Lecture 24: Poincaré duality

Definition

A topological manifold of dimension n , or a topological n-manifold, is a Hausdorff space in which each point has an open neighborhood homeomorphic to $\mathbb{R}^n$.

In this section, a manifold always means a topological manifold. We assume $n > 0$. For any point $x \in X$, there exists an open neighborhood U and a homeomorphism $\phi : U \to \mathbb{R}^n$. (U, ϕ) is called a chart around x.

Orientation

Definition

Let X be a n-manifold. $x \in X$ be a point. A generator of

$$\mathrm{H}_n(X, X - x) \simeq \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n - 0) \simeq \mathbb{Z}$$

is called a local orientation of X at x.

For any $x \in X$, there are two choices of local orientation at x. We obtain a two-sheet cover

$$\pi : \tilde{X} \to X$$

where

$$\tilde{X} = \{(x, \mu_x) | \mu_x \text{ is a local orientation of } X \text{ at } x\}.$$

Here π is the natural projection $(x, \mu_x) \to x$.

Definition

A (global) orientation of X is a section of $\pi : \tilde{X} \to X$, i.e., a continuous map

$$s : X \to \tilde{X}$$

such that

$$\pi \circ s = 1_X.$$

If an orientation exists, we say X is orientable.

Theorem

Let X be a connected manifold. Then X is orientable if and only if $\tilde{X}$ has two connected components. In particular, a connected orientable manifold has precisely two orientations.

Example

A simply connected manifold is orientable. This is because the covering of a simply connected space must be a trivial covering.

Proposition

Let X be connected non-orientable manifold. Then $\tilde{X}$ is connected orientable.

Proof.

X is non-orientable implies that $\tilde{X}$ has only one connnected component. Since $\pi : \tilde{X} \to X$ is a covering map, it is a local diffeomorphism and induces an isomorphism

$$\mathrm{H}_n(\tilde{X}, \tilde{X} - \tilde{x}) = \mathrm{H}_n(X, X - x), \quad x = \pi(\tilde{x}).$$

In particular, we have a canonical section

$$s : \tilde{X} \to \tilde{\tilde{X}}, \quad \tilde{x} = (x, \mu_x) \to (\tilde{x}, \mu_x).$$

This shows that $\tilde{X}$ is connected orientable. □

Fundamental class

Definition

Let X be an n-manifold. A fundamental class of X at a subspace $A \subset X$ is an element $s \in \mathrm{H}_n(X, X - A)$ whose image

$$\mathrm{H}_n(X, X - A) \to \mathrm{H}_n(X, X - x)$$

defines a local orientation for each $x \in A$. When $A = X$, $s \in \mathrm{H}_n(X)$ is called a fundamental clas of X.

Our next goal is to show that there exists a canonical fundamental class for each compact subset of an oriented manifold.

Example

Let $B \subset \mathbb{R}^n$ be a closed ball. Then

$$\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n - B) = \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n - x), \quad \forall x \in B.$$

A generator of $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n - B)$ defines a fundamental class of $\mathbb{R}^n$ at B.

Lemma

Let $U \subset \mathbb{R}^n$ be open. Then

$$\mathrm{H}_i(U) = 0, \quad \forall i \geq n.$$

Proof: Let $\alpha \in S_i(U)$ represent an element of $[\alpha] \in \mathrm{H}_i(U)$. Let $K \subset U$ be a compact subset such that $\mathrm{Supp}(\alpha) \in K$. Equip $\mathbb{R}^n$ with a CW structure in terms of small enough cubes such that

$$K \subset L \subset U$$

where L is a finite CW subcomplex.

We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}_{i+1}(\mathbb{R}^n, L) & \longrightarrow & \mathrm{H}_{i+1}(\mathbb{R}^n, U) \\
\downarrow & & \downarrow \\
\mathrm{H}_i(L) & \longrightarrow & \mathrm{H}_i(U)
\end{array}$$

By construction, $[\alpha] \in \mathrm{H}_i(U)$ lies in the image of $\mathrm{H}_i(L)$. But $\mathrm{H}_{i+1}(\mathbb{R}^n, L) \simeq \mathrm{H}_{i+1}^{cell}(\mathbb{R}^n, L) = 0$ for $i \geq n$.

$\square$

Lemma

Let $U \subset \mathbb{R}^n$ be open. Then the natural map

$$\mathrm{H}_n(\mathbb{R}^n, U) \to \prod_{x \in \mathbb{R}^n - U} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n - x)$$

is injective.

Proof: This is equivalent to the injectivity of

$$\tilde{\mathrm{H}}_{n-1}(U) \to \prod_{x \in \mathbb{R}^n - U} \tilde{\mathrm{H}}_{n-1}(\mathbb{R}^n - x).$$

Let α be a $(n-1)$-chain representing a class $[\alpha]_U$ in $\tilde{\mathrm{H}}_{n-1}(U)$ which is sent to zero in the above map.

We can choose a big open cube B and finite small closed cubes $D_1, \cdots, D_N$ such that D_i is not a subset of U and

$$\mathrm{Supp}(\alpha) \subset B - D_1 \cup \cdots \cup D_N \subset U.$$

Then α represents a class

$$[\alpha] \in \tilde{\mathrm{H}}_{n-1}(B - D_1 \cup \cdots \cup D_N) \simeq \mathrm{H}_n(B, B - D_1 \cup \cdots \cup D_N).$$

By assumption, it is mapped to zero

$$\tilde{\mathrm{H}}_{n-1}(B - D_1 \cup \cdots \cup D_N) \to \tilde{\mathrm{H}}_{n-1}(B - D_i)$$
$$\alpha \to 0$$

in each

$$\tilde{\mathrm{H}}_{n-1}(B - D_i) \simeq \mathrm{H}_n(B, B - D_i) \simeq \mathrm{H}_n(B, B - x_i), \quad x_i \in D_i - U.$$

We next show that

$$[\alpha] = 0 \quad \text{in} \quad \tilde{\mathrm{H}}_{n-1}(B - D_1 \cup \cdots \cup D_N)$$

hence $[\alpha]_U = 0$ in $\tilde{\mathrm{H}}_{n-1}(U)$. This would prove the required injectivity.

Consider the Mayer-Vietoris sequence

$$\tilde{\mathrm{H}}_n(V) \to \tilde{\mathrm{H}}_{n-1}(B - D_1 \cup \cdots \cup D_N) \to \tilde{\mathrm{H}}_{n-1}(B - D_2 \cup \cdots \cup D_N) \oplus \tilde{\mathrm{H}}_{n-1}(B - D_1)$$

where $V = (B - D_2 \cup \cdots \cup D_N) \cup (B - D_1)$ is open in $\mathbb{R}^n$.

We have $\tilde{\mathrm{H}}_n(V) = 0$. So

$$\mathrm{H}_{n-1}(B - D_1 \cup \cdots \cup D_N) \to \tilde{\mathrm{H}}_{n-1}(B - D_2 \cup \cdots \cup D_N) \oplus \tilde{\mathrm{H}}_{n-1}(B - D_1)$$

is an injection. It follows that

$$[\alpha] \text{ is zero in } \mathrm{H}_{n-1}(B - D_1 \cup \cdots \cup D_N)$$

if and only if its image in $\tilde{\mathrm{H}}_{n-1}(B - D_2 \cup \cdots \cup D_N)$ is zero.

Repeating this process, we find $[\alpha] = 0$. □

Theorem

Let X be a connected n-manifold. For any abelian group G,

$$\begin{cases} \mathrm{H}_i(X;G) = 0 & i > n \\ \mathrm{H}_n(X;G) = 0 & \text{if } X \text{ is noncompact.} \end{cases}$$

Proof: We prove the case for $G = \mathbb{Z}$. General G is similar.

Step 1: $X = U \subset \mathbb{R}^n$ is an open subset. This is proved before.

Step 2: $X = U \cup V$ where U open is homeomorphic to $\mathbb{R}^n$ and V open satisfies the statement in the theorem.

Consider the Mayer-Vietoris sequence

$$\tilde{\mathrm{H}}_i(U) \oplus \tilde{\mathrm{H}}_i(V) \to \tilde{\mathrm{H}}_i(U \cup V) \to \tilde{\mathrm{H}}_{i-1}(U \cap V) \to \tilde{\mathrm{H}}_{i-1}(U) \oplus \tilde{\mathrm{H}}_{i-1}(V)$$

For $i > n$, we find $\mathrm{H}_i(U \cap V) = 0$ by Step 1 since $V \cap U$ can be viewed as an open subset in $\mathbb{R}^n$.

Assume $X = U \cup V$ is not compact. We need to show

$$\tilde{\mathrm{H}}_{n-1}(U \cap V) \to \tilde{\mathrm{H}}_{n-1}(V)$$

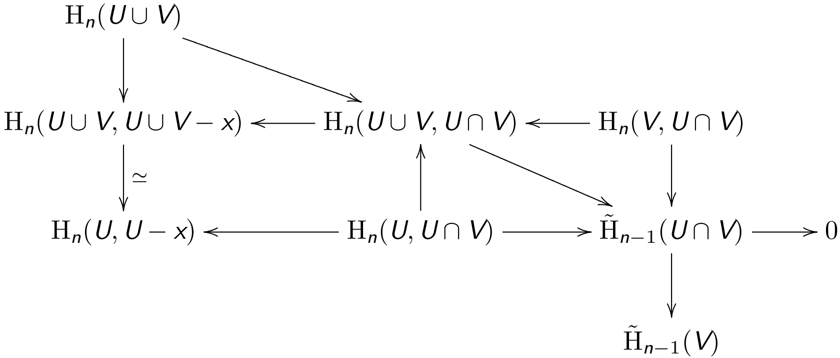
is injective.

Given $x \in X$, the noncompactness and connectedness of X implies that any simplex $\sigma : \Delta^n \to U \cup V$ is homotopic to another singular chain which does not meet x. This implies that

$$\mathrm{H}_n(U \cup V) \to \mathrm{H}_n(U \cup V, U \cup V - x)$$

is zero map for any $x \in X$.

Consider the commutative diagram, where $x \in U - U \cap V$



Let $\alpha \in \mathrm{H}_n(U, U \cap V)$ maps to $\ker(\tilde{\mathrm{H}}_{n-1}(U \cap V) \to \tilde{\mathrm{H}}_{n-1}(V))$. Diagram chasing implies that α maps to zero in $\mathrm{H}_n(U, U - x)$ for any $x \in U - U \cap V$. Since x is arbitrary, this implies $\alpha = 0$ by the previous lemma.

Step 3: General case. Let $\alpha \in S_i(X)$ representing a class in $\mathrm{H}_i(X)$. We can choose finite coordinate charts $U_1, \cdots, U_N$ such that

$$\mathrm{Supp}(\alpha) \subset U_1 \cup \cdots \cup U_N.$$

Then the class of α lies in the image of the map

$$\mathrm{H}_i(U_1 \cup \cdots \cup U_N) \to \mathrm{H}_i(X).$$

We only need to prove the theorem for $U_1 \cup \cdots \cup U_N$. This follows from Step 2 and induction on N. $\square$

Theorem

Let X be an oriented n-manifold, $K \subset X$ be compact. Then

(1) $\mathrm{H}_i(X, X - K) = 0$ for any $i > n$.

(2) The orientation of X defines a unique fundamental class of X at K.

In particular, if X is compact, then there exists a unique fundamental class of X associated to the orientation.

Proof

Step 1: K is a compact subset inside a cooridinate chart $U \simeq \mathbb{R}^n$.

$$\mathrm{H}_i(X, X-K) \simeq \mathrm{H}_i(U, U-K) \simeq \tilde{\mathrm{H}}_{i-1}(U-K) = 0 \quad i > n.$$

Take a big enough ball B such that $K \subset B \subset U$. The orientation of X at the local chart U determines an element of

$$\mathrm{H}_n(X, X-B) = \mathrm{H}_n(U, U-B)$$

which maps to the required fundamental class of X at K.

Step 2: $K = K_1 \cup K_2$ where $K_1, K_2, K_1 \cap K_2$ satisfy $(1)(2)$.

Using Mayer-Vietoris sequence

$$\mathrm{H}_{i+1}(X, X - K_1 \cap K_2) \to \mathrm{H}_i(X, X - K_1 \cup K_2) \to \mathrm{H}_i(X, X - K_1) \oplus \mathrm{H}_i(X, X - K_2) \to \mathrm{H}_i(X, X - K$$

we see K satisfies (1).

The unique fundamental classes at K_1 and K_2 map to the unique fundamental class at $K_1 \cap K_2$, giving rise to a unique fundamental class at $K_1 \cup K_2$ by the exact sequence

$$0 \to \mathrm{H}_n(X, X - K_1 \cup K_2) \to \mathrm{H}_n(X, X - K_1) \oplus \mathrm{H}_n(X, X - K_2) \to \mathrm{H}_n(X, X - K_1 \cap K_2)$$

Step 3: For arbitrary K, it is covered by a finite number of coordinates charts $\{U_i\}_{1\leq i\leq N}$. Let $K_i = K \cap U_i$. Then

$$K = K_1 \cup \cdots \cup K_N.$$

The theorem holds for K by induction on N and Step 1, 2. $\square$

Poincaré duality

Definition

Let $\mathcal{K}$ denote the set of compact subspaces of X. We define compactly supported cohomology of X by

$$\mathrm{H}_c^k(X) := \operatorname*{colim}_{K \in \mathcal{K}} \mathrm{H}^k(X, X - K)$$

where the colimit is taken with respect to the homomorphisms

$$\mathrm{H}^k(X, X - K_1) \to \mathrm{H}^k(X, X - K_2)$$

for $K_1 \subset K_2$ compact. In particular, if X is compact, then

$$\mathrm{H}_c^k(X) = \mathrm{H}^k(X).$$

Recall that a map is called proper if the pre-image of a compact set is compact.

The functorial structure of compactly supported cohomology is with respect to the proper maps: let $f : X \to Y$ be proper, then

$$f^* : \mathrm{H}_c^k(Y) \to \mathrm{H}_c^k(X).$$

Example

Let $X = \mathbb{R}^n$. Consider the sequence of compact subspaces $B_1 \subset B_2 \subset B_3 \subset \cdots$, where B_k is the closed ball of radius k. Any compact subspace is contained in some ball. Therefore

$$\mathrm{H}^i_c(\mathbb{R}^n) = \operatorname*{colim}_k \mathrm{H}^i(\mathbb{R}^n, \mathbb{R}^n - B_k) = \operatorname*{colim}_k \tilde{\mathrm{H}}^{i-1}(\mathbb{R}^n - B_k)$$

$$= \tilde{\mathrm{H}}^{i-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}.$$

Theorem

Let $X = U \cup V$ where U, V open. Then we have the Mayer-Vietoris exact sequence

$$\cdots \to \mathrm{H}_c^k(U \cap V) \to \mathrm{H}_c^k(U) \oplus \mathrm{H}_c^k(V) \to \mathrm{H}_c^k(X) \to \mathrm{H}_c^{k+1}(U \cap V) \to \cdots$$

Let X be an oriented n-manifold. For each compact K, let $\xi_K \in \mathrm{H}_n(X, X-K)$ be the fundamental class determined by the previous Theorem.

Taking the cap product we find

$$D_K : \mathrm{H}^p(X, X-K) \stackrel{\cap \xi_K}{\to} \mathrm{H}_{n-p}(X).$$

This passes to the colimit and induces a map

$$D : \mathrm{H}^p_c(X) \to \mathrm{H}_{n-p}(X).$$

Theorem (Poincaré Duality)

Let X be an oriented n-manifold. Then for any p,

$$D : \mathrm{H}_c^p(X) \to \mathrm{H}_{n-p}(X)$$

is an isomorphism. In particular, if X is compact then

$$\mathrm{H}^p(X) \simeq \mathrm{H}_{n-p}(X).$$

Proof

First we observe that if the theorem holds for open U, V and $U \cap V$, then the theorem holds for $U \cup V$.

This follows from Mayer-Vietoris sequence

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{H}_c^k(U \cap V) & \longrightarrow & \mathrm{H}_c^k(U) \oplus \mathrm{H}_c^k(V) & \longrightarrow & \mathrm{H}_c^k(U \cup V) & \longrightarrow & \mathrm{H}_c^{k+1}(U \cap V) & \longrightarrow \\
 & \downarrow D & & \downarrow D \oplus D & & \downarrow D & & \downarrow D & \\
\longrightarrow & \mathrm{H}_{n-k}(U \cap V) & \longrightarrow & \mathrm{H}_{n-k}(U) \oplus \mathrm{H}_{n-k}(V) & \longrightarrow & \mathrm{H}_{n-k}(U \cup V) & \longrightarrow & \mathrm{H}_{n-k-1}(U \cap V) & \longrightarrow
\end{array}$$

We prove a special case of the theorem when X has a finite open cover U_i such that any intersection of U_i's is homeomorphic to $\mathbb{R}^n$. This works for a large class of smooth manifolds where we can use distance to choose convex subset of local charts.

Then the theorem follows by the previous observation. □