^n(X)$.

We have the analogue of chain homotopy between cochain complexes. We leave the details to the readers. Proposition 15.16 holds for singular cochains as well.

Theorem 19.5. $\mathrm{H}^n(-; G)$ *defines a contra-variant functor*

$$\mathrm{H}^n(-; G) : \underline{\mathrm{h}\mathbf{Top}} \to \underline{\mathbf{Ab}}.$$

Theorem 19.6 (Dimension Axiom)**.** *If X is contractible, then*

$$\mathrm{H}^n(X; G) = \begin{cases} G & n = 0 \\ 0 & n > 0 \end{cases}$$

The proof is similar to Theorem 15.18.

Lemma 19.7. *Let G be a R-module and $0 \to A_1 \to A_2 \to A_3 \to 0$ be an exact sequence of R-modules. Then the following sequence is exact*

$$0 \to \mathrm{Hom}_R(A_3, G) \to \mathrm{Hom}_R(A_2, G) \to \mathrm{Hom}_R(A_1, G).$$

If A_3 is a free R-module (or more generally projective R-module), then the last morphism is also surjective.

Definition 19.8. Let G be an abelian group. Let $A \subset X$ be a subspace. We define the **relative singular cochain complex** with coefficient in G by

$$S^\bullet(X, A; G) := \mathrm{Hom}(S_\bullet(X)/S_\bullet(A), G).$$

Its cohomology is called the **relative singular cohomology**, denoted by $\mathrm{H}^\bullet(X, A; G)$.

Since $S_\bullet(X)/S_\bullet(A)$ is a free abelian group, we have a short exact sequence of cochain complex

$$0 \to S^\bullet(X, A; G) \to S^\bullet(X; G) \to S^\bullet(A; G) \to 0$$

which induces a long exact sequence of cohomology groups

$$0 \to \mathrm{H}^0(X, A; G) \to \mathrm{H}^0(X; G) \to \mathrm{H}^0(A; G) \to \mathrm{H}^1(X, A; G) \to \cdots.$$

Moreover, the connecting maps

$$\delta : \mathrm{H}^n(A, G) \to \mathrm{H}^{n+1}(X, A; G)$$

is natural in the same sense as that for homology.

Excision and Mayer-Vietoris sequence for cohomology are proved similarly as homology.

Theorem 19.9 (Excision). *Let $U \subset A \subset X$ be subspaces such that $\bar{U} \subset A^\circ$ (the interior of A). Then the inclusion $i : (X - U, A - U) \hookrightarrow (X, A)$ induces isomorphisms*

$$i^* : \mathrm{H}^n(X, A; G) \cong \mathrm{H}^n(X - U, A - U; G), \quad \forall n.$$

Theorem 19.10 (Mayer-Vietoris). *Let X_1, X_2 be subspaces of X and $X = X_1^\circ \cup X_2^\circ$. Then there is an exact sequence*

$$\cdots \to \mathrm{H}^n(X; G) \to \mathrm{H}^n(X_1; G) \oplus \mathrm{H}^n(X_2; G) \to \mathrm{H}^n(X_1 \cap X_2; G) \to \mathrm{H}^{n+1}(X; G) \to \cdots$$

Universal Coefficient Theorem for Cohomology

Definition 19.11. Let M, N be two R-modules. Let $P_\bullet \to M$ be a free R-module resolution of M:

$$\cdots P_n \to P_{n-1} \to \cdots P_1 \to P_0 \to M \to 0$$

is an exact sequence of R-modules and P_i's are free. We define the Ext group

$$\mathrm{Ext}_R^k(M, N) = \mathrm{H}^k(\mathrm{Hom}(P_\bullet, N))$$

and the Tor group

$$\mathrm{Tor}_k^R(M, N) = \mathrm{H}_k(P_\bullet \otimes_R N).$$

Note that

$$\mathrm{Ext}_R^0(M, N) = \mathrm{Hom}_R(M, N), \quad \mathrm{Tor}_0^R(M, N) = M \otimes_R N.$$

Ext and Tor are called the derived functors of Hom and $\otimes$. It is a classical result in homological algebra that $\mathrm{Ext}_R^k(M, N)$ and $\mathrm{Tor}_k^R(M, N)$ don't depend on the choice of resolutions of M. They are functorial with respect to both variables and Tor_k^R is symmetric in two variables

$$\mathrm{Tor}_k^R(M, N) = \mathrm{Tor}_k^R(N, M).$$

Moreover, for any short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

there are associated long exact sequences

$$\begin{aligned} 0 &\to \mathrm{Hom}_R(M_3, N) \to \mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N) \\ &\to \mathrm{Ext}^1_R(M_3, N) \to \mathrm{Ext}^1_R(M_2, N) \to \mathrm{Ext}^1_R(M_1, N) \\ &\to \mathrm{Ext}^2_R(M_3, N)) \to \mathrm{Ext}^2_R(M_2, N) \to \mathrm{Ext}^2_R(M_1, N) \to \cdots \end{aligned}$$

and

$$\begin{aligned} 0 &\to \mathrm{Hom}_R(N, M_1) \to \mathrm{Hom}_R(N, M_2) \to \mathrm{Hom}_R(N, M_3) \\ &\to \mathrm{Ext}^1_R(N, M_1) \to \mathrm{Ext}^1_R(N, M_2) \to \mathrm{Ext}^1_R(N, M_3) \\ &\to \mathrm{Ext}^2_R(N, M_1)) \to \mathrm{Ext}^2_R(N, M_2) \to \mathrm{Ext}^2_R(N, M_3) \to \cdots \end{aligned}$$

and

$$\begin{aligned} \cdots &\to \mathrm{Tor}^R_2(M_1, N) \to \mathrm{Tor}^R_2(M_2, N) \to \mathrm{Tor}^R_3(M_3, N) \\ &\to \mathrm{Tor}^R_1(M_1, N) \to \mathrm{Tor}^R_1(M_2, N) \to \mathrm{Tor}^R_1(M_3, N) \\ &\to M_1 \otimes_R N \to M_2 \otimes_R N \to M_3 \otimes_R N \to 0 \end{aligned}$$

Now we focus on the case of abelian groups $R = \mathbb{Z}$. For any abelian group M, let P_0 be a free abelian group such that $P_0 \to M$ is surjective. Let P_1 be its kernel. Then P_1 is also free and

$$0 \to P_1 \to P_0 \to M \to 0$$

defines a free resolution of abelian groups. This implies that

$$\mathrm{Ext}^k(M, N) = 0, \quad \mathrm{Tor}_k(M, N) = 0 \quad \text{for} \quad k \geq 2.$$

For abelian groups we will simply write

$$\boxed{\mathrm{Ext}(M, N) := \mathrm{Ext}^1_{\mathbb{Z}}(M.N), \quad \mathrm{Tor}(M, N) := \mathrm{Tor}^{\mathbb{Z}}_1(M, N)}.$$

Lemma 19.12. *If either M is free or N is divisible, then $\mathrm{Ext}(M, N) = 0$.*

Proposition 19.13. *Let $(C_\bullet, \partial)$ be a chain complex of free abelian groups, then there exists a split exact sequence*

$$0 \to \mathrm{Ext}(\mathrm{H}_{n-1}, G) \to \mathrm{H}^n(\mathrm{Hom}(C_\bullet, G)) \to \mathrm{Hom}(\mathrm{H}_n, G) \to 0$$

which induces isomorphisms

$$\mathrm{H}^n(\mathrm{Hom}(C_\bullet, G)) \cong \mathrm{Hom}(\mathrm{H}_n(C_\bullet), G) \oplus \mathrm{Ext}(\mathrm{H}_{n-1}(C_\bullet), G)$$

Proof. Let B_n be n-boundaries and Z_n be n-cycles, which are both free. We have exact sequences

$$0 \to B_n \to Z_n \to \mathrm{H}_n \to 0, \quad 0 \to Z_n \to C_n \to B_{n-1} \to 0.$$

This implies exact sequences

$$0 \to \mathrm{Hom}(\mathrm{H}_n, G) \to \mathrm{Hom}(Z_n, G) \to \mathrm{Hom}(B_n, G) \to \mathrm{Ext}(\mathrm{H}_n, G) \to 0$$

and the split exact sequence

$$0 \to \mathrm{Hom}(B_{n-1}, G) \to \mathrm{Hom}(C_n, G) \to \mathrm{Hom}(Z_n, G) \to 0.$$

Consider the commutative diagram with exact columns

$$
\begin{tikzcd}
0 & 0 \arrow[d] & \\
\mathrm{Hom}(Z_{n-1},G) \arrow[u] \arrow[r] & \mathrm{Hom}(B_{n-1},G) \arrow[r] \arrow[d] & \mathrm{Ext}(\mathrm{H}_{n-1},G) \\
\mathrm{Hom}(C_{n-1},G) \arrow[u] \arrow[r] & \mathrm{Hom}(C_n,G) \arrow[r] \arrow[d] & \mathrm{Hom}(C_{n+1},G) \\
\mathrm{Hom}(\mathrm{H}_n,G) \arrow[r] & \mathrm{Hom}(Z_n,G) \arrow[r] \arrow[d] & \mathrm{Hom}(B_n,G) \arrow[u] \\
 & 0 & 0 \arrow[u]
\end{tikzcd}
$$

Diagram chasing shows this implies a short exact sequence

$$0 \to \mathrm{Ext}(\mathrm{H}_{n-1},G) \to \mathrm{H}^n(\mathrm{Hom}(C_\bullet,G)) \to \mathrm{Hom}(\mathrm{H}_n,G) \to 0$$

which is also split due to the split of the middle column in the above diagram. □

Theorem 19.14 (Universal Coefficient Theorem for Cohomology)**.** *Let G be an abelian group and X be a topological space. Then for any $n \geq 0$, there exists a split exact sequence*

$$0 \to \mathrm{Ext}(\mathrm{H}_{n-1}(X),G) \to \mathrm{H}^n(X;G) \to \mathrm{Hom}(\mathrm{H}_n(X),G) \to 0$$

which induces isomorphisms

$$\mathrm{H}^n(X;G) \cong \mathrm{Hom}(\mathrm{H}_n(X),G) \oplus \mathrm{Ext}(\mathrm{H}_{n-1}(X),G).$$

Proof. Apply the previous Lemma to $C_\bullet = S_\bullet(X)$. □

Universal Coefficient Theorem for Homology

Definition 19.15. Let G be an abelian group. Let $A \subset X$ be a subspace. We define the **relative singular chain complex** with coefficient in G by

$$S_\bullet(X,A;G) := S_\bullet(X,A) \otimes_{\mathbb{Z}} G.$$

Its homology is called the **relative singular homology** with coefficient in G, denoted by $\mathrm{H}_\bullet(X,A;G)$. When $A = \emptyset$, we simply get the singular homology $\mathrm{H}_\bullet(X;G)$.

Similar long exact sequence for relative singular homologies follows from the short exact sequence

$$0 \to S_\bullet(A;G) \to S_\bullet(X;G) \to S_\bullet(X,A;G) \to 0.$$

Theorem 19.16 (Universal Coefficient Theorem for homology)**.** *Let G be an abelian group and X be a topological space. Then for any $n \geq 0$, there exists a split exact sequence*

$$0 \to \mathrm{H}_n(X) \otimes G \to \mathrm{H}_n(X;G) \to \mathrm{Tor}(\mathrm{H}_{n-1}(X),G) \to 0$$

which induces isomorphisms

$$\mathrm{H}_n(X;G) \cong (\mathrm{H}_n(X) \otimes G) \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(X),G).$$

The proof is similar to the cohomology case.

20 HUREWICZ THEOREM

Hurewicz Theorem connects homotopy groups with homology groups. Recall that

$$\tilde{\mathrm{H}}_n(S^n) = \mathbb{Z}.$$

Let us fix generators $i_n \in \tilde{\mathrm{H}}_n(S^n)$ which are compatible with the isomorphisms

$$\tilde{\mathrm{H}}_n(S^n) = \mathrm{H}_n(D^n, S^{n-1}) = \tilde{\mathrm{H}}_{n-1}(S^{n-1}).$$

Definition 20.1. For $n \geq 1$, the Hurewicz map is

$$\rho_n : \pi_n(X) \to \mathrm{H}_n(X) \quad \text{by sending} \quad [f : S^n \to X] \to f_*(i_n).$$

Proposition 20.2. *The Hurewicz map is a group homomorphism.*

Proof. Given $f, g : S^n \to X$ representing $[f], [g] \in \pi_n(X)$, their product in $\pi_n(X)$ is represented by

$$S^n \xrightarrow{\varphi} S^n \vee S^n \xrightarrow{f \vee g} X \vee X \to X.$$

Here the map φ shrinks the equator S^{n-1} of S^n to a point, and $S^n/S^{n-1} = S^n \vee S^n$. Apply $\mathrm{H}_n(-)$ we get

$$\mathrm{H}_n(S^n) \xrightarrow{\varphi_*} \mathrm{H}_n(S^n) \oplus \mathrm{H}_n(S^n) \xrightarrow{f_* \oplus g_*} \mathrm{H}_n(X) \oplus \mathrm{H}_n(X) \xrightarrow{\text{sum}} \mathrm{H}_n(X).$$

Observe $\varphi_* : \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n) \oplus \mathrm{H}_n(S^n)$ is the diagonal map $x \to x \oplus x$. It follows that

$$\rho_n([f][g]) = f_*(i_n) + g_*(i_n) = \rho_n(f) + \rho_n(g).$$

□

Given a group G, let $G_{\mathrm{ab}} = G/[G, G]$ denote its abelianization. The quotient map

$$G \to G_{\mathrm{ab}}$$

is called the abelianization homomorphism, which is an isomorphism if G is an abelian group.

Theorem 20.3 (Hurewicz Theorem). *Let X be a path-connected space which is $(n-1)$-connected ($n \geq 1$). Then the Hurewicz map*

$$\rho_n : \pi_n(X) \to \mathrm{H}_n(X)$$

is the abelianization homomorphism.

Explicitly, Hurewicz Theorem has the following two cases.

1°. If $n = 1$, then the Hurewicz map $\rho_1 : \pi_1(X) \to \mathrm{H}_1(X)$ induces an isomorphism

$$\pi_1(X)_{\mathrm{ab}} \xrightarrow{\cong} \mathrm{H}_1(X).$$

2°. If $n > 1$, then the Hurewicz map $\rho_n : \pi_n(X) \to \mathrm{H}_n(X)$ is an isomorphism.

Before we prove the Hurewicz Theorem, we first prepare some useful propositions.

Proposition 20.4. *Let $f : X \to Y$ be a weak homotopy equivalence. Then*

$$f_* : \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$$

is an isomorphism for all n.

Proof. We can assume f is a cofibration. Then $\pi_n(Y, X) = 0$ for all n.

Let $\sigma = \sum_i n_i \sigma_i$ represent an arbitrary element of $\mathrm{H}_n(Y, X)$ where

$$\sigma_i : \Delta^n \to Y, \quad \partial\sigma \in X.$$

We can use the simplexes of σ_i's to build up a finite CW complex K with a subcomplex L, and a map

$$f : K \to Y, \quad \varphi(L) \subset X$$

such that $[\sigma] = f_*[\gamma]$ is the image of an element $[\gamma] \in \mathrm{H}_n(K, L)$ under f. Since $X \hookrightarrow Y$ is an ∞-equivalence, f is homotopic relative L to a map g that sends K into X.

$$\begin{tikzcd} L \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\ K \arrow[ur, dashed, "g"] \arrow[r, bend left] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow"] & Y \end{tikzcd}$$

So $[\sigma] = g_*[\gamma] = 0$. It follows that $\mathrm{H}_n(Y, X) = 0$ for all n. This proves the proposition. $\square$

Proposition 20.5. *Let $Y = \bigvee S^n$ is a wedge of spheres ($n \geq 1$). Then*

$$\pi_n(Y) \to \tilde{\mathrm{H}}_n(Y)$$

is the abelianization homomorphism.

Proof. If $n > 1$, then $\pi_n(Y) = \tilde{\mathrm{H}}_n(Y) = \bigoplus \mathbb{Z}$. If $n = 1$. then $\pi_1(Y)$ is a free group, $\mathrm{H}_1(Y)$ is a free abelian group which is the abelianization of $\pi_1(Y)$.

$\square$

Proof of Hurewicz Theorem. We can assume X is a CW complex. Otherwise we replace X by a weak homotopic equivalent CW complex, which has the same homotopy and homology groups by Proposition 20.4. The construction of CW approximation also implies that we can assume

$$X^0 \subset X^1 \subset \cdots \subset X^{n-1} \subset X^n \subset \cdots \subset X$$

where $X^0 = X^1 = \cdots = X^{n-1}$ is a point. Since

$$\pi_n(X^{n+1}) = \pi_n(X), \quad \mathrm{H}_n(X^{n+1}) = \mathrm{H}_n(X),$$

we can further assume $X = X^{n+1}$. By assumption

$$X^n = \bigvee_i S^n$$

is a wedge of spheres. Let

$$\varphi : \coprod_\alpha S^n \to \bigvee_i S^n$$

be the gluing map for attaching $(n+1)$-cells. Using the cellular approximation, we can assume φ is based

$$\varphi : Y = \bigvee_\alpha S^n \to \bigvee_i S^n.$$

Let

$$Z = M_\varphi$$

be the reduced mapping cylinder of φ, which is homotopy equivalent to X^n. X is the cofiber of φ

$$X = Z/Y.$$

We have the push-out diagram

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[d, "j"'] & \star \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

where $\star$ is the base point. Replacing $\star$ by the reduced cone $C_\star Y$ and consider the push-out

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "j"'] & C_\star Y \arrow[d] \\ Z \arrow[r] & \tilde{X} \end{tikzcd}$$

Since j is an $(n-1)$-equivalence and i is an n-equivalence, Homotopy Excision Theorem (Theorem 14.1) implies

$$\pi_n(Z, Y) = \pi_n(\tilde{X}, C_\star Y) = \pi_n(X) \quad \text{if} \quad n > 1.$$

This implies the exact sequence

$$\pi_n(Y) \to \pi_n(Z) \to \pi_n(X) \to \pi_{n-1}(Y) = 0 \quad \text{if} \quad n > 1.$$

For the case $n = 1$, Seifert-van Kampen Theorem implies that $\pi_1(X)$ is the quotient of $\pi_1(Z)$ by the normal subgroup generated by the image of $\pi_1(Y)$.

On the other side, we have the homology exact sequence

$$\mathrm{H}_n(Y) \to \mathrm{H}_n(Z) \to \mathrm{H}_n(Z, Y) = \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(Y) = 0.$$

Now we consider the commutative diagram

$$\begin{tikzcd} \pi_n(Y) \arrow[r] \arrow[d] & \pi_n(Z) \arrow[r] \arrow[d] & \pi_n(X) \arrow[r] \arrow[d] & 0 \\ \mathrm{H}_n(Y) \arrow[r] & \mathrm{H}_n(Z) \arrow[r] & \mathrm{H}_n(X) \arrow[r] & 0 \end{tikzcd}$$

Proposition 20.5 implies that

$$\pi_n(Y) \to \mathrm{H}_n(Y) \quad \text{and} \quad \pi_n(Z) \to \mathrm{H}_n(Z)$$

are abelianization homomorphisms. Therefore $\pi_n(X) \to \mathrm{H}_n(X)$ is also the abelianization homomorphism.

□

Example 20.6. The homology of S^n and Hurewicz Theorem implies that

$$\pi_k(S^n) = \begin{cases} 0 & \text{if} \quad k < n \\ \mathbb{Z} & \text{if} \quad k = n. \end{cases}$$

In particular, the degree of a map $f : S^n \to S^n$ can be described by either homotopy or homology.

Hurewicz Theorem has a relative version as well.

Theorem 20.7. *Let (X, A) be a pair of path-connected spaces and A non-empty. Assume (X, A) is $(n-1)$-connected ($n \geq 2$) and A is simply-connected. Then*

$$\mathrm{H}_n(X, A) = 0 \quad \textit{for} \quad i < n$$

and the Hurewicz map

$$\pi_n(X, A) \to \mathrm{H}_n(X, A)$$

is an isomorphism.

Theorem 20.8 (Homology Whitehead Theorem)**.** *Let $f : X \to Y$ between simply connected CW complexes. Assume*

$$f_* : \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$$

is an isomorphism for each n. Then f is a homotopy equivalence.

Proof. We can assume X is a CW subcomplex of Y. Then

$$\mathrm{H}_n(Y, X) = 0 \quad \text{for all} \quad n.$$

By Hurewicz Theorem,

$$\pi_n(Y, X) = 0 \quad \text{for all} \quad n.$$

Therefore f is weak homotopy equivalence, hence a homotopy equivalence by Whitehead Theorem. □

Proposition 20.9. *Every simply connected and orientable closed 3-manifold is homotopy equivalent to S^3.*

Proof. Let X be a simply connected and orientable closed 3-manifold. Then

$$\mathrm{H}_0(X) = \mathbb{Z}, \quad \mathrm{H}_1(X) = \pi_1(X) = 0.$$

Since X is orientable, $\mathrm{H}_3(X) = \mathbb{Z}$ and Poincare duality holds (we will discuss in details later)

$$\mathrm{H}_2(X) = \mathrm{H}^1(X).$$

By the Universal Coefficient Theorem,

$$\mathrm{H}^1(X) = \mathrm{Hom}(\mathrm{H}_1(X), \mathbb{Z}) \oplus \mathrm{Ext}(\mathrm{H}_0(X), \mathbb{Z}) = 0.$$

So $\mathrm{H}_2(X) = 0$. By Hurewicz Theorem,

$$\pi_3(X) \to \mathrm{H}_3(X)$$

is an isomorphism. Let $f : S^3 \to X$ represent a generator of $\pi_3(X) = \mathbb{Z}$. Then

$$f_* : \mathrm{H}_\bullet(S^3) \to \mathrm{H}_\bullet(X)$$

are isomorphisms. It follows that f is a homotopy equivalence. □

Remark 20.10. The famous Poincare Conjecture asks that such X is homeomorphic to S^3.

21 SPECTRAL SEQUENCE

Motivation

Many applications of (co)homology theory are reduced to the computation

$$\mathrm{H}(C^\bullet, \delta)$$

of (co)homologies of certain (co)chain complexes. Usually the differential δ is complicated, making the (co)homology computation difficult. However, if we observe that "part" of the differential δ is simple, say

$$\delta = \delta_1 + \delta_2$$

while the computation of δ_1-cohomology is easier to perform, then we would like to use the δ_1-cohomology to approximate and compute the full δ-cohomology. This is the idea of spectral sequence.

Let us motivate this idea by a standard example. Consider the double complex

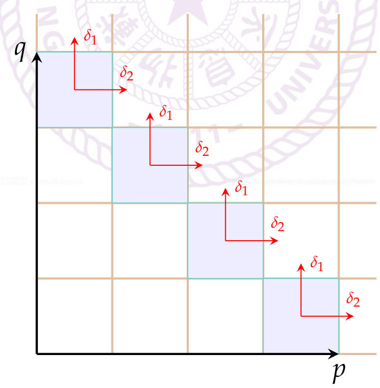
$$K = \bigoplus_{p,q \geq 0} K^{p,q}$$

which is equipped with two differentials

$$\begin{cases} \delta_1 : K^{p,q} \to K^{p,q+1} \\ \delta_2 : K^{p,q} \to K^{p+1,q} \end{cases}$$

such that

$$\delta_1^2 = \delta_2^2 = 0, \quad \delta_1\delta_2 + \delta_2\delta_1 = 0.$$



Consider the total complex

$$\mathrm{Tot}^\bullet(K), \quad \mathrm{Tot}^n(K) = \bigoplus_{p+q=n} K^{p,q}$$

with the differential

$$D = \delta_1 + \delta_2.$$

Our assumption on δ_1, δ_2 implies that

$$D^2 = 0.$$

Therefore $(\mathrm{Tot}^\bullet(K), D)$ indeed defines a cochain complex, and we are interested in

$$\mathrm{H}^\bullet(\mathrm{Tot}^\bullet(K), D).$$

Let x be a representative of an element in $\mathrm{H}^m(\mathrm{Tot}^\bullet(K), D)$. We can decompose x into

$$x = x_0 + x_1 + \cdots, \quad x_i \in K^{i,m-i}.$$

The cocycle condition $Dx = 0$ is equivalent to

$$\begin{cases} \delta_1 x_0 = 0 \\ \delta_2 x_0 = -\delta_1 x_1 \\ \delta_2 x_1 = -\delta_1 x_2 \\ \quad \vdots \end{cases}$$

Let us formally write

$$x_1 \text{``} = \text{''} - \delta_1^{-1}\delta_2 x_0, \quad x_2 \text{``} = \text{''} - \delta_1^{-1}\delta_2 x_1, \quad \cdots$$

Here the inverse δ_1^{-1} does not exist and this expression is only heuristic. Then we would solve

$$x \text{``} = \text{''} \frac{1}{1+\delta_1^{-1}\delta_2} x_0$$

while x_0 represents a cocycle for $(\mathrm{Tot}^\bullet(K), \delta_1)$. Intuitively, we treat δ_2 as a perturbation of δ_1 and

$$D = (\delta_1 + \delta_2) \text{``} = \text{''} \delta_1 (1 + \delta_1^{-1}\delta_2).$$

So

$$Dx \text{``} = \text{''} \delta_1(1+\delta_1^{-1}\delta_2) \frac{1}{1+\delta_1^{-1}\delta_2} x_0 \text{``} = \text{''} \delta_1 x_0 = 0.$$

The above discussion is of course vague and heuristic. But it motivates the following idea: we can construct a D-cocycle x by first looking at a δ_1-cocycle x_0 as a leading approximation, and then constructing $x_1, x_2, \cdots$ order by order using information from $\mathrm{H}^\bullet(\delta_1)$. This leads to the following statements

1°. If $\mathrm{H}^\bullet(\delta_1) = 0$, then $\mathrm{H}^\bullet(D) = 0$.

In fact, let x be a D-cocyle as above. Since $\delta_1 x_0 = 0$ and $\mathrm{H}^\bullet(\delta_1) = 0$, we can find

$$y_0 \in K^{0,m-1} \quad \text{such that} \quad x_0 = \delta_1 y_0.$$

Replacing x by $x - Dy_0$, we can assume $x_0 = 0$ so x starts from x_1. Then

$$Dx = 0 \Rightarrow \delta_1 x_1 = 0.$$

By the same reason, we can further kill x_1 to assume that x starts from x_2. Iterating this process, we can eventually find y such that

$$x = Dy.$$

So x is a D-coboundary. It follows that $\mathrm{H}^\bullet(D) = 0$.

2°. If $\mathrm{H}^\bullet(\delta_1) \neq 0$, then we need to understand

$$\text{whether} \quad \delta_1 x_{i+1} = -\delta_2 x_i \quad \text{is solvable.}$$

This puts extra condition on the initial data x_0 that allows to be an approximation of a D-cocycle.

For example, we want to solve

$$\delta_1 x_1 = -\delta_2 x_0.$$

Since

$$\delta_1(\delta_2 x_0) = -\delta_2 \delta_1 x_0 = 0,$$

we know $-\delta_2 x_0$ is δ_1-closed. The problem is whether this is δ_1-exact. We can view

$$\delta_2 : \mathrm{H}^\bullet(\delta_1) \to \mathrm{H}^\bullet(\delta_1)$$

as defining a cochain complex $(\mathrm{H}^\bullet(\delta_1), \delta_2)$, then the solvability of x_1 asks that the class $[x_0] \in \mathrm{H}^\bullet(\delta_1)$ is in fact δ_2-closed

$$\delta_2[x_0] = 0.$$

Therefore the "2nd"-order approximation of a D-cohomolpgy is an element in

$$\mathrm{H}^\bullet(\mathrm{H}^\bullet(\delta_1), \delta_2).$$

This will be called the E_2-page. Similarly, we will have E_3-page, E_4-page, etc, and eventually find the full description of D-cohomologies. Such process is the basic idea of spectral sequence.

Spectral sequence for filtered chain complex

Spectral sequences usually arise in two situations

1°. A $\mathbb{Z}$-filtration of a chain complex: a sequence of subcomplexes $\cdots \subset F_p \subset F_{p+1} \subset \cdots$.
2°. A $\mathbb{Z}$-filtration of a topological space: a family of subspaces $\cdots \subset X_p \subset X_{p+1} \subset \cdots$.

We first discuss the spectral sequence for chain complexes.

Definition 21.1. An **(ascending) filtration** of an R-module A is an increasing sequence of submodules

$$\cdots \subset F_p A \subset F_{p+1} A \subset \cdots$$

indexed by $p \in \mathbb{Z}$. We always assume that it is exhaustive and Hausdorff

$$\bigcup_p F_p A = A \quad \text{(exhaustive)}, \quad \bigcap_p F_p A = 0 \quad \text{(Hausdorff)}.$$

The filtration is **bounded** if $F_p A = 0$ for p sufficiently small and $F_p A = A$ for p sufficiently large. The **associated graded module** $\mathrm{Gr}^F_\bullet A$ is defined by

$$\mathrm{Gr}^F_\bullet(A) := \bigoplus_{p\in\mathbb{Z}} \mathrm{Gr}^F_p A, \quad \mathrm{Gr}^F_p A := F_p A / F_{p-1} A.$$

A **filtered chain complex** is a chain complex $(C_\bullet, \partial)$ together with an (ascending) filtration $F_p C_i$ of each C_i such that the differential preserves the filtration

$$\partial(F_p C_i) \subset F_p C_{i-1}.$$

In other words, we have an increasing sequence of subcomplexes $F_p C_\bullet$ of $C_\bullet$.

Remark 21.2. There is also the notion of a descending filtration. We will focus on the ascending case here.

A filtered chain complex induces a filtration on its homology

$$F_p \mathrm{H}_i(C_\bullet) = \mathrm{Im}(\mathrm{H}_i(F_p C_\bullet) \to \mathrm{H}_i(C_\bullet)).$$

In other words, an element $[\alpha] \in \mathrm{H}_i(C_\bullet)$ lies in $F_p \mathrm{H}_i(C_\bullet)$ if and only if there exists a representative $x \in F_p C_i$ such that $[\alpha] = [x]$. Its graded piece is given by

$$\mathrm{Gr}^F_p \mathrm{H}_i(C_\bullet) = \frac{\mathrm{Ker}(\partial : F_p C_i \to F_p C_{i-1})}{F_{p-1} C_i + \partial C_{i+1}}.$$

Notation 21.3. In this section, our notation of quotient means the quotient of the numerator by its intersection with the denominator, i.e., $\frac{A}{B} := \frac{A}{A\cap B}$.

Definition 21.4. Given a filtered R-module A, we define its **Rees module** as a submodule of $A[z, z^{-1}]$ by

$$A_F := \bigoplus_{p\in\mathbb{Z}} F_p A \, z^p \subset A[z, z^{-1}].$$

Our conditions for the filtration can be interpreted as follows

1°. increasing filtration: A_F is a $R[z]$-submodule of $A[z,z^{-1}]$ and $z : A_F \to A_F$ is injective.
2°. exhaustive: $A_F[z^{-1}] := A_F \otimes_{R[z]} R[z,z^{-1}]$ equals $A[z,z^{-1}]$.
3°. Hausdorff: $\bigcap\limits_{p\geq 0} z^p A_F = 0$ in $A[z,z^{-1}]$.

The associated graded module is given by

$$\mathrm{Gr}^F_\bullet(A) := A_F/zA_F.$$

Geometrically, we can think about $A[z,z^{-1}]$ as the space of algebraic sections of the trivial bundle on $\mathbb{C}^*$ with fiber A. Then A_F defines the extension of this bundle to $\mathbb{C}$, whose fiber at 0 is precisely $\mathrm{Gr}^F_\bullet(A)$.

Let $(C_\bullet, \partial, F_\bullet)$ be a filtered chain complex. Let us denote its Rees module by

$$C_F := \bigoplus_{p\in\mathbb{Z}} F_pC_\bullet\, z^p \subset C_\bullet[z,z^{-1}].$$

(C_F, ∂) is also a subcomplex of $(C_\bullet[z,z^{-1}], \partial)$. This defines a map on homologies

$$\mathrm{H}_\bullet(C_F,\partial) \to \mathrm{H}_\bullet(C_\bullet[z,z^{-1}],\partial) = \mathrm{H}_\bullet(C_\bullet,\partial)[z,z^{-1}].$$

The image of $\mathrm{H}_\bullet(C_F,\partial)$ defines a $\mathbb{C}[z]$-submodule of $\mathrm{H}_\bullet(C_\bullet,\partial)[z,z^{-1}]$. It induces a filtration on $\mathrm{H}_\bullet(C_\bullet,\partial)$ as described above. Our goal is to analyze the map

$$\varphi : \mathrm{H}_\bullet(C_F,\partial) \to \mathrm{H}_\bullet(C_\bullet,\partial)[z,z^{-1}]$$

in order to extract the information about this induced filtration on $\mathrm{H}_\bullet(C_\bullet,\partial)$.

Firstly

$$\mathrm{H}_\bullet(C_F,\partial) = \bigoplus_{p\in\mathbb{Z}} \mathrm{H}_\bullet(F_pC_\bullet,\partial)z^p.$$

However, the z-action

$$z : \mathrm{H}_\bullet(C_F,\partial) \to \mathrm{H}_\bullet(C_F,\partial)$$

may not be injective. Those elements that are annihilated by z^m for some finite m will be killed under φ. One way to kill such elements is to look at $\mathrm{im}(z^N)$ for N big enough. This motivates the following construction.

Let us define

$$E^r := \frac{\{x \in C_F | \partial x \in z^r C_F\}}{zC_F + z^{1-r}\partial C_F}.$$

E^r can be viewed as the r-th order approximation. E^r carries a differential

$$\partial_r : E^r \to E^r, \quad [x] \to z^{-r}[\partial x].$$

∂_r is indeed well-defined. In fact, let $z\alpha + z^{1-r}\partial\beta$ represent an element in $zC_F + z^{1-r}\partial C_F$. Then

$$\partial_r(z\alpha + z^{1-r}\partial\beta) = z^{1-r}\partial\alpha \quad \text{which is zero as a class in } E^r.$$

Obviously, $\partial_r^2 = 0$. We can define its homology by

$$\mathrm{H}(E^r,\partial_r) := \frac{\ker \partial_r}{\mathrm{im}\, \partial_r}.$$

Claim. *The homology of (E^r,∂_r) is precisely E^{r+1}*

$$\mathrm{H}(E^r,\partial_r) = E^{r+1}.$$

Proof. Assume $[x] \in \ker \partial_r$ in E^r. $\partial_r[x] = z^{-r}[\partial x] = 0$ implies the existence $\alpha, \beta \in C_F$ such that

$$\partial x = z^r(z\alpha + z^{1-r}\partial\beta) = z^{r+1}\alpha + z\partial\beta, \quad \partial\beta \in z^{r-1}C_F.$$

We have $\partial(x - z\beta) = z^{1+r}\alpha$, so $[x - z\beta]$ defines an element in E^{r+1}. This class does not depend on the choice of α, β. Therefore we have a natural map

$$f : \ker \partial_r \to E^{r+1}$$

which is clearly surjective.

Assume $[x] = \partial_r[y]$. Then there exists $u, v \in C_F$ such that

$$x = z^{-r}\partial y + zu + z^{1-r}\partial v.$$

So

$$f([x]) = [x - zu] = [z^{-r}\partial(y + zv)] = 0.$$

Therefore

$$\operatorname{im} \partial_r \subset \ker f.$$

On the other hand, assume $f([x]) = 0$. Then there exists $u, v \in C_F$ such that

$$x - z\beta = zu + z^{-r}\partial v, \quad \partial u = z^r\alpha.$$

We find $[x] = \partial_r[v]$. Hence

$$\ker f \subset \operatorname{im} \partial_r.$$

It follows that $\ker f = \operatorname{im} \partial_r$. This proves the claim.

□

We can describe (E^r, ∂_r) explicitly in terms of components. Let us denote

$$(C_F)_{p,q} := F_p C_{p+q}.$$

There is a natural identification

$$C_F = \bigoplus_{p,q \in \mathbb{Z}} (C_F)_{p,q}.$$

Similarly, we can decompose

$$E^r = \bigoplus_{p,q \in \mathbb{Z}} E^r_{p,q}$$

where

$$E^r_{p,q} = \frac{\{x \in F_p C_{p+q} | \partial x \in F_{p-r} C_{p+q-1}\}}{F_{p-1} C_{p+q} + \partial F_{p+r-1} C_{p+q+1}}.$$

The differential ∂_r acts on components by

$$\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1}, \quad x \to \partial x.$$

E^0 is given by

$$E^0 = C_F / zC_f, \quad E^0_{p,q} = \operatorname{Gr}^F_p C_{p+q} = \frac{F_p C_{p+q}}{F_{p-1} C_{p+q}}.$$

E^1 is given by

$$E^1 = \frac{\{x \in C_F | \partial x \in zC_F\}}{zC_F + \partial C_F} = \mathrm{H}(C_F / zC_F, \partial), \quad E^1_{p,q} = \mathrm{H}_{p+q}(\operatorname{Gr}^F_p C_\bullet).$$

If the filtration of C_i is bounded for each i, then for any p, q and $r >> 0$,

$$E^r_{p,q} = \frac{\{x \in F_p C_{p+q} | \partial x = 0\}}{F_{p-1}C_{p+q} + \partial C_{p+q+1}} = \operatorname{Gr}_p \mathrm{H}_{p+q}(C_\bullet).$$

In this case, we say $\{E^r\}_r$ converges to $\operatorname{Gr}_\bullet \mathrm{H}(C_\bullet)$ and write

$$E^\infty = \operatorname{Gr}_\bullet \mathrm{H}(C_\bullet).$$

Motivated by the above discussion, we now give the formal definition of spectral sequence.

Definition 21.5. A **spectral sequence** (of R-modules) consists of

- an R-module $E^r_{p,q}$ for any $p, q \in \mathbb{Z}$ and $r \geq 0$;
- a differential $\partial_r : E^r_{p,q} \to E^r_{p-r,q+r+1}$ such that $\partial_r^2 = 0$ and $E^{r+1} = \mathrm{H}(E^r, \partial_r)$.

A spectral sequence converges if for any p, q, we have

$$E^r_{p,q} = E^{r+1}_{p,q} = \cdots \quad \text{for} \quad r >> 0.$$

This limit will be denoted by $E^\infty_{p,q}$.

The following theorem follows directly from our discussion above.

Theorem 21.6. *There is an associated spectral sequence for any filtered chain complex* $(C_\bullet, \partial, F_\bullet)$ *where*

$$E^r_{p,q} = \frac{\{x \in F_p C_{p+q} | \partial x \in F_{p-r} C_{p+q-1}\}}{F_{p-1}C_{p+q} + \partial F_{p+r-1} C_{p+q+1}}$$

and

$$\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1}, \quad x \to \partial x.$$

The E^1*-page of the spectral sequence is*

$$E^1_{p,q} = \mathrm{H}_{p+q}(\operatorname{Gr}^F_p C_\bullet).$$

If the filtration of C_i *is bounded for each* i*, then the spectral sequence converges and*

$$E^\infty_{p,q} = \operatorname{Gr}_p \mathrm{H}_{p+q}(C_\bullet).$$

Spectral sequence for filtered cochain complex

The spectral sequence for filtered cochain complexes is similar. We will briefly summarize the result.

Definition 21.7. A **filtered cochain complex** is a cochain complex $(C^\bullet, d)$ with a (descending) filtration

$$\cdots \supset F_p C^i \supset F_{p+1} C^i \supset \cdots$$

of each C^i such that the differential preserves the filtration

$$d(F_p C^i) \subset F_p C^{i+1}.$$

In other words, we have a decreasing sequence of subcomplexes $F_p C^\bullet$ of $C^\bullet$.

The associated graded complex is

$$\operatorname{Gr}^F_p C^\bullet = F_p C^\bullet / F_{p+1} C^\bullet.$$

The convention for a special sequence in this case is

- an R-module $E^{p,q}_r$ for any $p, q \in \mathbb{Z}$ and $r \geq 0$;
- a differential $d_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$ such that $d_r^2 = 0$ and $E_{r+1} = \mathrm{H}(E_r, d_r)$.

Theorem 21.8. *There is an associated spectral sequence for any filtered cochain complex* $(C^\bullet, d, F_\bullet)$ *where*

$$E_r^{p,q} = \frac{\{x \in F_p C^{p+q} | dx \in F_{p+r} C^{p+q+1}\}}{F_{p+1} C^{p+q} + dF_{p-r+1} C^{p+q-1}}.$$

and

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}, \quad x \to dx.$$

The E_1-page of the spectral sequence is

$$E_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{Gr}_p^F C^\bullet).$$

If the filtration of C^i is bounded for each i, then the spectral sequence converges and

$$E_\infty^{p,q} = \mathrm{Gr}_p \mathrm{H}^{p+q}(C^\bullet).$$

Double complex

Let us come back to the double complex example discussed in the beginning

$$K = \bigoplus_{p,q \geq 0} K^{p,q}$$

which is equipped with two differentials

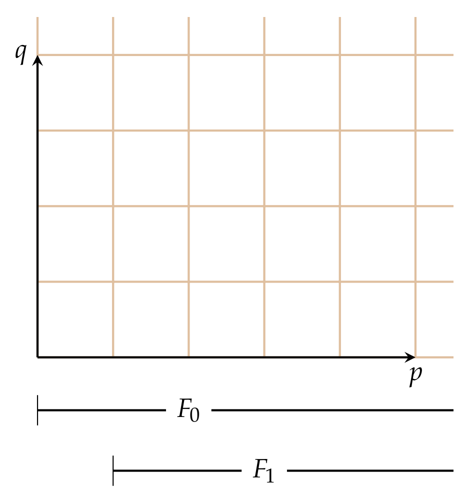
$$\begin{cases} \delta_1 : K^{p,q} \to K^{p,q+1} \\ \delta_2 : K^{p,q} \to K^{p+1,q} \end{cases}$$

We want to compute the cohomology of the total complex

$$\mathrm{H}^\bullet(\mathrm{Tot}^\bullet(K), D), \quad D = \delta_1 + \delta_2.$$

Let us define a descending filtration on K by

$$F_p K = \bigoplus_{m \geq p, n \geq 0} K^{m,n}.$$



This induces a descending filtration on $\mathrm{Tor}^\bullet(K)$ by

$$F_p \,\mathrm{Tor}^\bullet(K) := \mathrm{Tor}^\bullet(F_p K)$$

whose graded associated complex is

$$\mathrm{Gr}_p \operatorname{Tor}^\bullet(K) = \bigoplus_{q\geq 0} K^{p,q}.$$

The E_1 page of the spectral sequence is

$$E_1^{p,q} = \mathrm{H}_{\delta_1}^{p,q}(K), \quad d_1 = \delta_2.$$

Here $\mathrm{H}_{\delta_1}^{p,q}(K)$ is the δ_1-cohomology for the double complex K, which is again double graded.

The E_2 page of the spectral sequence is

$$E_2^{p,q} = \mathrm{H}_{\delta_2}^{p,q} \mathrm{H}_{\delta_1}(K).$$

An element of $E_r^{p,q}$ is represented by an $x_0 \in K^{p,q}$ that can be extended to a chain

$$x = x_0 + x_1 + \cdots + x_{r-1}, \quad x_i \in K^{p+i,q-i}$$

such that

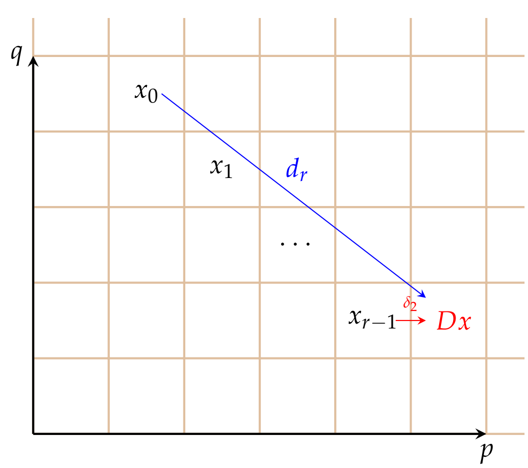
$$Dx \in K^{p+r+1,q-r}.$$

In other words, we can solve the following recursive equations up to x_{r-1}

$$\begin{cases} \delta_1 x_0 = 0 \\ \delta_2 x_0 = -\delta_1 x_1 \\ \delta_2 x_1 = -\delta_1 x_2 \\ \vdots \\ \delta_2 x_{r-2} = -\delta_1 x_{r-1}. \end{cases}$$

The corresponding differential for the E_r-page is

$$d_r[x_0] = [Dx] = [\delta_2 x_{r-1}].$$



Cellular chain complex revisited

Let X be a CW complex with cellular structure

$$X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(n)} \subset \cdots .$$

We define an ascending filtration on the singular chain complex $S_\bullet(X)$ by

$$F_p S_\bullet(X) = S_\bullet(X^{(p)}).$$

The E^0-page is

$$E^0_{p,q} = \mathrm{Gr}_p(S_{p+q}(X)) = \frac{S_{p+q}(X^{(p)})}{S_{p+q}(X^{(p-1)})} = S_{p+q}(X^{(p)}, X^{(p-1)}).$$

Therefore the E^1-page computes the relative homology

$$E^1_{p,q} = \mathrm{H}_{p+q}(X^{(p)}, X^{(p-1)}) = \begin{cases} C^{cell}_p(X) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

which gives precisely the cellular chains.

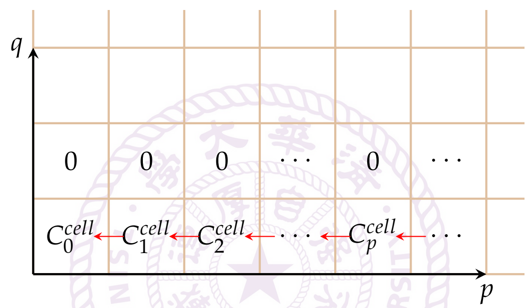


FIGURE 33. E^1-page

By chasing the definition, we find that the differential ∂_1 coincides with the cellular differential

$$\partial : C^{cell}_p(X) \to C^{cell}_{p-1}(X).$$

Therefore the E^2-page is

$$E^2_{p,q} = \begin{cases} \mathrm{H}^{cell}_p(X) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

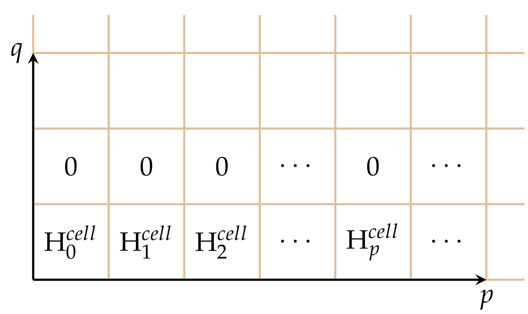


FIGURE 34. E^2-page

The shape of this E^2-page implies that

$$\partial_2 = \partial_3 = \cdots = 0, \quad \Longrightarrow E^2 = E^3 = \cdots = E^\infty.$$

This explains why the cellular homology computes the singular homology.

Leray-Serre spectral sequence

Let $\pi : E \to B$ be a Serre fibration with fiber F and base B.

$$\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B \end{tikzcd}$$

Assume B is a simply-connected CW complex. Then there is the Leray-Serre spectral sequence with E^2-page

$$E^2_{p,q} = \mathrm{H}_p(B) \otimes \mathrm{H}_q(F)$$

that converges to $\mathrm{Gr}_p \mathrm{H}_{p+q}(E)$.

The idea of this spectral sequence is that we can filter the singular chain complex of E such that it favors for the computation of singular homology along the fiber first. Explicitly, we can use the cellular structure

$$B^{(0)} \subset B^{(1)} \subset \cdots \subset B^{(n)} \subset \cdots$$

to obtain a filtration of topological spaces for E

$$E^{(0)} \subset E^{(1)} \subset \cdots \subset E^{(n)} \subset \cdots$$

where $E^{(n)}$ is given the pull-back

$$\begin{tikzcd} E^{(n)} \arrow[r] \arrow[d] & E \arrow[d] \\ B^{(n)} \arrow[r] & B \end{tikzcd}$$

We will not give the details here, but instead illustrate its use by some examples.

Example 21.9. Consider the fibration ($n \geq 2$)

$$\begin{tikzcd} \Omega S^n \arrow[r] & P\Omega^n \arrow[d] \\ & S^n \end{tikzcd}$$

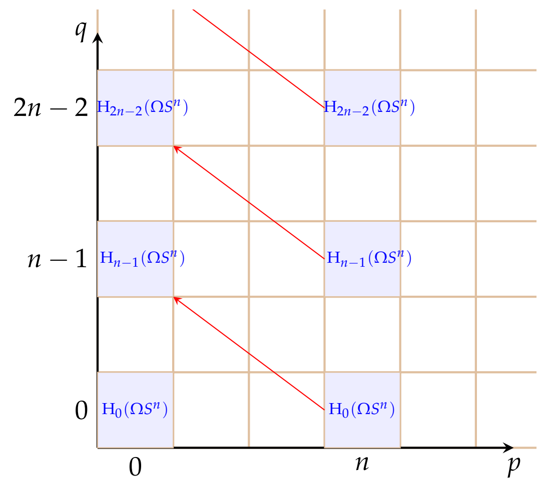
Here $P\Omega^n$ is the based path space of S^n. We have

$$\mathrm{H}_p(S^n) = \begin{cases} \mathbb{Z} & p = 0, n \\ 0 & p \neq 0, n \end{cases} \qquad \mathrm{H}_k(P\Omega^n) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k > 0 \end{cases}$$

To arrive at $\mathrm{H}_\bullet(P\Omega^n)$, the Leray-Serre spectral sequence must be of the form

$$E^2 = E^3 = \cdots = E^n$$

where the only non-zero terms are in the shaded locations as in the figure below.



Furthermore, the maps

$$d_n : \mathrm{H}_{(n-1)k}(\Omega S^n) \to \mathrm{H}_{(n-1)(k+1)}(\Omega S^n), \quad k \geq 0$$

must be isomorphisms in order to have $E^\infty = \operatorname{Gr} \mathrm{H}_\bullet(P\Omega^n) = \mathbb{Z}$. We conclude that

$$\mathrm{H}_i(\Omega S^n) = \begin{cases} \mathbb{Z} & i = k(n-1) \\ 0 & \text{otherwise} \end{cases}$$

Example 21.10. We illustrate Serre's approach to Hurewicz Theorem via spectral sequence.

Assume we have established Hurewicz Theorem for the $n = 1$ case $\pi_1 \to \mathrm{H}_1$. We prove by induction for the $n \geq 2$ case. Let $n \geq 2$ and X be a $(n-1)$-connected CW complex. Consider the fibration

$$\begin{tikzcd} \Omega X \arrow[r] & PX \arrow[d] \\ & X \end{tikzcd}$$

The E^2-page of the Leray-Serre spectral sequence is

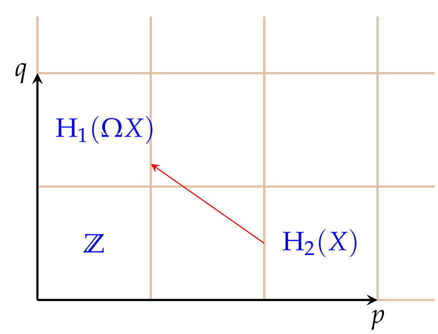


FIGURE 35. E^2-page

Since PX is contractible, the map

$$H_2(X) \to H_1(\Omega_X)$$

must be an isomorphism. This shows

$$H_2(X) = H_1(\Omega_X) = \pi_1(\Omega_X) = \pi_2(X) \quad (= 0 \text{ if } n > 2).$$

We can iterate this until we arrive at

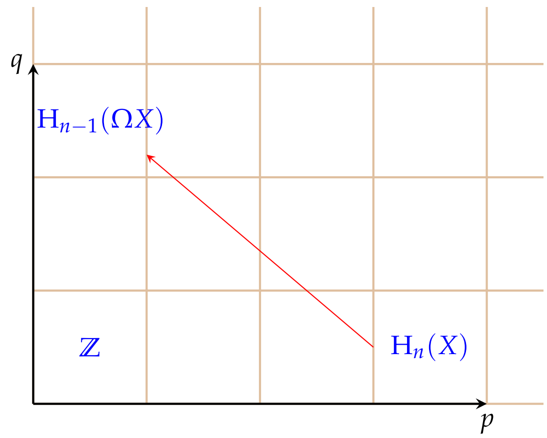


FIGURE 36. E^n-page

Again by the contractibility of PX, ∂_r must induce an isomorphism

$$H_n(X) = H_{n-1}(\Omega X) \overset{\text{induction}}{=} \pi_{n-1}(\Omega X) = \pi_n(X).$$

This is the Hurewicz isomorphism.

22 EILENBERG-ZILBER THEOREM AND KÜNNETH FORMULA

Eilenberg-Zilber Theorem

Definition 22.1. Let $(C_\bullet, \partial_C)$ and $(D_\bullet, \partial_D)$ be two chain complexes. We define their tensor product $C_\bullet \otimes D_\bullet$ to be the chain complex

$$(C_\bullet \otimes D_\bullet)_k := \sum_{p+q=k} C_p \otimes D_q$$

with the boundary map $\partial = \partial_{C \otimes D}$ given by

$$\partial(c_p \otimes d_q) := \partial_C(c_p) \otimes d_q + (-1)^p c_p \otimes \partial_D(d_q), \quad c_p \in C_p, d_q \in D_q.$$

This sign convention guarantees that

$$\partial^2 = 0.$$

Proposition 22.2. *Assume $C_\bullet$ is chain homotopy equivalent to $C'_\bullet$. Then $C_\bullet \otimes D_\bullet$ is chain homotopy equivalent to $C'_\bullet \otimes D_\bullet$.*

Proof. Assume $C_\bullet \underset{g}{\overset{f}{\rightleftarrows}} C'_\bullet$ define chain homotopy equivalence such that

$$1_{C'} - f \circ g = \partial_{C'} \circ s' + s' \circ \partial_{C'}$$
$$1_C - g \circ f = \partial_C \circ s + s \circ \partial_C$$

where

$$s : C_\bullet \to C_{\bullet+1}, \quad s' : C'_\bullet \to C'_{\bullet+1}.$$

Then our sign convention implies

$$1_{C' \otimes D} - (f \otimes 1_D) \circ (g \otimes 1_D) = \partial_{C' \otimes D} \circ (s' \otimes 1_D) + (s' \otimes 1_D) \circ \partial_{C' \otimes D}$$
$$1_{C \otimes D} - (g \otimes 1_D) \circ (f \otimes 1_D) = \partial_{C' \otimes D} \circ (s \otimes 1_D) + (s \otimes 1_D) \circ \partial_{C' \otimes D}$$

leaing to chain homotopy equivalence

$$C_\bullet \otimes D_\bullet \underset{g \otimes 1_D}{\overset{f \otimes 1_D}{\rightleftarrows}} C'_\bullet \otimes D_\bullet \ .$$

□

We would like to compare the following two functors

$$S_\bullet(- \times -), S_\bullet(-) \otimes S_\bullet(-) : \underline{\mathbf{Top}} \times \underline{\mathbf{Top}} \to \underline{\mathbf{Ch_\bullet}}$$

which send

$$X \times Y \to S_\bullet(X \times Y) \quad \text{and} \quad S_\bullet(X) \otimes S_\bullet(Y).$$

We first observe that there exists a canonical isomorphism

$$\mathrm{H}_0(X \times Y) \cong \mathrm{H}_0(X) \otimes \mathrm{H}_0(Y).$$

The following theorem of Eilenberg-Zilber says that such initial condition determines a natural homotopy equivalent between the above two functors which is unique up to chain homotopy.

Theorem 22.3 (Eilenberg-Zilber). *Then there exist natural transformations*

$$S_\bullet(-\times -) \underset{G}{\overset{F}{\rightleftarrows}} S_\bullet(-)\otimes S_\bullet(-)$$

which induce chain homotopy equivalence

$$S_\bullet(X\times Y) \underset{G}{\overset{F}{\rightleftarrows}} S_\bullet(X)\otimes S_\bullet(Y)$$

for every X, Y and the canonical isomorphism $\mathrm{H}_0(X\times Y)\cong \mathrm{H}_0(X)\otimes \mathrm{H}_0(Y)$. Such chain equivalence is unique up to chain homotopy. In particular, there are canonical isomorphisms

$$\mathrm{H}_n(X\times Y)=\mathrm{H}_n(S_\bullet(X)\otimes S_\bullet(Y)),\quad \forall n\geq 0.$$

*F, G will be called **Eilenberg-Zilber maps**.*

Proof. Observe that any map $\Delta^p \overset{(\sigma_x,\sigma_y)}{\to} X\times Y$ factors through

$$\Delta^p \overset{\delta_p}{\to} \Delta^p\times\Delta^p \overset{\sigma_x\times\sigma_y}{\to} X\times Y$$

where $\Delta^p \overset{\delta_p}{\to} \Delta^p\times\Delta^p$ is the diagonal map. This implies that a natural transformation F of the functor $S_\bullet(-\times -)$ is determined by its value on $\{\delta_p\}_{p\geq 0}$. Explicitly

$$F((\sigma_x,\sigma_y))=(\sigma_x\otimes\sigma_y)_*F(\delta_p).$$

Similarly, a natural transformation G of the functor $S_\bullet(-)\otimes S_\bullet(-)$ is determined by its value on $1_p\otimes 1_q$ where $1_p:\Delta^p\to\Delta^p$ is the identity map. Explicitly, for any $\sigma_x:\Delta^p\to X, \sigma_y:\Delta^q\to Y$, we have

$$G(\sigma_x\otimes\sigma_y)=(\sigma_x\times\sigma_y)_*G(1_p\otimes 1_q).$$

Therefore F and G are completely determined by

$$f_n:=F(\delta_n)\in\bigoplus_{p+q=n} S_p(\Delta^n)\otimes S_q(\Delta^n),\quad g_n:=\bigoplus_{p+q=n} G(1_p\otimes 1_q)\in\bigoplus_{p+q=n} S_n(\Delta^p\times\Delta^q).$$

We will use the same notations as in Definition 17.4. Then

$$f_n\circ g_n\in S_n(\Delta^n\times\Delta^n),\quad g_n\circ f_n\in\bigoplus_{p+q=n}(S_\bullet(\Delta^p)\otimes S_\bullet(\Delta^q))_n.$$

Let us denote the following chain complexes

$$C_n=\prod_{k\geq 0}(S_\bullet(\Delta^k)\otimes S_\bullet(\Delta^k))_{n+k},\quad D_n=\prod_{m\geq 0}\left(\bigoplus_{p+q=m} S_{n+p+q}(\Delta^p\times\Delta^q)\right)$$

with boundary map

$$\partial+\tilde\partial: C_n\to C_{n-1},\quad \partial+\tilde\partial: D_n\to D_{n-1}$$

as follows. ∂ is the usual boundary map of singular chain complexes

$$\partial:(S_\bullet(\Delta^k)\otimes S_\bullet(\Delta^k))_n\to(S_\bullet(\Delta^k)\otimes S_\bullet(\Delta^k))_{n-1},\quad \partial:\quad S_n(\Delta^p\times\Delta^q)\to S_{n-1}(\Delta^p\times\Delta^q).$$

$\tilde\partial$ is the map induced by composing with the face singular chain $\tilde\partial=\sum_k\partial\Delta^k\in\prod_k S_{k-1}(\Delta^k)$

$$\tilde\partial: S_p(\Delta^{k-1})\otimes S_q(\Delta^{k-1})\to S_p(\Delta^k)\otimes S_q(\Delta^k),\quad \sigma_p\otimes\sigma_q\to\tilde\partial\circ\sigma_p\otimes\tilde\partial\circ\sigma_q$$

and

$$\tilde\partial: S_n(\Delta^p\times\Delta^q)\to S_n(\Delta^{p+1}\times\Delta^q)\oplus S_n(\Delta^p\times\Delta^{q+1}),\quad \sigma_p\times\sigma_q\to(\tilde\partial\circ\sigma_p)\times\sigma_q+(-1)^{n-p}\sigma_p\times(\tilde\partial\circ\sigma_q).$$

Let $f = (f_n) \in C_0$ and $g = (g_n) \in D_0$. Then it can be checked that

$$F, G \text{ are chain maps} \iff f, g \text{ are 0-cycles in } C_\bullet, D_\bullet$$

and natural chain homotopy of F, G are given by 0-boundaries. We claim that

$$\mathrm{H}_n(C_\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}, \quad \mathrm{H}_n(D_\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}.$$

We sketch a proof here. In fact, there exists a spectral sequence with

$$E_1\text{-page} : \mathrm{H}(-, \partial)$$
$$E_2\text{-page} : \mathrm{H}(\mathrm{H}(-, \partial), \tilde{\partial})$$

and converging to $\partial + \tilde{\partial}$-homology. We need to use a stronger version of convergence than Theorem 21.8, which is guaranteed by the choice of direct product (so formal series is convergent) instead of direct sum in the definition of C_n and D_n. We leave this delicate issue to the reader.

For $C_\bullet$, the E_1-page $\mathrm{H}_\bullet(C_\bullet, \partial)$ is (using Proposition 22.2)

$$\mathrm{H}_n(C_\bullet, \partial) = \prod_{k \geq 0} \mathrm{H}_n(S_\bullet(\Delta^k) \otimes S_\bullet(\Delta^k)) = \begin{cases} \prod_{k \geq 0} \mathbb{Z} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

It is not hard to see that $\tilde{\partial}$ acts on this E_1-page as

$$\tilde{\partial} : \prod_{k \geq 0} \mathbb{Z} \to \prod_{k \geq 0} \mathbb{Z} \qquad (n_k)_{k \geq 0} \to (m_k)_{k \geq 0}$$
$$\text{where} \quad m_k = \frac{1}{2}(1 + (-1)^k) n_{k-1}.$$

In components, this can be represented by

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \cdots$$

whose $\tilde{\partial}$-homology is now $\mathbb{Z}$ concentrated at degree 0. It follows that $E_2 = E_3 = \cdots = E_\infty$ and therefore

$$\mathrm{H}_n(C_\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

The computation in the case of $D_\bullet$ is similar. This implies that the initial condition completely determines chain maps F, G up to chain homotopy.

Let us now analyze the composition $F \circ G$ and $G \circ F$. We similarly form the chain complexes

$$C'_n = \prod_{k \geq 0} S_{n+k}(\Delta^k \times \Delta^k), \quad D'_n := \prod_{m \geq 0} \bigoplus_{p+q=m} (S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q))_{n+p+q}$$

with boundary map $\partial + \tilde{\partial}$ defined similarly. Homology of $C'_\bullet$ controls natural chain maps of $S_\bullet(X \times Y)$ to itself up to chain homotopy, and similarly for $D'_\bullet$. We still have

$$\mathrm{H}_n(C'_\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}, \quad \mathrm{H}_n(D'_\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}.$$

It follows that $F \circ G$ and $G \circ F$ are both naturally chain homotopic to the identity map. The theorem follows. □

An explicit construction of G can be described as follows: given $\sigma_p : \Delta^p \to X, \sigma_q : \Delta^q \to Y$,

$$G(\sigma_p \otimes \sigma_q) : \Delta^p \times \Delta^q \to X \times Y$$

where we have to chop $\Delta^p \times \Delta^q$ into $p+q$-simplexes (similar to the decomposition of $\Delta^n \times \Delta^1 = \Delta^n \times I$ as in Proposition 15.16). This is the **shuffle product**.

An explicit construction of F can be given by the **Alexander-Whitney map** described as follows.

Definition 22.4. Given a singular n-simplex $\sigma : \Delta^n \to X$ and $0 \le p, q \le n$, we define

- the **front p-face** of σ to be the singular p-simplex
$$_p\sigma : \Delta^p \to X, \quad {}_p\sigma(t_0, \cdots, t_p) := \sigma(t_0, \cdots, t_p, 0, \cdots, 0)$$
- the **back q-face** of σ to be the singular q-simplex
$$\sigma_q : \Delta^q \to X, \quad \sigma_q(t_0, \cdots, t_q) := \sigma(0, \cdots, 0, t_0, \cdots, t_q).$$

Definition 22.5. Let X, Y be topological spaces. Let $\pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y$ be the projections. We define the **Alexander-Whitney map**

$$AW : S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$$

by the natural transformation given by the formula

$$AW(\sigma) := \sum_{p+q=n} {}_p(\pi_X \circ \sigma) \otimes (\pi_Y \circ \sigma)_q.$$

Theorem 22.6. *The Alexander-Whitney map is a chain homotopy equivalence.*

Proof. It can be checked that AW is a natural chain map which induces the canonical isomorphism

$$\mathrm{H}_0(X \times Y) \to \mathrm{H}_0(X) \otimes \mathrm{H}_0(Y).$$

Therefore AW is a chain homotopy equivalence by Eilenberg-Zilber Theorem. □

Künneth formula

Lemma 22.7. *Let $C_\bullet$ and $D_\bullet$ be chain complex of free abelian groups. Then*

$$\mathrm{H}_\bullet(C_\bullet \otimes D_\bullet) = \mathrm{H}_\bullet(C_\bullet \otimes \mathrm{H}_\bullet(D)).$$

Here $\mathrm{H}_\bullet(D)$ is viewed as a chain complex whose boundary map is zero.

Proof. Consider the exact sequence

$$0 \to Z_n \to D_n \to B_{n-1} \to 0$$

where Z_n are cycles in D_n and B_{n-1} are boundaries in D_{n-1}. Since $D_\bullet$'s are free, Z_n and B_{n-1} are also free abelian groups. Let $Z_\bullet, B_\bullet$ be chain complexes with zero boundary map. Then we have an exact sequence of chain complexes

$$0 \to Z_\bullet \to D_\bullet \to B_{\bullet-1} \to 0$$

whose associated long exact sequence of homologies splits into

$$0 \to B_n \to Z_n \to \mathrm{H}_n(D_\bullet) \to 0.$$

Tensoring with $C_\bullet$, we find an exact sequence

$$0 \to C_\bullet \otimes Z_\bullet \to C_\bullet \otimes D_\bullet \to C_\bullet \otimes B_{\bullet-1} \to 0$$

which induces a long exact sequence

$$\cdots \to \mathrm{H}_{n+1}(C_\bullet \otimes D_\bullet) \to \mathrm{H}_n(C_\bullet \otimes B_\bullet) \to \mathrm{H}_n(C_\bullet \otimes Z_\bullet) \to \mathrm{H}_n(C_\bullet \otimes D_\bullet) \to \mathrm{H}_{n-1}(C_\bullet \otimes B_\bullet) \to \cdots$$

Since $B_\bullet \to Z_\bullet$ are embeddings of free abelian groups, the maps

$$\mathrm{H}_n(C_\bullet \otimes B_\bullet) = \bigoplus_{p+q=n} \mathrm{H}_p(C_\bullet) \otimes B_q \to \mathrm{H}_n(C_\bullet \otimes Z_\bullet) = \bigoplus_{p+q=n} \mathrm{H}_p(C_\bullet) \otimes Z_q$$

are also injective. □

Theorem 22.8 (Algebraic Künneth formula). *Let $C_\bullet$ and $D_\bullet$ be chain complex of free abelian groups. Then there is a split exact sequence*

$$0 \to (\mathrm{H}_\bullet(C) \otimes \mathrm{H}_\bullet(D))_n \to \mathrm{H}_n(C_\bullet \otimes D_\bullet) \to \mathrm{Tor}(\mathrm{H}_\bullet(C), \mathrm{H}_\bullet(D))_{n-1} \to 0.$$

Here $\mathrm{Tor}(\mathrm{H}_\bullet(C), \mathrm{H}_\bullet(D))_k = \bigoplus_{p+q=k} Tor(\mathrm{H}_p(C), \mathrm{H}_q(D))$.

Proof. Consider the exact sequence

$$0 \to Z_n \to D_n \to B_{n-1} \to 0$$

where Z_n are cycles in D_n and B_{n-1} are boundaries in D_{n-1}. Since $D_\bullet$'s are free, Z_n and B_{n-1} are also free abelian groups. Let $Z_\bullet, B_\bullet$ be chain complexes with zero boundary map. Then we have an exact sequence of chain complexes

$$0 \to Z_\bullet \to D_\bullet \to B_{\bullet-1} \to 0$$

Tensoring with $C_\bullet$ and since $C_\bullet$'s are free, we find an exact sequence of chain complexes

$$0 \to C_\bullet \otimes Z_\bullet \to C_\bullet \otimes D_\bullet \to C_\bullet \otimes B_{\bullet-1} \to 0$$

which induces a long exact sequence

$$\cdots \to \mathrm{H}_{n+1}(C_\bullet \otimes D_\bullet) \to \mathrm{H}_n(C_\bullet \otimes B_\bullet) \to \mathrm{H}_n(C_\bullet \otimes Z_\bullet) \to \mathrm{H}_n(C_\bullet \otimes D_\bullet) \to \mathrm{H}_{n-1}(C_\bullet \otimes B_\bullet) \to \cdots$$

On the other hand, we have

$$0 \to B_q \to Z_q \to \mathrm{H}_q(D_\bullet) \to 0.$$

Tensoring with $\mathrm{H}_p(C_\bullet)$, we find

$$0 \to \mathrm{Tor}(\mathrm{H}_p(C_\bullet), \mathrm{H}_q(D_\bullet)) \to \mathrm{H}_p(C_\bullet) \otimes B_q \to \mathrm{H}_p(C_\bullet) \otimes Z_q \to \mathrm{H}_p(C_\bullet) \otimes \mathrm{H}_q(D_\bullet) \to 0.$$

Since B_q, Z_q's are free,

$$\mathrm{H}_p(C_\bullet) \otimes B_q = \mathrm{H}_p(C_\bullet \otimes B_q), \quad \mathrm{H}_p(C_\bullet) \otimes Z_q = \mathrm{H}_p(C_\bullet \otimes Z_q).$$

Summing over p, q, we find

$$0 \to \mathrm{Tor}(\mathrm{H}_\bullet(C), \mathrm{H}_\bullet(D))_n \to \mathrm{H}_n(C_\bullet \otimes B_\bullet) \to \mathrm{H}_n(C_\bullet \otimes Z_\bullet) \to (\mathrm{H}_\bullet(C) \otimes \mathrm{H}_\bullet(D))_n \to 0.$$

Combining with the above long exact sequence, we arrive at the required short exact sequence

$$0 \to (\mathrm{H}_\bullet(C) \otimes \mathrm{H}_\bullet(D))_n \to \mathrm{H}_n(C_\bullet \otimes D_\bullet) \to \mathrm{Tor}(\mathrm{H}_\bullet(C), \mathrm{H}_\bullet(D))_{n-1} \to 0.$$

□

Theorem 22.9 (Künneth formula). *For any topological spaces X, Y and $n \geq 0$, there is a split exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(X) \to \mathrm{H}_n(X \times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X), \mathrm{H}_q(Y)) \to 0.$$

Proof. This follows from the Eilenberg-Zilber Theorem and the algebraic Künneth formula.

□

23 CUP AND CAP PRODUCT

One of the key structure that distinguishes cohomology with homology is that cohomology carries an algebraic structure so $\mathrm{H}^\bullet(X)$ becomes a ring. This algebraic structure is called cup product. Moreover, $\mathrm{H}_\bullet(X)$ will be a module of $\mathrm{H}^\bullet(X)$, and this module structure is called cap product.

Let R be a commutative ring with unit. We have natural cochain maps

$$S^\bullet(X;R)\otimes_R S^\bullet(Y;R)\to \mathrm{Hom}(S_\bullet(X)\otimes S_\bullet(Y),R)\to S^\bullet(X\times Y;R)$$

where the first map sends $\varphi_p\in S^p(X;R),\eta_q\in S^q(X;R)$ to $\varphi_p\otimes\eta_q$ where

$$\varphi_p\otimes\eta_q:\sigma_p\otimes\sigma_q\to\varphi_p(\sigma_p)\cdot\eta_q(\sigma_q),\quad \sigma_p\in S_p(X),\quad \sigma_q\in S_q(X).$$

Here $\cdot$ is the product in R. The second map is dual (applying $\mathrm{Hom}(-,R)$) to the Alexander-Whitney map

$$AW:S_\bullet(X\times Y)\to S_\bullet(X)\otimes S_\bullet(Y).$$

This leads to a cochain map

$$S^\bullet(X;R)\otimes_R S^\bullet(Y;R)\to S^\bullet(X\times Y;R)$$

which further induces

$$\mathrm{H}^\bullet(X;R)\otimes_R \mathrm{H}^\bullet(Y;R)\to \mathrm{H}^\bullet(X\times Y;R).$$

Cup product

Definition 23.1. Let R be a commutative ring with unit. We define the **cup product** on cohomology groups

$$\cup:\mathrm{H}^p(X;R)\otimes_R \mathrm{H}^q(X;R)\to \mathrm{H}^{p+q}(X;R)$$

by the composition

$$\begin{tikzcd}
\mathrm{H}^\bullet(X;R)\otimes_R \mathrm{H}^\bullet(X;R) \arrow[r] \arrow[dr, "\cup"'] & \mathrm{H}^\bullet(X\times X;R) \arrow[d, "\Delta^*"] \\
 & \mathrm{H}^\bullet(X;R)
\end{tikzcd}$$

Here $\Delta:X\to X\times X$ is the diagonal map.

Alexander-Whitney map gives an explicit product formula

$$(\alpha\cup\beta)(\sigma)=\alpha({}_p\sigma)\cdot\beta(\sigma_q),\quad \alpha\in S^p(X;R),\beta\in S^q(X;R),\sigma:\Delta^{p+q}\to X.$$

Theorem 23.2. $\mathrm{H}^\bullet(X;R)$ *is a graded commutative ring with uint:*

1°. ***Unit***: *let* $1\in\mathrm{H}^0(X;R)$ *be represented by the cocyle which takes every singular* 0*-simplex to* $1\in R$*. Then*

$$1\cup\alpha=\alpha\cup 1=\alpha,\quad \forall\alpha\in\mathrm{H}^\bullet(X;R).$$

2°. ***Associativity:***

$$(\alpha\cup\beta)\cup\gamma=\alpha\cup(\beta\cup\gamma).$$

3°. ***Graded commutativity:***

$$\alpha\cup\beta=(-1)^{pq}\beta\cup\alpha,\quad \forall\alpha\in\mathrm{H}^p(X;R),\beta\in\mathrm{H}^q(X;R).$$

Proof. Unit of 1 is checked easily. Observe that the following two compositions of Eilenberg-Zilber maps are chain homotopic (similar to Eilenberg-Zilber Theorem)

$$S_\bullet(X \times Y \times Z) \to S_\bullet(X \times Y) \otimes S_\bullet(Z) \to S_\bullet(X) \otimes S_\bullet(Y) \otimes S_\bullet(Z)$$
$$S_\bullet(X \times Y \times Z) \to S_\bullet(X) \otimes S_\bullet(Y \times Z) \to S_\bullet(X) \otimes S_\bullet(Y) \otimes S_\bullet(Z).$$

Associativity follows from the commutative diagram (R is hidden for simplicity)

$$\begin{tikzcd}
\mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(X) \arrow[r] \arrow[d] & \mathrm{H}^\bullet(X \times X) \otimes \mathrm{H}^\bullet(X) \arrow[r, "(\Delta\times 1)^*"] \arrow[d] & \mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(X) \arrow[d] \\
\mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(X \times X) \arrow[r] \arrow[d, "(1\times\Delta)^*"] & \mathrm{H}^\bullet(X \times X \times X) \arrow[r, "(\Delta\times 1)^*"] \arrow[d, "(1\times\Delta)^*"'] & \mathrm{H}^\bullet(X \times X) \arrow[d, "\Delta^*"] \\
\mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(X) \arrow[r] & \mathrm{H}^\bullet(X \times X) \arrow[r, "\Delta^*"] & \mathrm{H}^\bullet(X)
\end{tikzcd}$$

Graded commutativity follows from the fact that the interchange map of tensor product of chain complexes

$$T : C_\bullet \otimes D_\bullet \to D_\bullet \otimes C_\bullet$$
$$c_p \otimes d_q \to (-1)^{pq} d_q \otimes c_p$$

is a chain isomorphism. Therefore the two chain maps

$$S_\bullet(X \times Y) \to S_\bullet(Y \times X) \to S_\bullet(Y) \otimes S_\bullet(X)$$
$$S_\bullet(X \times Y) \to S_\bullet(X) \times S_\bullet(Y) \xrightarrow{T} S_\bullet(Y) \otimes S_\bullet(X)$$

are chain homotopic, again by the uniqueness in Eilenberg-Zilber Theorem.

Set $Y = X$ we find the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(X) \arrow[r] \arrow[d, "T"] & \mathrm{H}^\bullet(X \times X) \arrow[d, "="] \\
\mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(X) \arrow[r] & \mathrm{H}^\bullet(X \times X).
\end{tikzcd}$$

which gives graded commutativity.

Alternately, all the above can be checked explicitly using Alexander-Whitney map □

Theorem 23.3. *Let $f : X \to Y$ be a continuous map. Then*

$$f^* : \mathrm{H}^\bullet(Y; R) \to \mathrm{H}^\bullet(X; R)$$

is a morphism of graded commutative rings, i.e. $f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$. In other words, $\mathrm{H}^\bullet(-)$ defines a functor from the category of topological spaces to the category of graded commutative rings.

Proof. The theorem follows from the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Delta"] & Y \arrow[d, "\Delta"] \\
X \times X \arrow[r, "f \times f"] & Y \times Y.
\end{tikzcd}$$

□

Theorem 23.4 (Künneth formula)**.** *Assumem R is a PID, and $\mathrm{H}_i(X;R)$ are finitely generated R-module, then there exists a split exact sequence of R-modules*

$$0 \to \bigoplus_{p+q=n} \mathrm{H}^p(X;R) \otimes \mathrm{H}^q(Y;R) \to \mathrm{H}^n(X \times Y;R) \to \bigoplus_{p+q=n+1} \mathrm{Tor}_1^R(\mathrm{H}^p(X;R), \mathrm{H}^q(Y;R)) \to 0.$$

In particular, if $\mathrm{H}^\bullet(X;R)$ or $\mathrm{H}^\bullet(Y;R)$ are free R-modules, we have an isomorphism of graded commutative rings

$$\mathrm{H}^\bullet(X \times Y;R) \cong \mathrm{H}^\bullet(X;R) \otimes_R \mathrm{H}^\bullet(Y;R).$$

Example 23.5. $\mathrm{H}^\bullet(S^n) = \mathbb{Z}[\eta]/\eta^2$ where $\eta \in \mathrm{H}^n(S^n)$ is a generator.

Example 23.6. Let $T^n = S^1 \times \cdots \times S^1$ be the n-torus. Then

$$\mathrm{H}^\bullet(T^n) \cong \mathbb{Z}[\eta_1, \cdots, \eta_n], \quad \eta_i\eta_j = -\eta_j\eta_i$$

is the exterior algebra with n generators. Each η_i corresponds a generator of $\mathrm{H}^1(S^1)$.

Proposition 23.7. *$\mathrm{H}^\bullet(\mathbb{CP}^n) = \mathbb{Z}[x]/x^{n+1}$, where $x \in \mathrm{H}^2(\mathbb{CP}^n)$ is a generator.*

Proof. We prove by induction n. We know that

$$\mathrm{H}^k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k = 2m \leq 2n \\ 0 & \text{otherwise} \end{cases}$$

Let x be a generator of $\mathrm{H}^2(\mathbb{CP}^n)$. We only need to show that x^k is a generator of $\mathrm{H}^{2k}(\mathbb{CP}^n)$ for each $k \leq n$. Using cellular chain complex, we know that for $k < n$

$$\mathrm{H}^{2k}(\mathbb{CP}^n) \to \mathrm{H}^{2k}(\mathbb{CP}^k)$$

is an isomorphism. By induction, this implies that x^k is a generator of $\mathrm{H}^{2k}(\mathbb{CP}^n)$ for $k < n$. Poincare duality theorem (which will be proved in the next section) implies that

$$\mathrm{H}^2(\mathbb{CP}^n) \otimes \mathrm{H}^{2n-2}(\mathbb{CP}^n) \xrightarrow{\cup} \mathrm{H}^{2n}(\mathbb{CP}^n)$$

is an isomorphism. This says that x^n is a generator of $\mathrm{H}^{2n}(\mathbb{CP}^n)$. This proves the proposition. □

Cap product

Definition 23.8. We define the evaluation map

$$\langle -, - \rangle : S^\bullet(X;R) \times_R S_\bullet(X;R) \to R$$

as follows: for $\alpha \in S^p(X;R), \sigma \in S_p(X), r \in R$,

$$\langle \alpha, \sigma \otimes r \rangle := \alpha(\sigma) \cdot r.$$

The evaluation map is compatible with boundary map and induces an evaluation map

$$\langle -, - \rangle : \mathrm{H}^p(X;R) \otimes_R \mathrm{H}_p(X;R) \to R.$$

This generalized to

$$S^\bullet(X;R) \otimes_R S_\bullet(X \times Y;R) \to S^\bullet(X;R) \otimes_R S_\bullet(X;R) \otimes_R S_\bullet(Y;R) \xrightarrow{\langle -,- \rangle \otimes 1} S_\bullet(Y;R)$$

which induces

$$\mathrm{H}^p(X;R) \otimes_R \mathrm{H}_{p+q}(X \times Y;R) \to \mathrm{H}_q(Y;R).$$

Definition 23.9. We define the **cap product**

$$\cap : \mathrm{H}^p(X;R) \otimes \mathrm{H}_{p+q}(X;R) \to \mathrm{H}_q(X;R)$$

by the composition

$$\begin{tikzcd}
\mathrm{H}^p(X;R) \otimes \mathrm{H}_{p+q}(X;R) \arrow[r, "1\otimes\Delta"] \arrow[dr, "\cap"] & \mathrm{H}^p(X;R) \otimes \mathrm{H}_{p+q}(X \times X;R) \arrow[d] \\
 & \mathrm{H}_q(X;R)
\end{tikzcd}$$

Theorem 23.10. *The cap product gives $\mathrm{H}_\bullet(X;R)$ a structure of $\mathrm{H}^\bullet(X;R)$-module.*

Theorem 23.11. *The cap product extends naturally to the relative case: for any pair $A \subset X$*

$$\cap : \mathrm{H}^p(X,A) \otimes \mathrm{H}_{p+q}(X,A) \to \mathrm{H}_q(X)$$
$$\cap : \mathrm{H}^p(X) \otimes \mathrm{H}_{p+q}(X,A) \to \mathrm{H}_q(X,A)$$

Proof. Since $S^\bullet(X,A) \subset S^\bullet(X)$, we have

$$\cap : S^\bullet(X,A) \times S_\bullet(X) \to S_\bullet(X).$$

We model the cap product on chains via the Alexander-Whitney map. Then

$$\cap : S^\bullet(X,A) \times S_\bullet(A) \to 0.$$

Therefore $\cap$ factors through

$$\cap : S^\bullet(X,A) \times \frac{S_\bullet(X)}{S_\bullet(A)} \to S_\bullet(X).$$

Passing to homology (cohomology) we find the first cap product. The second one is proved similarly using

$$\cap : S^\bullet(X) \times \frac{S_\bullet(X)}{S_\bullet(A)} \to \frac{S_\bullet(X)}{S_\bullet(A)}.$$

□

24 POINCARÉ DUALITY

Definition 24.1. A **topological manifold** of dimension n , or a topological n-manifold, is a Hausdorff space in which each point has an open neighborhood homeomorphic to $\mathbb{R}^n$.

In this section, a manifold always means a topological manifold. We assume $n > 0$. For any point $x \in X$, there exists an open neighborhood U and a homeomorphism $\phi : U \to \mathbb{R}^n$. (U, ϕ) is called a **chart** around x.

Orientation

Definition 24.2. Let X be a n-manifold. $x \in X$ be a point. A generator of

$$H_n(X, X - x) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong \mathbb{Z}$$

is called a **local orientation** of X at x.

For any $x \in X$, there are two choices of local orientation at x. We obtain a two-sheet cover

$$\pi : \tilde{X} \to X, \quad \text{where} \quad \tilde{X} = \{(x, \mu_x) | \mu_x \text{ is a local orientation of } X \text{ at } x\}.$$

Here π is the natural projection $(x, \mu_x) \to x$. $\tilde{X}$ is topologized as follows. Let U be a small open ball in X. Then for any $x \in U$, we have an isomorphism

$$H_n(X, X - U) \cong H_n(X, X - x)$$

which induces a set theoretical identification

$$\pi^{-1}(U) \cong U \times \mathbb{Z}_2.$$

Then we give a topology on $\tilde{X}$ by requiring all such identifications being homeomorphisms. In particular, $\pi : \tilde{X} \to X$ is a $\mathbb{Z}_2$-covering map.

Definition 24.3. A (global) **orientation** of X is a section of $\pi : \tilde{X} \to X$, i.e., a continuous map $s : X \to \tilde{X}$ such that $\pi \circ s = 1_X$. If an orientation exists, we say X is **orientable**.

Theorem 24.4. *Let X be a connected manifold. Then X is orientable if and only if $\tilde{X}$ has two connected components. In particular, a connected orientable manifold has precisely two orientations.*

Proof. Assume X is orientable. Let $s_1 : X \to \tilde{X}$ be a section defining an orientation. Since $\pi : \tilde{X} \to X$ is a double cover, we can define another section $s_2 : X \to \tilde{X}$ such that $\{s_1(x), s_2(x)\} = \pi^{-1}(x)$ for any $x \in X$. The covering property implies that s_2 is also continuous, hence defining another orientation. The sections s_1, s_2 lead to a diffeomorphism

$$\tilde{X} = X \times \mathbb{Z}_2 = X \coprod X.$$

Conversely, if $\tilde{X}$ has two connected component, then each one is diffeomorphic to X under the projection π and so defines an orientation. □

Example 24.5. A simply connected manifold is orientable. This is because the covering of a simply connected space must be a trivial covering.

Proposition 24.6. *Let X be connected non-orientable manifold. Then $\tilde{X}$ is connected orientable.*

Proof. X is non-orientable implies that $\tilde{X}$ has only one connnected component. Since $\pi : \tilde{X} \to X$ is a covering map, it is a local diffeomorphism and induces an isomorphism

$$H_n(\tilde{X}, \tilde{X} - \tilde{x}) = H_n(X, X - x), \quad x = \pi(\tilde{x}).$$

In particular, we have a canonical section

$$s : \tilde{X} \to \tilde{\tilde{X}}, \quad \tilde{x} = (x, \mu_x) \to (\tilde{x}, \mu_x).$$

This shows that $\tilde{X}$ is connected orientable. □

Lemma 24.7. *Let $U \subset \mathbb{R}^n$ be open. Then*

$$\mathrm{H}_i(U) = 0, \quad \forall i \geq n.$$

Proof. Let $\alpha \in S_i(U)$ represent an element of $[\alpha] \in \mathrm{H}_i(U)$. Let $K \subset U$ be a compact subset such that $\mathrm{Supp}(\alpha) \in K$. Equip $\mathbb{R}^n$ with a CW structure in terms of small enough cubes such that

$$K \subset L \subset U$$

where L is a finite CW subcomplex. We have a commutative diagram

$$\begin{tikzcd} \mathrm{H}_{i+1}(\mathbb{R}^n, L) \arrow[r] \arrow[d] & \mathrm{H}_{i+1}(\mathbb{R}^n, U) \arrow[d] \\ \mathrm{H}_i(L) \arrow[r] & \mathrm{H}_i(U) \end{tikzcd}$$

By construction, $[\alpha] \in \mathrm{H}_i(U)$ lies in the image of $\mathrm{H}_i(L)$. But $\mathrm{H}_{i+1}(\mathbb{R}^n, L) \cong \mathrm{H}^{cell}_{i+1}(\mathbb{R}^n, L) = 0$ for $i \geq n$.

□

Lemma 24.8. *Let $U \subset \mathbb{R}^n$ be open. Then the natural map*

$$\mathrm{H}_n(\mathbb{R}^n, U) \to \prod_{x \in \mathbb{R}^n - U} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n - x)$$

is injective.

Proof. This is equivalent to the injectivity of

$$\tilde{\mathrm{H}}_{n-1}(U) \to \prod_{x \in \mathbb{R}^n - U} \tilde{\mathrm{H}}_{n-1}(\mathbb{R}^n - x).$$

Let α be a $(n-1)$-chain representing a class $[\alpha]_U$ in $\tilde{\mathrm{H}}_{n-1}(U)$ which is sent to zero in the above map. We can choose a big open cube B and finite small closed cubes $D_1, \cdots, D_N$ such that D_i is not a subset of U and

$$\mathrm{Supp}(\alpha) \subset B - D_1 \cup \cdots \cup D_N \subset U.$$

Then α represents a class

$$[\alpha] \in \tilde{\mathrm{H}}_{n-1}(B - D_1 \cup \cdots \cup D_N) \cong \mathrm{H}_n(B, B - D_1 \cup \cdots \cup D_N).$$

By assumption, it is mapped to zero

$$\tilde{\mathrm{H}}_{n-1}(B - D_1 \cup \cdots \cup D_N) \to \tilde{\mathrm{H}}_{n-1}(B - D_i)$$
$$\alpha \to 0$$

in each $\tilde{\mathrm{H}}_{n-1}(B - D_i) \cong \mathrm{H}_n(B, B - D_i) \cong \mathrm{H}_n(B, B - x_i)$ where $x_i \in D_i - U$. We next show that $[\alpha] = 0$ in $\tilde{\mathrm{H}}_{n-1}(B - D_1 \cup \cdots \cup D_N)$, hence $[\alpha]_U = 0$ in $\tilde{\mathrm{H}}_{n-1}(U)$. This would prove the required injectivity.

Consider the Mayer-Vietoris sequence

$$\tilde{\mathrm{H}}_n(V) \to \tilde{\mathrm{H}}_{n-1}(B - D_1 \cup \cdots \cup D_N) \to \tilde{\mathrm{H}}_{n-1}(B - D_2 \cup \cdots \cup D_N) \oplus \tilde{\mathrm{H}}_{n-1}(B - D_1)$$

where $V = (B - D_2 \cup \cdots \cup D_N) \cup (B - D_1)$ is open in $\mathbb{R}^n$. By Lemma 24.7, $\tilde{\mathrm{H}}_n(V) = 0$. So

$$\mathrm{H}_{n-1}(B - D_1 \cup \cdots \cup D_N) \to \tilde{\mathrm{H}}_{n-1}(B - D_2 \cup \cdots \cup D_N) \oplus \tilde{\mathrm{H}}_{n-1}(B - D_1)$$

is an injection. It follows that $[\alpha]$ is zero in $\mathrm{H}_{n-1}(B - D_1 \cup \cdots \cup D_N)$ if and only if its image in $\tilde{\mathrm{H}}_{n-1}(B - D_2 \cup \cdots \cup D_N)$ is zero. Repeating this process, we find $[\alpha] = 0$. □

Fundamental class

Theorem 24.9. *Let X be a connected n-manifold. For any abelian group G, we have*

$$\begin{cases} \mathrm{H}_i(X;G) = 0 & i > n \\ \mathrm{H}_n(X;G) = 0 & \text{if } X \text{ is noncompact}. \end{cases}$$

Proof. We prove the case for $G = \mathbb{Z}$. General G is similar.

Step 1: $X = U \subset \mathbb{R}^n$ is an open subset. This is Lemma 24.7.

Step 2: $X = U \cup V$ where U open is homeomorphic to $\mathbb{R}^n$ and V open satisfies the statement in the theorem.

Consider the Mayer-Vietoris sequence

$$\tilde{\mathrm{H}}_i(U) \oplus \tilde{\mathrm{H}}_i(V) \to \tilde{\mathrm{H}}_i(U \cup V) \to \tilde{\mathrm{H}}_{i-1}(U \cap V) \to \tilde{\mathrm{H}}_{i-1}(U) \oplus \tilde{\mathrm{H}}_{i-1}(V)$$

For $i > n$, we find $\mathrm{H}_i(U \cap V) = 0$ by Step 1 since $V \cap U$ can be viewed as an open subset in $\mathbb{R}^n$. Assume $X = U \cup V$ is not compact. We need to show

$$\tilde{\mathrm{H}}_{n-1}(U \cap V) \to \tilde{\mathrm{H}}_{n-1}(V)$$

is injective. Given $x \in X$, the noncompactness and connectedness of X implies that any simplex $\sigma : \Delta^n \to U \cup V$ is homotopic to another singular chain which does not meet x. This implies that

$$\mathrm{H}_n(U \cup V) \to \mathrm{H}_n(U \cup V, U \cup V - x)$$

is zero map for any $x \in X$. Consider the commutative diagram, where $x \in U - U \cap V$

$$\begin{tikzcd}
\mathrm{H}_n(U \cup V) \arrow[d] \arrow[dr] & & \\
\mathrm{H}_n(U \cup V, U \cup V - x) \arrow[d, "\cong"] & \mathrm{H}_n(U \cup V, U \cap V) \arrow[l] \arrow[dr] & \mathrm{H}_n(V, U \cap V) \arrow[l] \arrow[d] \\
\mathrm{H}_n(U, U - x) & \mathrm{H}_n(U, U \cap V) \arrow[l] \arrow[u] \arrow[r] & \tilde{\mathrm{H}}_{n-1}(U \cap V) \arrow[r] \arrow[d] & 0 \\
& & \tilde{\mathrm{H}}_{n-1}(V) &
\end{tikzcd}$$

Let $\alpha \in \mathrm{H}_n(U, U \cap V)$ maps to $\ker(\tilde{\mathrm{H}}_{n-1}(U \cap V) \to \tilde{\mathrm{H}}_{n-1}(V))$. Diagram chasing implies that α maps to zero in $\mathrm{H}_n(U, U - x)$ for any $x \in U - U \cap V$. Since x is arbitrary, this implies $\alpha = 0$ by Lemma 24.8.

Step 3: General case. Let $\alpha \in S_i(X)$ representing a class in $\mathrm{H}_i(X)$. We can choose finite coordinate charts $U_1, \cdots, U_N$ such that $\mathrm{Supp}(\alpha) \subset U_1 \cup \cdots \cup U_N$. Then the class of α lies in the image of the map

$$\mathrm{H}_i(U_1 \cup \cdots \cup U_N) \to \mathrm{H}_i(X).$$

We only need to prove the theorem for $U_1 \cup \cdots \cup U_N$. This follows from Step 2 and induction on N. □

Definition 24.10. Let X be an n-manifold. A **fundamental class** of X at a subspace $A \subset X$ is an element $s \in \mathrm{H}_n(X, X - A)$ whose image

$$\mathrm{H}_n(X, X - A) \to \mathrm{H}_n(X, X - x)$$

defines a local orientation for each $x \in A$. When $A = X$, $s \in \mathrm{H}_n(X)$ is called a fundamental clas of X.

Theorem 24.11. *Let X be an oriented n-manifold, $K \subset X$ be compact subspace. Then*

(1) $\mathrm{H}_i(X, X - K) = 0$ *for any $i > n$.*
(2) *The orientation of X defines a unique fundamental class of X at K.*

In particular, if X is compact, then there exists a unique fundamental class of X associated to the orientation.

Proof.
Step 1: K is a compact subset inside a cooridinate chart $U \cong \mathbb{R}^n$. Then by Lemma 24.7

$$\mathrm{H}_i(X, X-K) \cong \mathrm{H}_i(U, U-K) \cong \tilde{\mathrm{H}}_{i-1}(U-K) = 0 \quad i > n.$$

Take a big enough ball B such that $K \subset B \subset U$. The orientation of X at the local chart U determines an element of $\mathrm{H}_n(X, X-B) = \mathrm{H}_n(U, U-B)$ which maps to the required fundamental class of X at K.

Step 2: $K = K_1 \cup K_2$ where $K_1, K_2, K_1 \cap K_2$ satisfy (1)(2). Using Mayer-Vietoris sequence

$$\cdots \mathrm{H}_{i+1}(X, X-K_1 \cap K_2) \to \mathrm{H}_i(X, X-K_1 \cup K_2) \to \mathrm{H}_i(X, X-K_1) \oplus \mathrm{H}_i(X, X-K_2) \to \mathrm{H}_i(X, X-K_1 \cap K_2) \to \cdots$$

we see K satisfies (1). The unique fundamental classes at K_1 and K_2 map to the unique fundamental class at $K_1 \cap K_2$, giving rise to a unique fundamental class at $K_1 \cup K_2$ by the exact sequence

$$0 \to \mathrm{H}_n(X, X-K_1 \cup K_2) \to \mathrm{H}_n(X, X-K_1) \oplus \mathrm{H}_n(X, X-K_2) \to \mathrm{H}_n(X, X-K_1 \cap K_2)$$

Step 3: For arbitrary K, it is covered by a finite number of coordinates charts $\{U_i\}_{1 \le i \le N}$. Let $K_i = K \cap U_i$. Then $K = K_1 \cup \cdots \cup K_N$. The theorem holds for K by induction on N and Step 1, 2. □

Poincaré duality

Definition 24.12. Let $\mathcal{K}$ denote the set of compact subspaces of X. We define **compactly supported cohomology** of X by

$$\mathrm{H}_c^k(X) := \operatorname*{colim}_{K \in \mathcal{K}} \mathrm{H}^k(X, X-K)$$

where the colimit is taken with respect to the homomorphisms

$$\mathrm{H}^k(X, X-K_1) \to \mathrm{H}^k(X, X-K_2)$$

for $K_1 \subset K_2$ compact. In particular, if X is compact, then $\mathrm{H}_c^k(X) = \mathrm{H}^k(X)$.

Recall that a map is called proper if the pre-image of a compact set is compact. The functorial structure of compactly supported cohomology is with respect to the proper maps: let $f : X \to Y$ be proper, then

$$f^* : \mathrm{H}_c^k(Y) \to \mathrm{H}_c^k(X).$$

Example 24.13. Let $X = \mathbb{R}^n$. Consider the sequence of compact subspaces $B_1 \subset B_2 \subset B_3 \subset \cdots$, where B_k is the closed ball of radius k. Any compact subspace is contained in some ball. Therefore

$$\mathrm{H}_c^i(\mathbb{R}^n) = \operatorname*{colim}_k \mathrm{H}^i(\mathbb{R}^n, \mathbb{R}^n - B_k) = \operatorname*{colim}_k \tilde{\mathrm{H}}^{i-1}(\mathbb{R}^n - B_k) = \tilde{\mathrm{H}}^{i-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}.$$

Theorem 24.14. *Let $X = U \cup V$ where U, V open. Then we have the Mayer-Vietoris exact sequence*

$$\cdots \to \mathrm{H}_c^k(U \cap V) \to \mathrm{H}_c^k(U) \oplus \mathrm{H}_c^k(V) \to \mathrm{H}_c^k(X) \to \mathrm{H}_c^{k+1}(U \cap V) \to \cdots$$

Let X be an oriented n-manifold. For each compact K, let $\xi_K \in \mathrm{H}_n(X, X-K)$ be the fundamental class determined by Theorem 24.11. Taking the cap product we find

$$D_K : \mathrm{H}^p(X, X-K) \stackrel{\cap \xi_K}{\to} \mathrm{H}_{n-p}(X).$$

This passes to the colimit and induces a map

$$D : \mathrm{H}_c^p(X) \to \mathrm{H}_{n-p}(X).$$

Theorem 24.15 (Poincaré Duality)**.** *Let X be an oriented n-manifold. Then for any p,*

$$D : \mathrm{H}_c^p(X) \to \mathrm{H}_{n-p}(X)$$

is an isomorphism. In particular, if X is compact then $\mathrm{H}^p(X) \cong \mathrm{H}_{n-p}(X)$.

Proof. We prove the theorem for all open subset U of X.

Step 1: If the theorem holds for open U, V and $U \cap V$, then the theorem holds for $U \cup V$.

This follows from Mayer-Vietoris sequence and the commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{H}_c^k(U\cap V) \arrow[r] \arrow[d, "D"] & \mathrm{H}_c^k(U) \oplus \mathrm{H}_c^k(V) \arrow[r] \arrow[d, "D\oplus D"] & \mathrm{H}_c^k(U\cup V) \arrow[r] \arrow[d, "D"] & \mathrm{H}_c^{k+1}(U\cap V) \arrow[r] \arrow[d, "D"] & \cdots \\
{} \arrow[r] & \mathrm{H}_{n-k}(U\cap V) \arrow[r] & \mathrm{H}_{n-k}(U) \oplus \mathrm{H}_{n-k}(V) \arrow[r] & \mathrm{H}_{n-k}(U\cup V) \arrow[r] & \mathrm{H}_{n-k-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

Step 2: Let $U_1 \subset U_2 \subset \cdots$ and $U = \cup_i U_i$. Assume the theorem holds for U_i, then it holds for U.

This follows from the isomorphism

$$\mathrm{H}_c^k(U) = \operatorname*{colim}_i \mathrm{H}_c^k(U_i), \quad \mathrm{H}_{n-k}(U) = \operatorname*{colim}_i \mathrm{H}_{n-k}(U_i).$$

Step 3: The theorem holds for an open U contained in a coordinate chart.

This follows by expressing U as a countable union of convex subsets of $\mathbb{R}^n$.

Step 4: For any open U.

By Step 2, 3 and Zorn's lemma, there is a maximal open subset U of X for which the theorem is true. By Step 1, U must be the same as X. $\square$

25 LEFSCHETZ FIXED POINT THEOREM

In this section X will be an oriented connected compact n-dim manifold. $[X]$ its fundamental class.

Intersection form

Poincaré duality gives an isomorphism

$$\mathrm{H}^i(X) \overset{\cap[X]}{\cong} \mathrm{H}_{n-i}(X).$$

The cup product on cohomology has a geometric meaning under Poincaré duality as follows. Let Y, Z be two oriented closed submanifold of X. Assume $\dim(Y) = i, \dim(Z) = j$, and Y intersects Z transversely so that their intersection $Y \cap Z$ is manifold of dimension $i + j - n$. $Y \cap Z$ has an induced orientation. Let $[Y]^* \in \mathrm{H}^{n-i}(X)$ be the Poincaré dual of the fundamental class $[Y] \in \mathrm{H}_i(X)$. Then

$$[Y]^* \cup [Z]^* = [Y \cap Z]^*.$$

Therefore the cup product is interpreted as intersection under Poincaré duality.

An important case is when Y and Z have complementary dimension, i.e. $i + j = n$ so that $Y \cap Z$ is a finite set of points, whose signed sum gives the intersection number of Y and Z.

Definition 25.1. We define the intersection pairing

$$\langle -, - \rangle : \mathrm{H}_i(X) \times \mathrm{H}_{n-i}(X) \to \mathrm{H}_0(X) \cong \mathbb{Z}.$$

Equivalently, we have the pairing on cohomology

$$\langle -, - \rangle : \mathrm{H}^i(X) \times \mathrm{H}^{n-i}(X) \to \mathrm{H}^n(X) \overset{[X]}{\cong} \mathbb{Z}.$$

The intersection pairing is non-degenerate when torsion elements are factored out. In particular

$$\mathrm{H}^i(X; \mathbb{Q}) \times \mathrm{H}^{n-i}(X; \mathbb{Q}) \to \mathbb{Q}$$

is a non-degenerate pairing.

Example 25.2. $T^2 = S^1 \times S^1$. $Y_1 = S^1 \times \{1\}, Y_2 = \{1\} \times S^1$. $Y_1 \cap Y_2$ is a point. This is dual to the ring structure $\mathrm{H}^\bullet(T^2) = \mathbb{Z}[\eta_1, \eta_2]$, where η_i is dual to Y_i.

Lefschetz Fixed Point Theorem

Let us consider the diagonal $\Delta \subset X \times X$. Let $\{e_i\}$ be a basis of $\mathrm{H}_\bullet(X; \mathbb{Q})$, consisting of elements of pure degree. Let e^i be its dual basis of $\mathrm{H}_\bullet(X; \mathbb{Q})$ such that

$$\left\langle e_i, e^j \right\rangle = \delta_i^j.$$

First we observe that

$$[\Delta] \in \mathrm{H}_n(X \times X; \mathbb{Q}) \cong \bigoplus_p \mathrm{H}_p(X; \mathbb{Q}) \otimes \mathrm{H}_{n-p}(X; \mathbb{Q})$$

is given by

$$[\Delta] = \sum_i (-1)^{\deg(e_i)} e_i \otimes e^i.$$

This can be checked by intersecting with a basis of $\mathrm{H}_\bullet(X \times X; \mathbb{Q})$.

Let $f : X \to X$ be a smooth map. Let

$$\Gamma_f := \{(x, f(x)) | x \in X\} \subset X \times X$$

be the graph of f. Let $\alpha \in \mathrm{H}_p(X), \beta \in \mathrm{H}_{n-p}(X)$. From the geometry of graph, we find

$$[\Gamma_f] \cdot \alpha \times \beta = f_* \alpha \cdot \beta.$$

Applying this to $[\Delta]$, we find

$$[\Gamma_f] \cdot [\Delta] = \sum_i (-1)^{|e_i|} f_* e_i \cdot e^i = \sum_p (-1)^p \operatorname{Tr}(f_* : \mathrm{H}_p(X; \mathbb{Q}) \to \mathrm{H}_p(X; \mathbb{Q})).$$

Definition 25.3. We define the **Lefschetz number** of f by

$$L(f) := \sum_p (-1)^p \operatorname{Tr}(f_* : \mathrm{H}_p(X; \mathbb{Q}) \to \mathrm{H}_p(X; \mathbb{Q})).$$

When Γ_f and Δ intersects transversely,

$$\sharp \mathrm{Fix}(f) = [\Gamma_f] \cdot [\Delta]$$

gives a signed count of fixed points of the map f. This gives the Lefschetz Fixed Point Theorem

$$\sharp \mathrm{Fix}(f) = L(f).$$

In particular, if the right hand side is not zero, there must exist a fixed point of f.

Example 25.4. Let n be even. Then any map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ has a fixed point. In fact,

$$f^* : \mathrm{H}^\bullet(\mathbb{CP}^n) \to \mathrm{H}^\bullet(\mathbb{CP}^n)$$

is a ring map. Let $x \in \mathrm{H}^2(\mathbb{CP}^n)$ be a generator, let $f^*(x) = kx$ for some $k \in \mathbb{Z}$. Then

$$\sum_p (-1)^p \operatorname{Tr}(f_*|_{\mathrm{H}_p(\mathbb{CP}^n; \mathbb{Q})}) = \sum_{i=0}^n k^n$$

is an odd number, hence not zero. By Lefschetz Fixed Point Theorem, f must have a fixed point.

Example 25.5. The Lefschetz number of the identity map $\mathrm{id} : X \to X$ is precisely the Euler characteristic

$$L(\mathrm{id}) = \chi(X).$$

Consider the sphere S^2, and the map

$$f : S^2 \to S^2, \quad x \to \frac{x+v}{|x+v|}, \quad v = (0, 0, 1/2).$$

f has two fixed points: north and south pole, and f is homotopy to the identify. We find

$$\chi(S^2) = L(\mathrm{id}) = L(f) = 2.$$

Example 25.6. Consider a compact connected Lie group G. Let $g \in G$ which is not identity but close to identity. Then multiplication by g has no fixed point, and it is hompotopic to the identity map. We find

$$\chi(G) = 0.$$