

INTRODUCTION TO ALGEBRAIC TOPOLOGY

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ABSTRACT. To be continued. This is course note for Algebraic Topology in Spring 2018 at Tsinghua university.

Course References:

- (1) Hatcher: Algebraic Topology
- (2) Bott and Tu: Differential forms in algebraic topology.
- (3) May: A Concise Course in Algebraic Topology
- (4) Spanier: Algebraic Topology.

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1. CATEGORY AND FUNCTOR

Category.

Definition 1.1. A category \mathcal{C} consists of

- (1) a class of *objects*: $\text{Obj}(\mathcal{C})$
- (2) *morphisms*: a set $\text{Hom}_{\mathcal{C}}(A, B), \forall A, B \in \text{Obj}(\mathcal{C})$. An element $f \in \text{Hom}(A, B)$ will be denoted by

$$A \xrightarrow{f} B \quad \text{or} \quad f : A \rightarrow B.$$

- (3) *composition*:

$$\begin{aligned} \text{Hom}(A, B) \times \text{Hom}(B, C) &\rightarrow \text{Hom}(A, C), \forall A, B, C \in \text{Obj}(\mathcal{C}) \\ f \times g &\rightarrow g \circ f \end{aligned}$$

satisfying the following axioms

- (1) *associativity*: $h \circ (g \circ f) = (h \circ g) \circ f$ for any $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$.
- (2) *identity*: $\forall A \in \text{Obj}(\mathcal{C}), \exists 1_A \in \text{Hom}(A, A)$ called the identity element, such that

$$f \circ 1_A = f = 1_B \circ f, \quad \forall A \xrightarrow{f} B.$$

A category is called *small* if its objects form a set.

Definition 1.2. A morphism $f : A \rightarrow B$ is called an **equivalence/invertible** if $\exists g : B \rightarrow A$ such that

$$f \circ g = 1_B, \quad g \circ f = 1_A.$$

Two objects A, B are called **equivalent** if there exists an equivalence $f : A \rightarrow B$.

Definition 1.3. A category where all morphisms are equivalences is called a **groupoid**.

Definition 1.4. A **subcategory** $\mathcal{C}' \subset \mathcal{C}$ is a category such that

- $\text{Obj}(\mathcal{C}') \subset \text{Obj}(\mathcal{C})$
- $\text{Hom}_{\mathcal{C}'}(A, B) \subset \text{Hom}_{\mathcal{C}}(A, B), \forall A, B \in \text{Obj}(\mathcal{C}')$
- composition coincides.

\mathcal{C}' is called a full subcategory of \mathcal{C} if $\text{Hom}_{\mathcal{C}'}(A, B) = \text{Hom}_{\mathcal{C}}(A, B), \forall A, B \in \text{Obj}(\mathcal{C}')$.

Definition 1.5. Let \sim be an equivalence relation defined on each $\text{Hom}(A, B)$, $A, B \in \text{Obj}(\mathcal{C})$ satisfying

$$f_1 \sim f_2, g_1 \sim g_2 \implies g_1 \circ f_1 \sim g_2 \circ f_2.$$

Then we define the quotient category $\mathcal{C}' = \mathcal{C} / \sim$ by

- $\text{Obj}(\mathcal{C}') = \text{Obj}(\mathcal{C})$
- $\text{Hom}_{\mathcal{C}'}(A, B) = \text{Hom}_{\mathcal{C}}(A, B) / \sim, \forall A, B \in \text{Obj}(\mathcal{C}')$

Example 1.6. We will frequently use the following categories.

- Set: the category of set.
- Vect: the category of vector spaces.
- Group: the category of groups.
- Ab: the category of abelian groups.
- Ring: the category of rings.

Vect \subset Set is a subcategory, and Ab \subset Group is a full subcategory.

The main object of our interest is the category of topological spaces Top

- objects of Top are topological spaces.
- morphism $f : X \rightarrow Y$ is a continuous map.

Definition 1.7. Given $X, Y \in \text{Top}$, $f_0, f_1 : X \rightarrow Y$ are said to be homotopic, denoted by $f_0 \simeq f_1$, if

$$\exists F : X \times I \rightarrow Y, \text{ such that } F|_{X \times 0} = f_0, F|_{X \times 1} = f_1. \quad I = [0, 1].$$

Homotopy defines an equivalence relation on Top. We denote its quotient category by

$$\underline{\text{hTop}} = \text{Top} / \simeq.$$

We also denote

$$\text{Hom}_{\underline{\text{hTop}}}(X, Y) = [X, Y].$$

Definition 1.8. Two topological spaces X, Y are said to have the **same homotopy type** (or homotopy equivalent) if they are equivalent in hTop.

There is also a relative version as follows.

Definition 1.9. Let $A \subset X \in \text{Top}$, $f_0, f_1 : X \rightarrow Y$ such that $f_0|_A = f_1|_A : A \rightarrow Y$. We say f_0 is homotopic to f_1 relative to A , denoted by

$$f_0 \simeq f_1 \text{ rel } A$$

if there exists $F : X \times I \rightarrow Y$ such that

$$F|_{X \times 0} = f_0, \quad F|_{X \times 1} = f_1, \quad F|_{A \times t} = f_0|_A, \forall t \in I.$$

Functor.

Definition 1.10. Let \mathcal{C}, \mathcal{D} be two categories. A **covariant functor** (or **contravariant functor**) $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of

- $F : \text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{D}), A \rightarrow F(A)$

- $\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B)), \forall A, B \in \text{Obj}(\mathcal{C})$. We denote by

$$A \xrightarrow{f} B \implies F(A) \xrightarrow{F(f)} F(B)$$

(or $\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(B), F(A)), \forall A, B \in \text{Obj}(\mathcal{C})$, denoted by $A \xrightarrow{f} B \implies F(B) \xrightarrow{F(f)} F(A)$)

satisfying

- $F(g \circ f) = F(g) \circ F(f)$ (or $F(g \circ f) = F(f) \circ F(g)$) for any $A \xrightarrow{f} B \xrightarrow{g} C$
- $F(1_A) = 1_{F(A)}, \forall A \in \text{Obj}(\mathcal{C})$.

F is called **faithful** (or **full**) if $\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B))$ is injective (or surjective) $\forall A, B \in \text{Obj}(\mathcal{C})$.

Example 1.11. $\forall X \in \text{Obj}(\mathcal{C})$,

$$\text{Hom}(X, -) : \mathcal{C} \rightarrow \underline{\text{Set}}, \quad A \rightarrow \text{Hom}(X, A)$$

defines a covariant functor. Similarly $\text{Hom}(-, X)$ defines a contravariant functor. A functor $F : \mathcal{C} \rightarrow \underline{\text{Set}}$ of such type is called *representable*.

Example 1.12. Let G be an abelian group. Given $X \in \underline{\text{Top}}$, we will study its n -th cohomology $H^n(X; G)$. It defines a functor

$$H^n(-; G) : \underline{\text{hTop}} \rightarrow \underline{\text{Set}}, \quad X \rightarrow H^n(X; G)$$

We will see that this functor is representable by the Eilenberg-MacLane space if we work with the subcategory of CW-complexes.

Example 1.13. We define a contravariant functor

$$\text{Fun} : \underline{\text{Top}} \rightarrow \underline{\text{Ring}}, \quad X \rightarrow \text{Fun}(X) = \text{Hom}(X, \mathbb{R})$$

$\text{Fun}(X)$ are continuous real functions on X . A classical theorem of Gelfand-Kolmogoroff says that two compact Hausdorff spaces X, Y are homeomorphic if and only if $\text{Fun}(X), \text{Fun}(Y)$ are ring isomorphic.

Proposition 1.14. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. $f : A \rightarrow B$ is an equivalence. Then $F(f) : F(A) \rightarrow F(B)$ is also an equivalence.

Natural transformation.

Definition 1.15. Let \mathcal{C}, \mathcal{D} be two categories. $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be two functors. A **natural transformation** $\tau : F \rightarrow G$ consists of morphisms

$$\tau = \{\tau_A : F(A) \rightarrow G(A) | \forall A \in \text{Obj}(\mathcal{C})\}$$

such that the following diagram commutes for any $A, B \in \text{Obj}(\mathcal{C})$

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow \tau_A & & \downarrow \tau_B \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

τ is called **natural equivalence** if τ_A is an equivalence for any $A \in \text{Obj}(\mathcal{C})$. We write $F \simeq G$.

Definition 1.16. Two categories \mathcal{C}, \mathcal{D} are called **isomorphic** if $\exists F : \mathcal{C} \rightarrow \mathcal{D}, G : \mathcal{D} \rightarrow \mathcal{C}$ such that $F \circ G = 1_{\mathcal{D}}, G \circ F = 1_{\mathcal{C}}$. They are called **equivalent** if $\exists F : \mathcal{C} \rightarrow \mathcal{D}, G : \mathcal{D} \rightarrow \mathcal{C}$ such that $F \circ G \simeq 1_{\mathcal{D}}, G \circ F \simeq 1_{\mathcal{C}}$

Proposition 1.17. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be an equivalence of categories. Then F is fully faithful.

Definition 1.18. Let \mathcal{C} be a small category, and \mathcal{D} be a category. We define the functor category $\text{Fun}(\mathcal{C}, \mathcal{D})$

- objects: covariant functors from \mathcal{C} to \mathcal{D}
- morphism: natural transformations between two functors (which is indeed a set since \mathcal{C} is small).

2. FUNDAMENTAL GROUPOID

Path connected component.

Definition 2.1. Let $X \in \underline{\text{Top}}$. A map $\gamma : I \rightarrow X$ is called a path from $\gamma(0)$ to $\gamma(1)$. We denote γ^{-1} be the path from $\gamma(1)$ to $\gamma(0)$ defined by $\gamma^{-1}(t) = \gamma(1 - t)$. We denote $i_{x_0} : I \rightarrow X$ be the constant map to $x_0 \in X$.

Let us introduce an equivalence relation on X by

$$x_0 \sim x_1 \iff \exists \text{ a path from } x_0 \text{ to } x_1.$$

We denote the quotient space

$$\pi_0(X) = X / \sim$$

which is the set of path connected components of X .

Proposition 2.2. $\pi_0 : \underline{\text{hTop}} \rightarrow \underline{\text{Set}}$ defines a covariant functor.

As a consequence, $\pi_0(X) \cong \pi_0(Y)$ if X, Y are homotopy equivalent.

Path category/fundamental groupoid.

Definition 2.3. Let $\gamma : I \rightarrow X$ be a path. We define the path class of γ

$$[\gamma] = \{ \tilde{\gamma} : I \rightarrow X \mid \tilde{\gamma} \simeq \gamma \text{ rel } \partial I = \{0, 1\} \}$$

Definition 2.4. Let $\gamma_1, \gamma_2 : I \rightarrow X$ such that $\gamma_1(1) = \gamma_2(0)$. We define

$$\gamma_2 \star \gamma_1 : I \rightarrow X$$

by

$$\gamma_2 \star \gamma_1(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq 1/2 \\ \gamma_2(2t - 1) & 1/2 \leq t \leq 1. \end{cases}$$

\star is not associative for strict paths. However, \star defines an associative composition on path classes.

Theorem 2.5. Let $X \in \underline{\text{Top}}$. We define a category $\Pi_1(X)$ as follows:

- $\text{Obj}(\Pi_1(X)) = X$.
- $\text{Hom}_{\Pi_1(X)}(x_0, x_1) = \text{path classes from } x_0 \text{ to } x_1$.
- $1_{x_0} = i_{x_0}$.

Then $\Pi_1(X)$ defines a category which is in fact a groupoid. The inverse of $[\gamma]$ is given by $[\gamma^{-1}]$. $\Pi_1(X)$ is called the **fundamental groupoid** of X .

Let \mathcal{C} be a groupoid. Let $A \in \text{Obj}(\mathcal{C})$, then

$$\text{Aut}_{\mathcal{C}}(A) := \text{Hom}_{\mathcal{C}}(A, A)$$

forms a group. For any $f : A \rightarrow B$, it induces a group isomorphism

$$\begin{aligned} Ad_f : \text{Aut}_{\mathcal{C}}(A) &\rightarrow \text{Aut}_{\mathcal{C}}(B) \\ g &\rightarrow f \circ g \circ f^{-1}. \end{aligned}$$

This naturally defines a functor

$$\begin{aligned} \mathcal{C} &\rightarrow \underline{\text{Group}} \\ A &\rightarrow \text{Aut}_{\mathcal{C}}(A) \\ f &\rightarrow Ad_f \end{aligned}$$

Specialize this to topological spaces, we find a functor

$$\boxed{\Pi_1(X) \rightarrow \underline{\text{Group}}}.$$

Definition 2.6. Let $x_0 \in X$, the group

$$\pi_1(X, x_0) := \text{Aut}_{\Pi_1(X)}(x_0)$$

is called the fundamental group of the pointed space (X, x_0) .

Theorem 2.7. Let X be path connected. Then for $x_0, x_1 \in X$, they have group isomorphism

$$\pi_1(X, x_0) \cong \pi_1(X, x_1).$$

Let $f : X \rightarrow Y$ be a continuous map. It defines a functor

$$\begin{aligned} \Pi_1(f) : \Pi_1(X) &\rightarrow \Pi_1(Y) \\ x &\rightarrow f(x) \\ [\gamma] &\rightarrow [f \circ \gamma]. \end{aligned}$$

Then Π_1 defines a functor

$$\boxed{\Pi_1 : \underline{\text{Top}} \rightarrow \underline{\text{Groupoid}}}, \quad X \rightarrow \Pi_1(X)$$

from the category Top to the category Groupoid of groupoids. Here morphisms in Groupoid are given by natural transformations.

Proposition 2.8. Let $f, g : X \rightarrow Y$ be maps which are homotopic by $F : X \times I \rightarrow Y$. Let us define path classes

$$\tau_{x_0} = [F|_{x_0 \times I}] \in \text{Hom}_{\Pi_1(Y)}(f(x_0), g(x_0)).$$

Then τ defines a natural transformation

$$\tau : \Pi_1(f) \Longrightarrow \Pi_1(g).$$

This proposition can be pictured by the following diagram

$$X \begin{array}{c} \xrightarrow{f} \\ \Downarrow F \\ \xrightarrow{g} \end{array} Y \implies \Pi_1(X) \begin{array}{c} \xrightarrow{\Pi_1(f)} \\ \Downarrow \tau \\ \xrightarrow{\Pi_1(g)} \end{array} \Pi_1(Y)$$

The following theorem is a formal consequence of the above proposition

Theorem 2.9. Let $f : X \rightarrow Y$ be a homotopy equivalence. Then

$$\Pi_1(f) : \Pi_1(X) \rightarrow \Pi_1(Y)$$

is an equivalence of categories. In particular, it induces group isomorphisms

$$\pi_1(X, x_0) \cong \pi_1(Y, f(x_0)),$$

3. COVERING AND FIBRATION

Covering.

Definition 3.1. Let $p : E \rightarrow B$ be continuous. A *trivialization* of p over an open $U \subset B$ is a homeomorphism $\varphi : p^{-1}(U) \rightarrow U \times F$ over U , i.e., the following diagram commutes

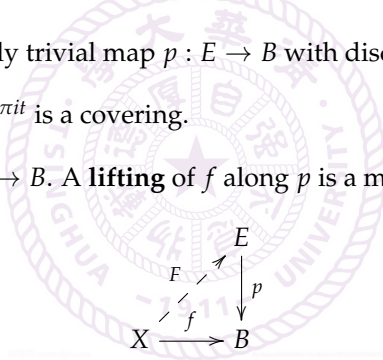
$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ & \searrow & \swarrow \\ & U & \end{array}$$

p is called *locally trivial* if there exists an open cover \mathcal{U} of B such that p has a trivialization over each open $U \in \mathcal{U}$. Such p is also called a **fiber bundle** and F is called the fiber.

Definition 3.2. A **covering** is a locally trivial map $p : E \rightarrow B$ with discrete fiber F .

Example 3.3. $ex : \mathbb{R}^1 \rightarrow S^1, t \rightarrow e^{2\pi it}$ is a covering.

Definition 3.4. Let $p : E \rightarrow B, f : X \rightarrow B$. A **lifting** of f along p is a map $F : X \rightarrow E$ such that $p \circ F = f$



Lemma 3.5. Let $p : E \rightarrow B$ be a covering. Let

$$D = \{(x, x) \in E \times E | x \in E\}$$

$$Z = \{(x, y) \in E \times E | p(x) = p(y)\}.$$

Then $D \subset Z$ is open and closed.

Theorem 3.6 (Uniqueness of lifting). Let $p : E \rightarrow B$ be a covering. Let $F_0, F_1 : X \rightarrow E$ be two liftings of f . Suppose X is connected and F_0, F_1 agree somewhere. Then $F_0 = F_1$.

Proof. Consider the map $\tilde{F} = (F_0, F_1) : X \rightarrow Z$. $\tilde{F}(X) \cap D \neq \emptyset$. The above lemma implies $\tilde{F}(X) \subset D$.

□

fibration.

Definition 3.7. A map $p : E \rightarrow B$ is said to have the **homotopy lifting property** (HLP) with respect to X if for any maps $\tilde{f} : X \rightarrow E$ and $F : X \times I \rightarrow B$ such that $p \circ \tilde{f} = F|_{X \times 0}$, there exists a lifting \tilde{F} of F along p such

that $\tilde{F}|_{X \times 0} = \tilde{f}$, i.e., the following diagram commutes

$$\begin{array}{ccc} X \times 0 & \xrightarrow{\tilde{f}} & E \\ \downarrow & \nearrow \exists \tilde{F} & \downarrow p \\ X \times I & \xrightarrow{F} & B \end{array}$$

Definition 3.8. A map $p : E \rightarrow B$ is called a **fibration** (or **Hurewicz fibration**) if p has HLP for any space.

Theorem 3.9. A covering is a fibration.

Corollary 3.10. Let $p : E \rightarrow B$ be a fibration. Then for any path $\gamma : I \rightarrow B$ and $e \in E$ such that $p(e) = \gamma(0)$, there exists a unique path $\tilde{\gamma} : I \rightarrow E$ that lifts γ and $\tilde{\gamma}(0) = e$.

Proof. Apply HLP to $X = \text{pt}$.

$$\begin{array}{ccc} 0 & \xrightarrow{e} & E \\ \downarrow & \nearrow \tilde{\gamma} & \downarrow p \\ I & \xrightarrow{\gamma} & B \end{array}$$

□

Corollary 3.11. Let $p : E \rightarrow B$ be a covering. Then $\Pi_1(E) \rightarrow \Pi_1(B)$ is a faithful functor. In particular, the induced map $\pi_1(E, e) \rightarrow \pi_1(B, p(e))$ is injective.

Transport functor.

Let $p : E \rightarrow B$ be a covering. Let $\gamma : I \rightarrow B$ be a path in B from b_1 to b_2 . It defines a map

$$\begin{aligned} T_\gamma : p^{-1}(b_1) &\rightarrow p^{-1}(b_2) \\ e_1 &\rightarrow \tilde{\gamma}(1) \end{aligned}$$

where $\tilde{\gamma}$ is a lift of γ with initial condition $\tilde{\gamma}(0) = e_1$.

Assume $[\gamma_1] = [\gamma_2]$ in B . HLP implies that $T_{\gamma_1} = T_{\gamma_2}$. We find a well-defined map

$$\begin{aligned} T : \text{Hom}_{\Pi_1(B)}(b_1, b_2) &\rightarrow \text{Hom}_{\text{Set}}(p^{-1}(b_1), p^{-1}(b_2)) \\ [\gamma] &\rightarrow T_{[\gamma]} \end{aligned}$$

Proposition 3.12. The following data

$$\begin{aligned} T : \Pi_1(B) &\rightarrow \underline{\text{Set}} \\ b &\rightarrow p^{-1}(b) \\ [\gamma] &\rightarrow T_{[\gamma]}. \end{aligned}$$

defines a functor, called the **transport functor**. In particular, we have a well-defined map

$$\pi_1(B, b) \rightarrow \text{Aut}(p^{-1}(b)).$$

Proposition 3.13. Let $p : E \rightarrow B$ be a covering, E path connected. Let $e \in E, b = p(e) \in B$. Then the action of $\pi_1(B, b)$ on $p^{-1}(b)$ is transitive, whose stabilizer at e is $\pi_1(E, e)$. In other words,

$$p^{-1}(b) \cong \pi_1(B, b) / \pi_1(E, e)$$

as a coset space.

Lifting Criterion.

Theorem 3.14 (Lifting Criterion). *Let $p : E \rightarrow B$ be a covering. $f : X \rightarrow B$ for X path connected and locally path connected. Let $e \in E, x_0 \in X$ such that $f(x_0) = p(e)$. Then there exists a lift F of f with $F(x_0) = e$ if and only if*

$$f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e)).$$

Proof. If such F exists, $f_*(\pi_1(X, x_0)) = p_*F_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e))$. Conversely, consider the product

$$\begin{array}{ccc} \tilde{E} & \longrightarrow & E \\ \tilde{p} \downarrow & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

Let $\tilde{e} = (e, x_0) \in \tilde{E}$. Then \tilde{p} is a covering and we have a commuting diagram of functors

$$\begin{array}{ccc} \Pi_1(X) & & \\ \downarrow & \searrow T & \\ \Pi_1(B) & \xrightarrow{T} & \underline{\text{Set}} \end{array}$$

which induces a natural map

$$\pi_1(X, x_0) \rightarrow \pi_1(B, b) \rightarrow \text{Aut}(p^{-1}(b)).$$

The condition $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e))$ says that $\pi_1(X, x_0)$ stabilizes \tilde{e} . This implies

$$\pi_1(\tilde{E}, \tilde{e}) \cong \pi_1(X, x_0).$$

Since X is locally path connected, \tilde{E} is also locally path connected. Then path connected components and connected components of \tilde{E} coincide. Let \tilde{X} be the (path) connected component of \tilde{E} containing \tilde{e} , then $\pi_1(\tilde{E}, \tilde{e}) \cong \pi_1(\tilde{X}, \tilde{e})$ implies that $\tilde{p} : \tilde{X} \rightarrow X$ is a covering with fiber a single point, hence a homeomorphism. Its inverse defines a continuous map $X \rightarrow \tilde{E}$ whose composition with $\tilde{E} \rightarrow E$ gives F . \square

4. $\pi_1(S^1)$ AND APPLICATIONS

G-principal covering.

Definition 4.1. Let G be a discrete group. An action $G \times X \rightarrow X$ is called **properly discontinuous** if $\forall x \in X, \exists$ open neighborhood U of x such that

$$g(U) \cap U = \emptyset, \quad \forall g \neq 1 \in G.$$

We define the orbit space X/G by the quotient X/\sim where $x \sim g(x)$ for any $x \in X, g \in G$.

Proposition 4.2. *Assume G acts properly discontinuously on X , then the quotient map $X \rightarrow X/G$ is a covering.*

Definition 4.3. A left (right) **G-principal covering** is a covering $p : E \rightarrow B$ with a left (right) properly discontinuous G -action on E over B

$$\begin{array}{ccc} E & \xrightarrow{g} & E, \quad \forall g \in G \\ & \searrow p & \swarrow p \\ & & B \end{array}$$

such that the induced map $E/G \rightarrow B$ is a homeomorphism.

Example 4.4. $ex : \mathbb{R}^1 \rightarrow S^1$ is a \mathbb{Z} -principal covering for the action $n : t \rightarrow t + n, \forall n \in \mathbb{Z}$.

Example 4.5. $S^n \rightarrow \mathbb{R}P^n \cong S^n/\mathbb{Z}_2$ is a \mathbb{Z}_2 -principal covering.

Proposition 4.6. Let $p : E \rightarrow B$ be a G -principal covering. Then transportation commutes with G -action, i.e.,

$$T_{[\gamma]} \circ g = g \circ T_{[\gamma]}, \quad \forall g \in G, \gamma \text{ a path in } B.$$

Theorem 4.7. Let $p : E \rightarrow B$ be a G -principal covering, E path connected, $e \in E, b = p(e)$. Then we have an exact sequence of groups

$$1 \rightarrow \pi_1(E, e) \rightarrow \pi_1(B, b) \rightarrow G \rightarrow 1.$$

In other words, $\pi_1(E, e)$ is a normal subgroup of $\pi_1(B, b)$ and $G = \pi_1(B, b)/\pi_1(E, e)$.

Proof. Let $F = p^{-1}(b)$. The previous proposition implies that $\pi_1(B, b)$ -action and G -action on F commute. It induces a $\pi_1(B, b) \times G$ -action on F . Consider its stabilizer at e and two projections

$$\begin{array}{ccc} & \text{Stab}_e(\pi_1(B, b) \times G) & \\ & \swarrow \text{pr}_1 & \searrow \text{pr}_2 \\ \pi_1(B, b) & & G \end{array}$$

pr_1 is an isomorphism and pr_2 is an epimorphism with $\ker(\text{pr}_2) = \text{Stab}_e(\pi_1(B, b)) = \pi_1(E, e)$. \square

Apply this theorem to the covering $ex : \mathbb{R}^1 \rightarrow S^1$, we find a group isomorphism

$$\text{deg} : \boxed{\pi_1(S^1) \rightarrow \mathbb{Z}}$$

which is called the **degree map**.

Applications.

Definition 4.8. $i : A \subset X$ be a subspace. A continuous map $r : X \rightarrow A$ is called a **retraction** if $r \circ i = 1_A$. It is called a **deformation retraction** if furthermore $i \circ r \simeq 1_X$. We say A is a (deformation) retract of X if such a (deformation) retraction exists.

Proposition 4.9. If $i : A \subset X$ is a retract, then $r_* : \pi_1(A) \rightarrow \pi_1(X)$ is injective.

Corollary 4.10. Let D^2 be the unit disk in \mathbb{R}^2 . Then its boundary S^1 is not a retract of D^2 .

Theorem 4.11 (Brouwer fixed point Theorem). Let $f : D^2 \rightarrow D^2$. Then there exists $x \in D^2$ such that $f(x) = x$.

Proof. Assume f has no fixed point. Let l_x be the ray starting from $f(x)$ pointing toward x . Then

$$D^2 \rightarrow S^1, \quad x \rightarrow l_x \cap \partial D^2$$

is a retraction of $\partial D^2 = S^1 \subset D^2$. Contradiction. \square

Theorem 4.12 (Fundamental Theorem of Algebra). Let $f(x) = x^n + c_1x^{n-1} + \cdots + c_n$ be a polynomial with $c_i \in \mathbb{C}, n > 0$. Then there exists $a \in \mathbb{C}$ such that $f(a) = 0$.

Proof. Assume f has no root in \mathbb{C} . Define a homotopy

$$F : S^1 \times I \rightarrow S^1, \quad F(e^{2\pi i\theta}, t) = \frac{f(\tan(\frac{\pi t}{2})e^{2\pi i\theta})}{|f(\tan(\frac{\pi t}{2})e^{2\pi i\theta})|}.$$

Then $\text{deg}(F|_{S^1 \times 0}) = 0$ and $\text{deg}(F|_{S^1 \times 1}) = n$. Contradiction. \square

Theorem 4.13 (Borsuk-Ulam). Let $f : S^2 \rightarrow \mathbb{R}^2$. Then $\exists x \in S^2$ such that $f(x) = f(-x)$.

Proof. Assume $f(x) \neq f(-x), \forall x \in S^2$. Define

$$\rho : S^2 \rightarrow S^1, \quad \rho(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

Let D^2 be the upper hemi-sphere of S^2 . It defines a homotopy between constant map and $\rho|_{\partial D^2} : S^1 \rightarrow S^1$, hence $\deg(\rho|_{\partial D^2}) = 0$. On the other hand, $\rho|_{\partial D^2}$ is antipode-preserving: $\rho|_{\partial D^2}(-x) = -\rho|_{\partial D^2}(x)$, hence $\deg(\rho|_{\partial D^2})$ is odd. Contradiction. \square

Corollary 4.14 (Ham Sandwich Theorem). *Let A_1, A_2 be two bounded regions of positive areas in \mathbb{R}^2 . Then there exists a line which cuts each A_i into half of equal areas.*

Proof. Let $A_1, A_2 \subset \mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$. Given $u \in S^2$, let P_u be the plane passing the origin and perpendicular to the unit vector u . Let $A_i(u) = \{p \in A_i | p \cdot u \leq 0\}$. Define the map

$$f : S^2 \rightarrow \mathbb{R}^2, \quad f_i(u) = \text{Area}(A_i(u)).$$

By Borsuk-Ulam, $\exists u$ such that $f(u) = f(-u)$. The intersection $\mathbb{R}^2 \times \{1\} \cap P_u$ gives the required line. \square

5. CLASSIFICATION OF COVERING

Definition 5.1. The universal cover of B is a covering map $p : E \rightarrow B$ with E simply connected.

Theorem 5.2. *Assume B is path connected and locally path connected. Then universal cover of B exists if and only if B is semi-locally simply connected space.*

Definition 5.3. We define the category $\text{Cov}(B)$ of coverings of B

- objects are covering maps
- a morphism between two coverings $p_1 : E_1 \rightarrow B$ and $p_2 : E_2 \rightarrow B$ is a map $f : E_1 \rightarrow E_2$ such that the following diagram commutes

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & \searrow p_1 & \swarrow p_2 \\ & B & \end{array}$$

Definition 5.4. Let B be connected. We define $\text{Cov}_0(B) \subset \text{Cov}(B)$ to be the subcategory whose objects consist of coverings of B which are connected spaces.

Proposition 5.5. *Let B be connected and locally path connected. Then any morphism in $\text{Cov}_0(B)$ is a covering map.*

Definition 5.6. We define the orbit category $\text{Orb}(G)$

- objects consist of (left) coset G/H , where H is a subgroup of G
- morphisms are G -equivariant maps: $G/H_1 \rightarrow G/H_2$.

A morphism $\rho : G/H_1 \rightarrow G/H_2$ is equivalent to an element $\gamma \in G$ such that $H_1 \subset \gamma H_2 \gamma^{-1}$. Then

$$\rho(gH_1) = g\gamma H_2.$$

In particular, G/H_1 and G/H_2 are equivalent if and only if H_1 and H_2 are conjugate subgroups of G .

For convenience, we also introduce the following category

Definition 5.7. We define the category $G\text{-Set}$

- objects consist of sets with G -action
- morphisms are G -equivariant set maps.

Given a covering $p : E \rightarrow B, b \in B$, we find

$$p^{-1}(b) \in \pi_1(B, b)\text{-}\underline{\text{Set}}.$$

Proposition 5.8. *Assume B is path connected and locally path connected. Let $p_1, p_2 \in \text{Cov}(B)$. Then*

$$\text{Hom}_{\text{Cov}(B)}(p_1, p_2) \cong \text{Hom}_{\pi_1(B, b)\text{-}\underline{\text{Set}}}(p_1^{-1}(b), p_2^{-1}(b))$$

Proof. This is a consequence of Lifting Criterion and the Theorem of Uniqueness of lifting. \square

Definition 5.9. Let B be path connected and $p : E \rightarrow B$ be a connected covering. *deck transformation* (or *covering transformation*) of p is a homeomorphism $f : E \rightarrow E$ such that $p \circ f = p$. Let $\text{Aut}(p)$ denote the group of deck transformation.

Note that $\text{Aut}(p)$ acts freely on E by the Uniqueness of Lifting.

Proposition 5.10. *Let B be path connected and $p : E \rightarrow B$ be a connected covering. Then $\text{Aut}(p)$ acts properly discontinuous on E .*

We find that the universal cover E is a $\pi_1(B, b)$ -principal covering.

Corollary 5.11. *Assume B is path connected, locally path connected. Let $p : E \rightarrow B$ be a connected covering, $e \in E, b = p(e) \in B, G = \pi_1(B, b), H = \pi_1(E, e)$. Then*

$$\text{Aut}(p) \cong N_G(H)/H$$

where $N_G(H)$ is the normalizer of H in G .

Proof. By the above proposition,

$$\text{Aut}(p) \cong \text{Hom}_{G\text{-}\underline{\text{Set}}}(G/H, G/H) = N_G(H)/H.$$

\square

Theorem 5.12. *Assume B is path connected, locally path connected and semi-locally simply connected. $b \in B$. Then there exists an equivalence of categories*

$$\boxed{\text{Cov}(B) \simeq \pi_1(B, b)\text{-}\underline{\text{Set}}}.$$

Proof. Let us denote $\pi_1 = \pi_1(B, b)$. Let $\tilde{p} : \tilde{B} \rightarrow B$ be a fixed universal cover of B and $\tilde{b} \in \pi^{-1}(b)$ chosen.

We define the following functors

$$\text{Cov}(B) \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \pi_1\text{-}\underline{\text{Set}}.$$

Let $p : E \rightarrow B$ be a covering, we define

$$F(p) = p^{-1}(b).$$

Let $S \in \pi_1\text{-}\underline{\text{Set}}$, we define

$$G(S) = \tilde{B} \times_{\pi_1} S = \tilde{B} \times S / \sim, \quad \text{where } (e \cdot g, s) \sim (e, g \cdot s), \forall e \in \tilde{B}, s \in S, g \in \pi_1.$$

Here $e \cdot g$ represents the (right) π_1 -action on \tilde{B} . Then we have natural equivalences

$$F \circ G \xrightarrow{\eta} 1, \quad G \circ F \xrightarrow{\tau} 1.$$

Here η is the natural equivalence

$$\eta_S \in \text{Hom}_{\pi_1\text{-Set}}(F \circ G(S), S), \quad \eta_S(e, s) = g \cdot s \quad \text{if } e = \tilde{b} \cdot g.$$

τ is the natural equivalence

$$\tau_E \in \text{Hom}_{\text{Cov}(B)}(p', p) \cong \text{Hom}_{\pi_1\text{-Set}}(p^{-1}(b), p^{-1}(b)), \quad p' : \tilde{B} \times_{\pi_1} p^{-1}(b) \rightarrow B,$$

which is determined by the identity map in $\text{Hom}_{\pi_1\text{-Set}}(p^{-1}(b), p^{-1}(b))$. \square

If we restrict the above theorem to connected coverings, we find an equivalence of categories

$$\boxed{\text{Cov}_0(B) \simeq \text{Orb}(\pi_1(B, b))}.$$

The universal cover $\tilde{B} \rightarrow B$ corresponds to the orbit $\pi_1(B, b)$. For the orbit $\pi_1(B, b)/H$, it corresponds to

$$E = \tilde{B}/H \rightarrow B.$$

We have the following commuting diagram

$$\begin{array}{ccccc} \pi_1(B, b) & \longrightarrow & \tilde{\pi}_1(B, b)/H & \Longrightarrow & \tilde{B} & \xrightarrow{f} & \tilde{B}/H \\ & \searrow & \swarrow & & \searrow & & \swarrow \\ & & 1 & & & & B \end{array}$$

A more intrinsic formulation is as follows. Given a covering $p : E \rightarrow B$, we obtain a transport functor

$$T_p : \Pi_1(B) \rightarrow \underline{\text{Set}}.$$

Given a commuting diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ p_1 \searrow & & \swarrow p_2 \\ & B & \end{array}$$

we find a natural transformation

$$\tau : T_{p_1} \Longrightarrow T_{p_2}, \quad \tau = \{f : p_1^{-1}(b) \rightarrow p_2^{-1}(b) \mid b \in B\}.$$

The above structure can be summarized by a functor

$$\boxed{T : \text{Cov}(B) \rightarrow \text{Fun}(\Pi_1(B), \underline{\text{Set}})}.$$

Theorem 5.13. *Assume B is path connected, locally path connected and semi-locally simply connected. Then*

$$T : \text{Cov}(B) \rightarrow \text{Fun}(\Pi_1(B), \underline{\text{Set}})$$

is an equivalence of categories.

6. SEIFERT-VAN KAMPEN THEOREM

Product.

Definition 6.1. Let \mathcal{C} be a category, $\{A_\alpha\}_{\alpha \in I}$ be a set of objects in \mathcal{C} . Their **product** is an object A in \mathcal{C} together with $\pi_\alpha : A \rightarrow A_\alpha$ satisfying the following universal property: for any X in \mathcal{C} and $f_\alpha : X \rightarrow A_\alpha$, there exists a unique morphism $f : X \rightarrow A$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\exists! f} & A \\ & \searrow f_\alpha & \downarrow \pi_\alpha \\ & & A_\alpha \end{array}$$

The universal property implies that the product is unique up to equivalence if it exists. We denote it by

$$\prod_{\alpha \in I} A_\alpha.$$

Example 6.2.

- Let $S_\alpha \in \underline{\text{Set}}$. $\prod_{\alpha} S_\alpha = \{(s_\alpha) | s_\alpha \in S_\alpha\}$ is the Cartesian product.
- Let $X_\alpha \in \underline{\text{Top}}$. $\prod_{\alpha} X_\alpha$ is the Cartesian product with induced product topology.
- Let $G_\alpha \in \underline{\text{Group}}$. $\prod_{\alpha} G_\alpha$ is the Cartesian product with induced group structure.

Coproduct.

Definition 6.3. Let \mathcal{C} be a category, $\{A_\alpha\}_{\alpha \in I}$ be a set of objects in \mathcal{C} . Their **coproduct** is an object A in \mathcal{C} together with $i_\alpha : A_\alpha \rightarrow A$ satisfying the following universal property: for any X in \mathcal{C} and $f_\alpha : A_\alpha \rightarrow X$, there exists a unique morphism $f : A \rightarrow X$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xleftarrow{\exists! f} & A \\ & \swarrow f_\alpha & \uparrow i_\alpha \\ & & A_\alpha \end{array}$$

The universal property implies that the product is unique up to equivalence if it exists. We denote it by

$$\coprod_{\alpha \in I} A_\alpha.$$

Example 6.4.

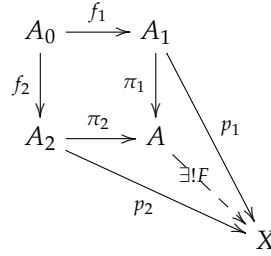
- Let $X_\alpha \in \underline{\text{Top}}$. $\coprod_{\alpha} X_\alpha$ is the disjoint union of topological spaces.
- Let $G_\alpha \in \underline{\text{Group}}$. $\coprod_{\alpha} G_\alpha$ is the free product of groups.

Pushout.

Definition 6.5. Let \mathcal{C} be a category. Given $f_1 : A_0 \rightarrow A_1, f_2 : A_0 \rightarrow A_2$, their **pushout** is an object A together with $\pi_1 : A_1 \rightarrow A, \pi_2 : A_2 \rightarrow A$ such that

- $\pi_1 \circ f_1 = \pi_2 \circ f_2$.
- $p_i : A_i \rightarrow X$ in \mathcal{C} such that $p_1 \circ f_1 = p_2 \circ f_2$, there exists a unique $F : A \rightarrow X$ such that $p_i = F \circ \pi_i$.

It can be described by the following diagram



The universal property implies that the pushout is unique up to equivalence if it exists. We denote it by

$$A_1 \coprod_{A_0} A_2.$$

Example 6.6.

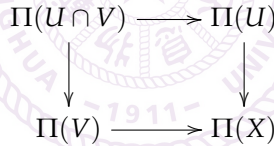
- Let $j_1 : X_0 \rightarrow X_1, j_2 : X_0 \rightarrow X_2$ in Top. Their pushout is the quotient of $X_1 \coprod X_2$ by identifying $j_1(y) \sim j_2(y), y \in X_0$. It glues X_1, X_2 along X_0 using j_1, j_2 .
- Let $\rho_1 : H \rightarrow G_1, \rho_2 : H \rightarrow G_2$ in Group, then

$$G_1 \coprod_H G_2 = (G_1 * G_2) / N$$

where $G_1 * G_2$ is the free product and N is the normal subgroup generated by $\rho_1(h)\rho_2^{-1}(h), h \in H$.

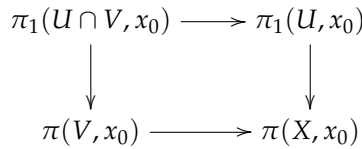
Seifert-van Kampen Theorem.

Theorem 6.7 (Seifert-van Kampen Theorem, Groupoid version). *Let $X = U \cup V$ where $U, V \subset X$ are open. Then the following diagram*



is a pushout in the category Groupoid.

Corollary 6.8 (Seifert-van Kampen Theorem). *Let $X = U \cup V$ where $U, V \subset X$ are open and $U, V, U \cap V$ are path connected. Let $x_0 \in U \cap V$. Then the following diagram*



is a pushout in the category Group.

7. PATH SPACE AND HOMOTOPY FIBER

Path space and loop space.

Definition 7.1. Let $X, Y \in \text{Top}$, we let $C(X, Y) \in \text{Top}$ denote the set of continuous maps from X to Y with the compact open topology. It is also denoted Y^X . For $A \subset X, B \subset Y$, we denote the subspace

$$C(X, A; Y, B) = \{f \in C(X, Y) | f(A) \subset B\}.$$

Theorem 7.2 (Exponential Correspondence). *Let Y be locally compact Hausdorff. Then the evaluation map $C(Y, Z) \times Y \rightarrow Z$ is continuous and we have*

$$\text{Hom}_{\underline{\text{Top}}}(X \times Y, Z) = \text{Hom}_{\underline{\text{Top}}}(X, C(Y, Z)).$$

If furthermore X is Hausdorff, then

$$C(X \times Y, Z) = C(X, C(Y, Z)).$$

Definition 7.3. Let $X \in \underline{\text{Top}}$, we define

- **free path space** $PX = C(I, X)$ and **based path space** $P_x X = C(I, 0; X, x)$;
- **free loop space** $\mathcal{L}X = C(S^1, X)$ and **based loop space** $\Omega_x X = C(S^1, 1; X, x)$ or simply ΩX .

We denote the two maps

$$\begin{array}{ccc} PX & \xrightarrow{p_1} & X \\ p_0 \downarrow & & \\ X & & \end{array}$$

where $p_0(\gamma) = \gamma(0)$ is the start point and $p_1(\gamma) = \gamma(1)$ is the end point of the path γ . It induces

$$p = (p_0, p_1) : PX \rightarrow X \times X.$$

Theorem 7.4. *Let $X \in \underline{\text{Top}}$.*

- (1) $p : PX \rightarrow X \times X$ is a fibration.
- (2) The map $p_0 : PX \rightarrow X$ is a fibration whose fiber at x_0 is $P_{x_0} X$.
- (3) The map $p_1 : P_{x_0} X \rightarrow X$ is a fibration whose fiber at x_0 is $\Omega_{x_0} X$.
- (4) $p_0 : PX \rightarrow X$ is homotopy equivalence. $P_{x_0} X$ is contractible.

Proof. (1) We need to prove the HLP of the diagram

$$\begin{array}{ccc} Y \times \{0\} & \longrightarrow & X^I \\ \downarrow & \nearrow ? & \downarrow p \\ Y \times I & \longrightarrow & X \times X \end{array}$$

Since I is locally compact Hausdorff, this is equivalent to the extension problem

$$\begin{array}{ccc} Y \times \{0\} \times I \cup Y \times I \times \partial I & \longrightarrow & X \\ \downarrow & \nearrow ? & \\ Y \times I \times I & & \end{array}$$

which is easily solved by observing that $Y \times \{0\} \times I \cup Y \times I \times \partial I$ is a deformation retract of $Y \times I \times I$.

(2) follows from the composition of two fibrations

$$\begin{array}{ccc} P_X & \longrightarrow & X \times X \\ & \searrow & \downarrow \\ & & X \end{array}$$

(3) follows from the pull-back diagram

$$\begin{array}{ccc} P_{x_0}X & \longrightarrow & PX \\ \downarrow & & \downarrow \\ X & \xrightarrow{x_0 \times \text{id}} & X \times X. \end{array}$$

(4) follows from retracting the path. □

Definition 7.5. Let $f : X \rightarrow Y$. We define the **mapping path space** P_f by the pull-back diagram

$$\begin{array}{ccc} P_f & \longrightarrow & Y^I \\ \downarrow & & \downarrow p_1 \\ X & \xrightarrow{f} & Y \end{array}$$

An element of P_f is a pair (x, γ) where γ is a path in Y that ends at $f(x)$.

Let $\iota : X \hookrightarrow P_f$ represent the constant paths and $q_0 : P_f \rightarrow Y$ be the start point of the path. We have

$$\begin{array}{ccc} X & \xrightarrow{\iota} & P_f \\ & \searrow f & \downarrow q_0 \\ & & Y \end{array}$$

Theorem 7.6. $\iota : X \rightarrow P_f$ is strong deformation retract (hence homotopy equivalence) and $q_0 : P_f \rightarrow Y$ is a fibration. In particular, any map $f : X \rightarrow Y$ is a composition of a homotopy equivalence with a fibration.

Proof. The pull-back diagram

$$\begin{array}{ccc} P_f & \longrightarrow & Y^I \\ \downarrow & & \downarrow \\ Y \times X & \xrightarrow{\text{id} \times f} & Y \times Y \end{array}$$

implies that $P_f \rightarrow Y \times X$ is a fibration. Since $Y \times X \rightarrow Y$ is also a fibration, its composition q_1 is a fibration. □

This theorem says that in \mathbf{hTop} , every map is equivalent to a fibration.

Fiber homotopy.

Definition 7.7. Let $p_1 : E_1 \rightarrow B$ and $p_2 : E_2 \rightarrow B$ be two fibrations. A fiber map from p_1 to p_2 is a map $f : E_1 \rightarrow E_2$ such that $p_1 = p_2 \circ f$:

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & \searrow p_1 & \swarrow p_2 \\ & & B \end{array}$$

Two fiber maps $f_0, f_1 : p_1 \rightarrow p_2$ are said to be **fiber homotopic**

$$f_0 \simeq_B f_1$$

if there exists a homotopy $F : E_1 \times I \rightarrow E_2$ from f_0 to f_1 such that $F(-, t)$ is a fiber map for each $t \in I$. $f : p_1 \rightarrow p_2$ is a **fiber homotopic equivalence** if there exists $g : p_2 \rightarrow p_1$ such that both $f \circ g$ and $g \circ f$ are fiber homotopic to identity maps.

Proposition 7.8. Let $p_1 : E_1 \rightarrow B$ and $p_2 : E_2 \rightarrow B$ be two fibrations and $f : E_1 \rightarrow E_2$ be a fiber map. Assume $f : E_1 \rightarrow E_2$ is a homotopy equivalence, then f is a fiber homotopy equivalence. In particular, $f : p_1^{-1}(b) \rightarrow p_2^{-1}(b)$ is a homotopy equivalence for any $b \in B$.

Proof. We only need to prove that for any fiber map $f : E_1 \rightarrow E_2$ which is a homotopy equivalence, there is a fiber map $g : E_2 \rightarrow E_1$ such that $g \circ f \simeq_B 1$. In fact, such a g is also a homotopy equivalence and we can find $h : E_1 \rightarrow E_2$ such that $h \circ g \simeq_B 1$. Then $f \simeq_B h \circ g \circ f \simeq_B h$, which implies $f \circ g \simeq_B 1$ as well.

Let $g : E_2 \rightarrow E_1$ be a homotopy inverse of f , so $g = f^{-1}$ in \mathbf{hTop} . We first show that we can choose a homotopy class of g such that g is a fiber map. In fact, consider the diagram

$$\begin{array}{ccc} & & E_1 \\ & \nearrow g & \downarrow p_1 \\ E_2 & \xrightarrow{p_1 \circ g} & B \\ & \searrow p_2 & \\ & & \end{array}$$

Since $g \circ p_1 = g \circ f \circ p_2$ is homotopic to p_2 and p_1 is a fibration, we can lift the above homotopy to a homotopy from g to $g' : E_2 \rightarrow E_1$ which lifts p_2 . Then g' is a fiber map as required.

We further reduce the problem to prove the following

“Claim”: Let $p : E \rightarrow B$ be a fibration and $f : E \rightarrow E$ is a fiber map that is homotopic to 1_E , then there is a fiber map $h : E \rightarrow E$ such that $h \circ f \simeq_B 1$.

In fact, let $f : E_1 \rightarrow E_2$ as in the proposition, $g : E_2 \rightarrow E_1$ be a fiber map such that $g \circ f \simeq 1$ as chosen above. The “Claim” implies that we can find a fiber map $h : E_1 \rightarrow E_1$ such that $h \circ g \circ f \simeq_B 1$. The fiber map $\tilde{g} = h \circ g$ has the required property that $\tilde{g} \circ f \simeq_B 1$.

Now we prove the “Claim”. Let F be a homotopy from f to 1_E and $G = p \circ F$. Since p is fibration, we can construct a homotopy H that starts from 1_E and lifts G . Here is the picture

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ \Downarrow F & & \Downarrow H \\ E & \xrightarrow{1_E} & E \\ \parallel & & \parallel \\ E & \xrightarrow{p} & B \\ \Downarrow G & & \Downarrow G \\ E & \xrightarrow{p} & B \end{array} \quad \begin{array}{ccc} E & \xrightarrow{1_E} & E \\ \Downarrow H & & \Downarrow h \\ E & \xrightarrow{h} & E \\ \parallel & & \parallel \\ E & \xrightarrow{p} & B \\ \Downarrow G & & \Downarrow G \\ E & \xrightarrow{p} & B \end{array}$$

Combining these two homotopy we find a homotopy \tilde{F} from $h \circ f$ to 1_E that lifts the following homotopy

$$\tilde{G} : E \times I \rightarrow B, \quad \tilde{G}(-, t) = \begin{cases} G(-, 2t) & 0 \leq t \leq 1/2 \\ G(-, 2-2t) & 1/2 \leq t \leq 1 \end{cases}$$

Here is the picture

$$\begin{array}{ccc} E & \xrightarrow{h \circ f} & E \\ \Downarrow \tilde{F} & & \Downarrow p \\ E & \xrightarrow{1_E} & E \\ \parallel & & \parallel \\ E & \xrightarrow{p} & B \\ \Downarrow \tilde{G} & & \Downarrow p \\ E & \xrightarrow{p} & B \end{array}$$

It is easy to see that we can construct a homotopy $K : E \times I \times I \rightarrow B$ such that

$$K(-, u, 0) = \tilde{G}(-, u), \quad K(-, u, 1) = p(-) = K(-, 0, t) = K(-, 1, t), \quad \forall u, t \in I.$$

Since p is a fibration, we can find a lift $\tilde{K} : E \times I \times I \rightarrow E$ of K such that

$$\tilde{K}(-, u, 0) = \tilde{F}(-, u).$$

Then we have the following fiber homotopy

$$h \circ f = \tilde{K}(-, 0, 0) \simeq_B \tilde{K}(-, 0, 1) \simeq_B \tilde{K}(-, 1, 1) \simeq_B \tilde{K}(-, 1, 0) = 1_E.$$

□

Homotopy fiber.

Definition 7.9. Let $f : X \rightarrow Y$, we define its **homotopy fiber** over $y \in Y$ to be the fiber of $P_f \rightarrow Y$ over y .

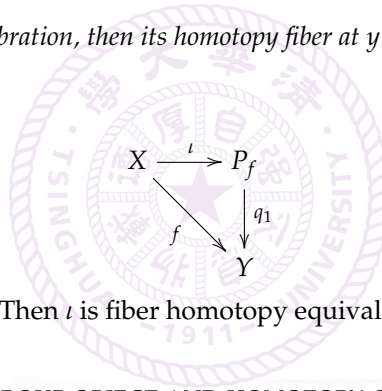
If Y is path connected, then all homotopy fibers are homotopic equivalent since $P_f \rightarrow Y$ is a fibration. In this case we will usually write the following diagram

$$\begin{array}{ccc} F & \longrightarrow & X \\ & & \downarrow f \\ & & Y \end{array}$$

where F denotes the homotopy fiber.

Proposition 7.10. If $f : X \rightarrow Y$ is a fibration, then its homotopy fiber at y is homotopy equivalent to $f^{-1}(y)$.

Proof. We have



where ι is a homotopy equivalence. Then ι is fiber homotopy equivalence. □

8. GROUP OBJECT AND HOMOTOPY GROUP

Definition 8.1. We define the category $\underline{\text{Top}}_*$ of pointed topological space where

- An object (X, x_0) is a topological space X with a based point $x_0 \in X$
- morphisms are based continuous maps that map based point to based point.

Definition 8.2. Let $X, Y \in \underline{\text{Top}}_*$ be two pointed spaces. A based homotopy between two based maps $f_0, f_1 : X \rightarrow Y$ is a homotopy between f_0, f_1 relative to the base points. We denote $[X, Y]_0$ to be based homotopy classes of based maps. We define the category $\underline{\text{hTop}}_*$ by the quotient of $\underline{\text{Top}}_*$ where

$$\text{Hom}_{\underline{\text{hTop}}_*}(X, Y) = [X, Y]_0.$$

The loop space defines a functor

$$\Omega : \underline{\text{Top}}_* \rightarrow \underline{\text{Top}}_*, \quad X \rightarrow \Omega X$$

where ΩX is based at the constant loop to the base point of X . It is easy to see that it also defines

$$\Omega : \underline{\text{hTop}}_* \rightarrow \underline{\text{hTop}}_*.$$

Definition 8.3. Let \mathcal{C} be a category with finite product and terminal object \star . A **group object** in \mathcal{C} is an object G in \mathcal{C} together with morphisms

$$\mu : G \times G \rightarrow G, \quad \eta : G \rightarrow G, \quad \epsilon : \star \rightarrow G$$

such that the following diagrams commute

(1) associativity:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{1 \times \mu} & G \times G \\ \mu \times 1 \downarrow & & \downarrow \mu \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

(2) unit:

$$\begin{array}{ccccc} G \times \star & \xrightarrow{1 \times \epsilon} & G \times G & \xleftarrow{\epsilon \times 1} & \star \times G \\ & \searrow & \downarrow \mu & \swarrow & \\ & & G & & \end{array}$$

(3) inverse

$$\begin{array}{ccccc} G & \xrightarrow{1 \times \eta} & G \times G & \xleftarrow{\eta \times 1} & G \\ \downarrow & & \downarrow \mu & & \downarrow \\ \star & \xrightarrow{\epsilon} & G & \xleftarrow{\epsilon} & \star \end{array}$$

μ is called the multiplication, η is called the inverse, ϵ is called the unit.

Example 8.4.

- Group objects in \mathbf{Set} are groups.
- Group objects in \mathbf{Top} are topological groups.
- Group objects in \mathbf{hTop} are called H-groups.

Proposition 8.5. Let \mathcal{C} be a category with finite product and terminal object. Let G be a group object. Then

$$\mathbf{Hom}(-, G) : \mathcal{C} \rightarrow \mathbf{Group}$$

defines a contravariant functor from \mathcal{C} to \mathbf{Group} .

In the category \mathbf{Top}_\star and \mathbf{hTop}_\star , product exists and is given by

$$(X, x_0) \times (Y, y_0) = (X \times Y, x_0 \times y_0).$$

Initial objects and terminal objects are a single pointed space.

Theorem 8.6. Let $X \in \mathbf{Top}_\star$. Then ΩX is a group object in \mathbf{hTop}_\star .

Corollary 8.7. For any $X, Y \in \mathbf{Top}_\star$, $[Y, \Omega X]_0$ forms a group.

Definition 8.8. Let $(X, x_0) \in \mathbf{Top}_\star$. We define its **suspension** ΣX by the quotient of $X \times I$

$$\boxed{\Sigma X = X \times I / X \times \partial I \cup x_0 \times I}.$$

It defines functors

$$\Sigma : \mathbf{Top}_\star \rightarrow \mathbf{Top}_\star, \quad \mathbf{hTop}_\star \rightarrow \mathbf{hTop}_\star.$$

Example 8.9. $\Sigma S^n \cong S^{n+1}$ are homeomorphic for any $n \geq 0$.

Definition 8.10. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be two functors. (F, G) is called **adjoint pair** if there are isomorphisms

$$\tau : \text{Hom}_{\mathcal{D}}(FX, Y) \cong \text{Hom}_{\mathcal{C}}(X, GY), \quad \forall X \in \mathcal{C}, Y \in \mathcal{D}$$

which are natural for all X, Y . In other words, τ defines a natural equivalence between two functors

$$\text{Hom}_{\mathcal{D}}(F-, -), \text{Hom}_{\mathcal{C}}(-, G-) : \mathcal{C}^{op} \times \mathcal{D} \rightarrow \underline{\text{Set}}.$$

$F(G)$ is called the left (right) adjoint of $G(F)$, denoted by $F \dashv G$.

Example 8.11. Let Y be locally compact Hausdorff, then $- \times Y$ is left adjoint to $C(Y, -)$.

Proposition 8.12. (Σ, Ω) is an adjoint pair in $\underline{\text{Top}}_*$ and $\underline{\text{hTop}}_*$.

Definition 8.13. Let $(X, x_0) \in \underline{\text{Top}}_*$. We define the n -th **homotopy group**

$$\pi_n(X, x_0) = [S^n, X]_0.$$

Sometimes we simply denote it by $\pi_n(X)$.

For $n \geq 1$, we know that

$$\pi_n(X) = [\Sigma S^{n-1}, X]_0 = [S^{n-1}, \Omega X]$$

which is a group since ΩX is a group object.

Proposition 8.14. $\pi_n(X)$ is abelian if $n \geq 2$.

Proposition 8.15. Let X be path connected. There is a natural functor

$$T : \Pi_1(X) \rightarrow \underline{\text{Group}}$$

which sends x_0 to $\pi_n(X, x_0)$. In particular, there is a natural action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ and all $\pi_n(X, x_0)$'s are isomorphic for different choices of x_0 .

Proposition 8.16. Let $f : X \rightarrow Y$ be homotopy equivalence. Then

$$f_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0))$$

is a group isomorphism.

9. EXACT PUPPE SEQUENCE

Definition 9.1. A sequence of maps of sets with base points

$$(A, a_0) \xrightarrow{f} (B, b_0) \xrightarrow{g} (C, c_0)$$

is said to be **exact** at B if $\text{im}(f) = \ker(g)$ where $\text{im}(f) = f(A), \ker(g) = g^{-1}(c_0)$. A sequence

$$\cdots \rightarrow A_{n+1} \rightarrow A_n \rightarrow A_{n-1} \rightarrow \cdots$$

is called an exact sequence if it is exact at every A_i .

Definition 9.2. A sequence of maps in $\underline{\text{hTop}}_*$

$$\cdots \rightarrow X_{n+1} \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots$$

is called exact if for any $Y \in \underline{\text{hTop}}_*$, the following sequence of pointed sets is exact

$$\cdots \rightarrow [Y, X_{n+1}]_0 \rightarrow [Y, X_n]_0 \rightarrow [Y, X_{n-1}]_0 \rightarrow \cdots$$

Definition 9.3. Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a map in $\underline{\text{Top}}_*$. We define its homotopy fiber F_f in $\underline{\text{hTop}}_*$ by the pull-back diagram

$$\begin{array}{ccc} F_f & \longrightarrow & P_{y_0}Y \\ \pi \downarrow & & \downarrow p_1 \\ X & \xrightarrow{f} & Y. \end{array} \quad F_f = \{(x, \gamma) \in X \times PY \mid \gamma(0) = y_0, \gamma(1) = f(x)\}$$

Note that F_f is precisely the fiber of $P_f \rightarrow Y$ over y_0 . We have the following commutative diagram

$$\begin{array}{ccc} f^{-1}(y_0) & \longrightarrow & F_f \\ \downarrow & & \downarrow \\ X & \xrightarrow{\iota} & P_f \\ & \searrow f & \swarrow \\ & & Y \end{array} \quad \begin{array}{ccc} f^{-1}(y_0) & \longrightarrow & F_f \\ \downarrow & \nearrow \pi & \\ X & & \end{array}$$

When f is a fibration, ι is a fiber homotopy equivalence, hence $f^{-1}(y_0) \rightarrow F_f$ is a homotopy equivalence.

Lemma 9.4. *The sequence*

$$F_f \xrightarrow{\pi} X \xrightarrow{f} Y$$

is exact at X in $\underline{\text{hTop}}_$.*

Proof. We first observe that $f \circ \pi$ factors through $P_{y_0}Y$ which is contractible. Therefore $f \circ \pi$ is null homotopy. Let $Z \in \underline{\text{hTop}}_*$. Consider

$$[Z, F_f]_0 \xrightarrow{\pi_*} [Z, X]_0 \xrightarrow{f_*} [Z, Y]_0.$$

Since $f \circ \pi$ is null homotopic, we have $\text{im } \pi_* \subset \ker f_*$.

Let $g : Z \rightarrow X$ such that $[g]_0 \in \ker f_*$. Let G be a homotopy of $f \circ g$ to the trivial map. G defines a lifting

$$\begin{array}{ccc} & & P_{y_0}Y \\ & \nearrow G & \downarrow p_1 \\ Z & \xrightarrow{g} & X \xrightarrow{f} Y \end{array}$$

By the definition of pull-back, the pair (G, g) defines a map to F_f such that the following diagram commutes

$$\begin{array}{ccc} & F_f & \longrightarrow P_{y_0}Y \\ & \nearrow & \downarrow \pi \\ Z & \xrightarrow{g} & X \xrightarrow{f} Y \end{array} \quad \begin{array}{ccc} & & \downarrow p_1 \\ & & Y \end{array}$$

This implies $[g]_0 \in \text{im } \pi_*$. Therefore $\ker f_* \subset \text{im } \pi_*$. □

The fiber of F_f over x_0 is precisely ΩY . We find the following sequence of pointed maps

$$\Omega X \xrightarrow{\Omega f} \Omega Y \rightarrow F_f \xrightarrow{\pi} X \xrightarrow{f} Y.$$

Lemma 9.5. *The sequence $\Omega X \xrightarrow{\Omega f} \Omega Y \rightarrow F_f \xrightarrow{\pi} X \xrightarrow{f} Y$ is exact in $\underline{\text{hTop}}_*$.*

Proof. We construct the following diagram in $\underline{\text{hTop}}_*$ with all vertical arrows homotopy equivalences

$$\begin{array}{ccccccc}
 \Omega X & \longrightarrow & \Omega Y & \longrightarrow & F_f & \xrightarrow{\pi} & X & \xrightarrow{f} & Y \\
 \downarrow j' & & \downarrow j & & \downarrow & & \downarrow & & \downarrow \\
 F_{\pi'} & \xrightarrow{\pi''} & F_{\pi} & \xrightarrow{\pi'} & F_f & \xrightarrow{\pi} & X & \xrightarrow{f} & Y
 \end{array}$$

Since $F_f \xrightarrow{\pi} X$ is a fibration with fiber ΩY , we have a commutative diagram

$$\begin{array}{ccc}
 \Omega Y = \pi^{-1}(x_0) & \xrightarrow{j} & F_{\pi} \\
 & \searrow & \downarrow \\
 & & F_f
 \end{array}$$

where j is a homotopy equivalence. This explains the second square above.

Similarly, the fiber of the fibration $F_{\pi} \rightarrow F_f$ is ΩX . We find the following diagram

$$\begin{array}{ccc}
 \Omega X & & \\
 \downarrow j' & \searrow k & \\
 F_{\pi'} & \xrightarrow{\pi''} & F_{\pi}
 \end{array}$$

$F_{\pi'}$ is the homotopy fiber, and j' is homotopy equivalence as before. However, the following diagram

$$\begin{array}{ccc}
 \Omega X & \xrightarrow{\Omega f} & \Omega Y \\
 \searrow k & & \downarrow j \\
 & & F_{\pi}
 \end{array}$$

is NOT commutative in $\underline{\text{Top}}_*$.

However, it is easy to see that $j \circ \Omega f$ is homotopic to k , so this diagram is commutative in $\underline{\text{hTop}}_*$. Therefore

$$\begin{array}{ccc}
 \Omega X & \xrightarrow{\Omega f} & \Omega Y \\
 \downarrow j' & & \downarrow j \\
 F_{\pi'} & \xrightarrow{\pi''} & F_{\pi}
 \end{array}$$

is commutative in $\underline{\text{hTop}}_*$.

The lemma follows from the above commutative diagram in $\underline{\text{hTop}}_*$ and that j, j' are homotopy equivalence. \square

Lemma 9.6. Let $X_1 \rightarrow X_2 \rightarrow X_3$ be exact in $\underline{\text{hTop}}_*$, then so is $\Omega X_1 \rightarrow \Omega X_2 \rightarrow \Omega X_3$.

Proof. Use the fact that Ω is right adjoint to the suspension Σ . \square

The following Theorem is a direct consequence of the above Lemmas.

Theorem 9.7 (Exact Puppe Sequence). Let $f : X \rightarrow Y$ in $\underline{\text{Top}}_*$. Then the following sequence is exact in $\underline{\text{hTop}}_*$

$$\cdots \rightarrow \Omega^2 Y \rightarrow \Omega F_f \rightarrow \Omega X \rightarrow \Omega Y \rightarrow F_f \rightarrow X \rightarrow Y.$$

Theorem 9.8. Let $\pi : E \rightarrow B$ be a map in Top_* . Assume π is fibration whose fiber over the base point is F . Then we have the following exact sequence of homotopy groups

$$\cdots \rightarrow \pi_n(F) \rightarrow \pi_n(E) \rightarrow \pi_n(B) \rightarrow \pi_{n-1}(F) \rightarrow \cdots \rightarrow \pi_0(E) \rightarrow \pi_0(B)$$

where all homotopy groups are understood to have based points on the relevant spaces.

Proof. Since π is a fibration, F is homotopy equivalent to F_π . Apply $[S^0, -]$ to the Puppe Sequence. □

The following proposition gives a criterion for fibration

Theorem 9.9. Let $p : E \rightarrow B$ with B paracompact Hausdorff. Assume there exists an open cover $\{U_\alpha\}$ of B such that $p^{-1}(U_\alpha) \rightarrow U_\alpha$ is a fibration. Then p is a fibration.

Corollary 9.10. Let $p : E \rightarrow B$ be a fiber bundle with B paracompact Hausdorff. Then p is a fibration.

Proposition 9.11. If $i < n$, then $\pi_i(S^n) = 0$.

Example 9.12. We have the Hopf fibration $S^3 \rightarrow S^2$ with fiber S^1 . Its associated exact sequence of homotopy groups implies

$$\pi_2(S^2) \cong \mathbb{Z}, \quad \pi_n(S^3) \cong \pi_n(S^2) \text{ for } n \geq 3.$$

10. COFIBRATION

Cofibration.

Definition 10.1. A map $i : A \rightarrow X$ is said to have the **homotopy extension property** (HEP) with respect to Y if for any maps $f : X \rightarrow Y$ and $F : A \rightarrow Y^I$ such that $p_0 \circ F = f \circ i$, there exists a map $\tilde{F} : X \rightarrow Y^I$ such that the following diagram commutes

$$\begin{array}{ccc} Y & \xleftarrow{f} & X \\ p_0 \uparrow & \exists \tilde{F} \nearrow & \uparrow i \\ Y^I & \xleftarrow{F} & A \end{array}$$

Definition 10.2. A map $i : A \rightarrow X$ is called a **cofibration** if it has HEP for any spaces.

The notion of cofibration is dual to that of the fibration. Fibration is defined by the HLP of the diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & E \\ \downarrow & \exists \tilde{F} \nearrow & \downarrow p \\ Y \times I & \xrightarrow{F} & B \end{array}$$

If we reverse the arrows and observe that $Y \times I$ is dual to the path space Y^I via the adjointness of $(-) \times I$ and $(-)^I$, we arrive at HEP.

Definition 10.3. Let $f : A \rightarrow X$. We define its **mapping cylinder** M_f by the push-out

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & A \times I \\ f \downarrow & & \downarrow \\ X \times \{0\} & \longrightarrow & M_f \end{array}$$

The HEP of $i : A \rightarrow X$ is equivalent to the property of filling the commutative diagram

$$\begin{array}{ccc} M_i & \longrightarrow & Y \\ \downarrow & \nearrow & \\ X \times I & & \exists? \end{array}$$

It is enough to consider $Y = M_i$ to check cofibration by the universal property of push-out. .

Proposition 10.4. Let $i : A \rightarrow X$ and $j : M_i \rightarrow X \times I$ be the above map. Then i is a cofibration if and only there exists $r : X \times I \rightarrow M_i$ such that $r \circ j = 1_{M_i}$.

Proposition 10.5. Let $i : A \rightarrow X$ be a cofibration. Then i is a homeomorphism to its image (i.e. embedding). If furthermore X is Hausdorff. Then i has closed image (i.e. closed embedding).

Proof. Use the retraction in the previous proposition

$$\begin{array}{ccc} M_i & \xrightarrow{1_{M_i}} & M_i \\ \downarrow & \nearrow & \\ X \times I & & \exists? \end{array}$$

□

Lemma 10.6. Let A be a closed subspace of X . Then the inclusion map $i : A \subset X$ is a cofibration if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Proof. If i is closed embedding, then M_i is homeomorphic to the subspace $X \times \{0\} \cup A \times I$ of $X \times I$. □

Remark 10.7. This lemma still holds if we only assume A is a subspace without closeness condition. It can be shown that if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$, then M_i is again homeomorphic to the subspace $X \times \{0\} \cup A \times I$ of $X \times I$. This homeomorphism may fail without the assumption of the existence of retract.

Example 10.8. The inclusion $S^{n-1} \hookrightarrow D^n$ is a cofibration.

Definition 10.9. Let A be a subspace of X . We say (X, A) is **cofibered** if the inclusion $A \subset X$ is a cofibration.

Definition 10.10. Let A be a subspace of X . A is called a **neighborhood deformation retract (NDR)** if there exists a continuous map $u : X \rightarrow I$ with $A = u^{-1}(0)$ and a homotopy $H : X \times I \rightarrow X$ such that

$$\begin{cases} H(x, 0) = x & \forall x \in X \\ H(a, t) = a & \text{if } (a, t) \in A \times I \\ H(x, 1) \in A & \text{if } u(x) < 1 \end{cases}$$

Note that if A is a NDR of X , then A is a strong deformation retract of the open subset $u^{-1}([0, 1))$ of X .

Theorem 10.11. Let A be a closed subspace of X . Then the following conditions are equivalent

- (1) (X, A) is a cofibered pair.
- (2) A is a NDR of X .
- (3) $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.
- (4) $X \times \{0\} \cup A \times I$ is a strong deformation retract of $X \times I$.

Proposition 10.12. Let $i : A \rightarrow X$ be a cofibration, $f : A \rightarrow B$ is a map. Consider the push-out

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ i \downarrow & & \downarrow j \\ X & \longrightarrow & Y \end{array}$$

Then $j : B \rightarrow Y$ is also a cofibration.

Definition 10.13. Let $i : A \rightarrow X, j : A \rightarrow Y$ be cofibrations. A map $f : X \rightarrow Y$ is called a cofiber map if the following diagram commutes

$$\begin{array}{ccc} & A & \\ i \swarrow & & \searrow j \\ X & \xrightarrow{f} & Y \end{array}$$

A cofiber homotopy between two cofiber maps $f, g : X \rightarrow Y$ is a homotopy of cofiber maps between f and g . Cofiber homotopy equivalence is defined similarly.

Proposition 10.14. Let $i : A \rightarrow X, j : A \rightarrow Y$ be cofibrations. Let $f : X \rightarrow Y$ be a cofiber map. Assume f is a homotopy equivalence. Then f is a cofiber homotopy equivalence.

Let $f : A \rightarrow X$ be a map. Consider the diagram of mapping cylinder

$$\begin{array}{ccc} A & \xrightarrow{i_0} & A \times I \\ f \downarrow & & \downarrow \\ X & \longrightarrow & M_f \end{array}$$

There is a natural commuting diagram

$$\begin{array}{ccc} & A & \\ i_1 \swarrow & & \searrow f \\ M_f & \xrightarrow{r} & X \end{array}$$

Here $i_1(a) = (a, 1), r(a, t) = f(a), r(x, 0) = x$. It is easy to see that r is a homotopy equivalence. Moreover, A is a closed subspace of M_f and $M_f \times \{0\} \cup A \times I$ is a retract of $M_f \times I$. Therefore i_1 is a cofibration.

We arrive at the dual result of fibrations: any map $f : A \rightarrow X$ can be factored as $f = r \circ i_1$ where i_1 is a cofibration and r is a homotopy equivalence. Moreover, if f is a cofibration, then $r : M_f \rightarrow X$ is a cofiber homotopy equivalence.

Cofiber exact sequence.

Now we work with the category $\underline{\text{Top}}_*$ and $\underline{\text{hTop}}_*$.

Definition 10.15. Let $(X, x_0) \in \underline{\text{Top}}_*$. We define its **cone** in $\underline{\text{Top}}_*$ by

$$CX = X \wedge I = X \times I / X \times \{0\} \cup x_0 \times I.$$

Given $f : X \rightarrow Y$ in $\underline{\text{Top}}_*$, we define its **homotopy cofiber** C_f by the push-out

$$\begin{array}{ccc} X & \xrightarrow{i_1} & CX \\ f \downarrow & & \downarrow \\ Y & \xrightarrow{j} & C_f \end{array}$$

where $i_1(x) = (x, 1)$.

The closed embedding i_1 is a cofibration. Therefore $j : Y \rightarrow C_f$ is also a cofibration. Note that the quotient of C_f by Y is precisely ΣX . We can extend the above maps by

$$X \longrightarrow Y \longrightarrow C_f \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma C_f \longrightarrow \Sigma^2 X \longrightarrow \dots$$

Definition 10.16. A sequence of maps in $\underline{\text{hTop}}_*$

$$\dots \rightarrow X_{n+1} \rightarrow X_n \rightarrow X_{n-1} \rightarrow \dots$$

is called **co-exact** if for any $Y \in \underline{\text{hTop}}_*$, the following sequence of pointed sets is exact

$$\dots \rightarrow [X_{n-1}, Y]_0 \rightarrow [X_n, Y]_0 \rightarrow [X_{n+1}, Y]_0 \rightarrow \dots$$

Theorem 10.17 (Co-exact Puppe Sequence). *Let $f : X \rightarrow Y$ in $\underline{\text{Top}}_*$. The following sequence is co-exact in $\underline{\text{hTop}}_*$*

$$X \longrightarrow Y \longrightarrow C_f \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma C_f \longrightarrow \Sigma^2 X \longrightarrow \dots$$

Proposition 10.18. *Let $i : A \rightarrow X$ be a cofibration. Then the natural map*

$$\bar{r} : C_f \rightarrow X/A$$

is a homotopy equivalence. In other words, cofiber is homotopy equivalent to the homotopy cofiber.

Theorem 10.19. *Let $i : A \rightarrow X$ be a cofibration. The following sequence is co-exact in $\underline{\text{hTop}}_*$*

$$A \longrightarrow X \longrightarrow X/A \longrightarrow \Sigma A \longrightarrow \Sigma X \longrightarrow \Sigma(X/A) \longrightarrow \Sigma^2 A \longrightarrow \dots$$

11. CW COMPLEX

D^n denotes the n -**disk** and $e^n = D^n - \partial D^n = D^n - S^{n-1}$ denotes the open disk called n -**cell**.

Definition 11.1. A **cell decomposition** of a space X is a family $\mathcal{E} = \{e_\alpha^n \mid \alpha \in J_n\}$ of subspaces of X such that each e_α^n is a n -cell and we have a disjoint union of sets

$$X = \coprod e_\alpha^n.$$

The n -**skeleton** of X is the subspace

$$X^n = \coprod_{\alpha \in J_m, m \leq n} e_\alpha^m.$$

Definition 11.2. A **CW complex** is a pair (X, \mathcal{E}) of a Hausdorff space X with a cell decomposition such that

- (1) **Characteristic map:** for each n -cell e_α^n , there is a characteristic map $\Phi_{e_\alpha^n} : D^n \rightarrow X$ such that the restriction of $\Phi_{e_\alpha^n}$ to $D^n - S^{n-1}$ is a homeomorphism to e_α^n and $\Phi_{e_\alpha^n}(S^{n-1}) \subset X^{n-1}$.

- (2) **Closure finiteness:** for any cell $e \in \mathcal{E}$ the closure \bar{e} intersects only a finite number of other cells in \mathcal{E} .
 (3) **Weak topology:** a subset $A \subset X$ is closed if and only if $A \cap \bar{e}$ is closed in \bar{e} for each $e \in \mathcal{E}$.

We say X is n -dim CW complex if the maximal dimension of cells in \mathcal{E} is n (n could be ∞).

Note that the Hausdorff property of X implies that $\bar{e} = \Phi_e(D^n)$ for each cell $e \in \mathcal{E}$. The surjective map $\Phi_e : D^n \rightarrow \bar{e}$ is a quotient since D^n is compact and \bar{e} is Hausdorff. Let us denote the full characteristic maps

$$\Phi : \coprod_{e \in \mathcal{E}} D^n \xrightarrow{\coprod \Phi_e} X.$$

Then the weak topology implies that Φ is a quotient map. This implies the following proposition.

Proposition 11.3. *Let (X, \mathcal{E}) be a CW complex. Then $f : X \rightarrow Y$ is continuous if and only if $f \circ \Phi_e$ is continuous for each $e \in \mathcal{E}$.*

Proposition 11.4. *Let (X, \mathcal{E}) be a CW complex. Then any compact subspace of X meets only finite many cells in \mathcal{E} .*

Example 11.5. $\mathbb{R}^n, S^n, CP^n, HP^n, S^\infty, CP^\infty, HP^\infty$.

Definition 11.6. A subcomplex (X', \mathcal{E}') of the CW complex (X, \mathcal{E}) is a closed subspace $X' \subset X$ with a cell decomposition $\mathcal{E}' \subset \mathcal{E}$. We will just write $X' \subset X$ when the cell decomposition is clear. We will also write $X' = |\mathcal{E}'|$. Equivalently, a subcomplex is described by a subset $\mathcal{E}' \subset \mathcal{E}$ such that

$$e_1 \in \mathcal{E}', e_2 \in \mathcal{E}, \bar{e}_1 \cap e_2 \neq \emptyset \implies e_2 \in \mathcal{E}'.$$

Example 11.7. The n -skeleton X^n is a subcomplex of X of dimension $\leq n$.

Definition 11.8. Given $f : S^{n-1} \rightarrow X$. Consider the push-out

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ D^n & \xrightarrow{\Phi_f} & D^n \amalg_f X \end{array}$$

We say $D^n \amalg_f X$ is obtained by attaching an n -cell to X . Φ_f is called the characteristic map of the attached n -cell. More generally, if we have a set of maps $f_\alpha : S^{n-1} \rightarrow X$, the push-out

$$\begin{array}{ccc} \coprod_\alpha S^{n-1} & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \coprod_\alpha D^n & \xrightarrow{\Phi_f} & (\coprod D^n) \amalg_f X \end{array} \quad f = \coprod f_\alpha$$

is called X with n -cells attached.

Proposition 11.9. *Let (X, \mathcal{E}) be a CW complex, and $\mathcal{E} = \coprod \mathcal{E}^n$ where \mathcal{E}^n is the set of n -cells. Then the diagram*

$$\begin{array}{ccc} \coprod_{e \in \mathcal{E}^n} S^{n-1} & \xrightarrow{\partial \Phi^n} & X^{n-1} \\ \downarrow & & \downarrow \\ \coprod_{e \in \mathcal{E}^n} D^n & \xrightarrow{\Phi^n} & X^n \end{array} \quad \Phi^n = \coprod_{e \in \mathcal{E}^n} \Phi_e$$

is a push-out. In particular, X^n is obtained from X^{n-1} by attaching n -cells in X .

Proof. This follows from the fact that X^{n-1} is a closed subspace of X^n and the weak topology. □

The converse is also true. The next proposition can be viewed as an alternate definition of CW complex.

Proposition 11.10. *Suppose we have a sequence of spaces*

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \dots \subset X^n \subset X^{n+1} \subset \dots$$

where X^n is obtained from X^{n-1} by attaching n -cells. Let $X = \bigcup_{n \geq 0} X^n$ be the union with the weak topology: $A \subset X$ is closed if and only if $A \cap X^n$ is closed in X^n for each n . Then X is a CW complex.

Proof. The nontrivial part is to show that X is Hausdorff. □

Definition 11.11. Let A be a subspace of X . A CW decomposition of (X, A) consists of a sequence

$$A = X^{-1} \subset X^0 \subset X^1 \subset \dots \subset X$$

such that X^n is obtained from X^{n-1} by attaching n -cells and X carries the weak topology with respect to the subspaces X^n . The pair (X, A) is called a **relative CW complex**.

Note that for a relative CW complex (X, A) , A itself may not have any cell structures.

Proposition 11.12. *Let (X, A) be a relative CW complex. Then $A \subset X$ is a cofibration.*

Proof. $S^{n-1} \hookrightarrow D^n$ is a cofibration, and cofibration is preserved under push-out and compositions. □

Corollary 11.13. *Let X be a CW complex and X' be a CW subcomplex. Then $X' \rightarrow X$ is a cofibration.*

Proof. (X, X') is a relative CW complex. □

Proposition 11.14. *Let X, Y be CW complexes. X is locally compact. Then $X \times Y$ is a CW complex*

12. WHITEHEAD THEOREM

Relative homotopy group.

Definition 12.1. We define the category $\underline{\text{TopP}}$ of topological pairs where an object (X, A) is a topological space X with a subspace A , and morphisms $(X, A) \rightarrow (Y, B)$ are continuous maps $f : X \rightarrow Y$ such that $f(A) \subset B$. A homotopy between two maps $f_1, f_2 : (X, A) \rightarrow (Y, B)$ is a homotopy $F : X \times I \rightarrow Y$ between f_0, f_1 such that $F|_{X \times \{t\}}(A) \subset B$ for any $t \in I$.

The quotient category of $\underline{\text{TopP}}$ by homotopy of maps is denoted by $\underline{\text{hTopP}}$. The pointed versions are defined similarly and denoted by $\underline{\text{TopP}}_*$ and $\underline{\text{hTopP}}_*$. Morphisms in $\underline{\text{hTopP}}$ and $\underline{\text{hTopP}}_*$ are denoted by

$$[(X, A), (Y, B)], \quad [(X, A), (Y, B)]_0.$$

Lemma 12.2. *Let $f : (X, A) \rightarrow (Y, B)$. Let $\bar{f} = f|_A$. Then the sequence*

$$(X, A) \rightarrow (Y, B) \rightarrow (C_f, C_{\bar{f}})$$

is co-exact in $\underline{\text{hTopP}}_$. When $A = B = \text{point}$, this recovers the co-exactness of homotopy cofiber.*

Theorem 12.3. Let $f : (X, A) \rightarrow (Y, B)$. Let $\bar{f} = f|_A$. Then the sequence

$$(X, A) \rightarrow (Y, B) \rightarrow (C_f, C_{\bar{f}}) \rightarrow \Sigma(X, A) \rightarrow \Sigma(Y, B) \rightarrow \Sigma(C_f, C_{\bar{f}}) \rightarrow \Sigma^2(X, A) \rightarrow \dots$$

is co-exact in $\underline{\text{hTopP}}_*$. This generalizes the co-exact Puppe sequence to the pair case.

Definition 12.4. Let $(X, A) \in \underline{\text{TopP}}_*$. We define the **relative homotopy group** $\pi_n(X, A)$

$$\pi_n(X, A) = [(D^n, S^{n-1}), (X, A)]_0.$$

We will also write $\pi_n(X, A; x_0)$ when we want to specify the base point.

Note that for $n \geq 2$

$$(D^n, S^{n-1}) \simeq \Sigma^{n-1}(D^1, S^0),$$

therefore $\pi_n(X, A)$ is a group for $n \geq 2$ by the adjunct pair (Σ, Ω) .

Lemma 12.5. $f : (D^n, S^{n-1}) \rightarrow (X, A)$ is zero in $\pi_n(X, A)$ if and only if f is homotopic rel S^{n-1} to a map whose image lies in A .

This lemma can be summarized by the following diagram

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & A \\ \downarrow & \nearrow g & \downarrow \\ D^n & \xrightarrow{f} & X \end{array}$$

Here g maps D^n to A and $g \simeq f \text{ rel } S^{n-1}$.

Theorem 12.6. Let $B \subset A \subset X$ in $\underline{\text{Top}}_*$. Then there is a long exact sequence

$$\dots \rightarrow \pi_n(A, B) \xrightarrow{i_*} \pi_n(X, B) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A, B) \dots \rightarrow \pi_0(X)$$

Here the boundary map ∂ sends $f \in [(D^n, S^{n-1}), (X, A)]_0$ to its restriction to $S^{n-1} = D^{n-1}/S^{n-2}$ viewed as

$$\partial f : (D^{n-1}, S^{n-2}) \rightarrow (A, B)$$

where ∂f sends the whole S^{n-2} to the base point in B .

Proof. We prove the case for $A = B = \text{base point } x_0 \in X$. Consider

$$f : (S^0, \{0\}) \rightarrow (S^0, S^0).$$

Let $\bar{f} = f|_{\{0\}} : \{0\} \rightarrow S^0$. It is easy to see that

$$(C_f, C_{\bar{f}}) \simeq (D^1, S^0).$$

Since $\Sigma^n(S^0) = S^n$, $\Sigma(D^n, S^{n-1}) = (D^{n+1}, S^n)$, the co-exact Puppe sequence

$$(S^0, \{0\}) \rightarrow (S^0, S^0) \rightarrow (D^1, S^0) \rightarrow (S^1, \{0\}) \rightarrow (S^1, S^1) \rightarrow (D^2, S^1) \rightarrow (S^2, \{0\}) \rightarrow \dots$$

implies the exact sequence

$$\dots \rightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \dots \rightarrow \pi_0(X)$$

□

Definition 12.7. A pair (X, A) is called **n-connected** ($n \geq 0$) if $\pi_0(A) \rightarrow \pi_0(X)$ is surjective and $\pi_k(X, A; x_0) = 0$ for any $1 \leq k \leq n$, $x_0 \in A$.

From the long exact sequence

$$\cdots \rightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \cdots \rightarrow \pi_0(X)$$

we see that (X, A) is n -connected if and only if for any $x_0 \in A$

$$\begin{cases} \pi_r(A, x_0) \rightarrow \pi_r(X, x_0) \text{ is bijective for } r < n \\ \pi_n(A, x_0) \rightarrow \pi_n(X, x_0) \text{ is surjective} \end{cases}$$

Definition 12.8. A map $f : X \rightarrow Y$ is called an **n -equivalence** ($n \geq 0$) if for any $x_0 \in X$

$$\begin{cases} f_* : \pi_r(X, x_0) \rightarrow \pi_r(Y, f(x_0)) \text{ is bijective for } r < n \\ f_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0)) \text{ is surjective} \end{cases}$$

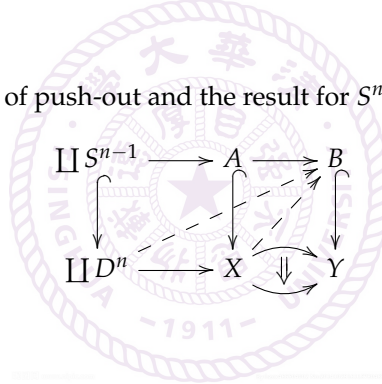
f is called **weak homotopy equivalence** or **∞ -equivalence** if f is n -equivalence for any $n \geq 0$.

Example 12.9. For any $n \geq 0$, the pair (D^{n+1}, S^n) is n -connected.

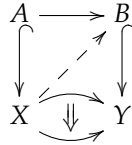
CW complex.

Lemma 12.10. Let X be obtained from A by attaching n -cells. Let (Y, B) be a pair such that $\pi_n(Y, B; b) = 0, \forall b \in B$ if $n \geq 1$ or $\pi_0(B) \rightarrow \pi_0(Y)$ surjective if $n = 0$. Then any map from $(X, A) \rightarrow (Y, B)$ is homotopic rel A to a map from X to B .

Proof. Apply the universal property of push-out and the result for $S^{n-1} \hookrightarrow D^n$.



Theorem 12.11. Let (X, A) be a relative CW complex with relative dimension $\leq n$. Let (Y, B) be n -connected ($0 \leq n \leq \infty$). Then any map from (X, A) to (Y, B) is homotopic relative to A to a map from X to B .



Proof. Apply the previous Lemma to

$$A \subset X^0 \subset X^1 \subset \cdots \subset X^n = X$$

and observe that all embeddings are cofibrations. □

Proposition 12.12. Let $f : X \rightarrow Y$ be a weak homotopy equivalence, P be a CW complex. Then

$$f_* : [P, X] \rightarrow [P, Y]$$

is a bijection.

Proof. We can assume f is an embedding and (Y, X) is ∞ -connected. Otherwise replace Y by M_f .

Surjectivity follows from the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ P & \xrightarrow{\quad} & Y \\ & \searrow & \downarrow \\ & & Y \end{array}$$

Injectivity follows from the diagram (observe $P \times I, P \times \partial I$ are CW complexes)

$$\begin{array}{ccc} P \times \partial I & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ P \times I & \xrightarrow{\quad} & Y \\ & \searrow & \downarrow \\ & & Y \end{array}$$

□

Theorem 12.13 (Whitehead Theorem). *A map between CW complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

Proof. Let $f : X \rightarrow Y$ be a weak homotopy equivalence between CW complexes. We have bijections

$$f_* : [X, X]_0 \rightarrow [X, Y]_0, \quad f_* : [Y, X] \rightarrow [Y, Y]_0.$$

Let $g \in [Y, X]_0$ such that $f_*[g] = 1_Y$. Then $g \circ f \simeq 1_Y$. On the other hand,

$$f_*[f \circ g] = [f \circ g \circ f] \simeq [f \circ 1] = [f] = f_*[1_X]$$

we find $[f \circ g] = 1_X$. Therefore f is a homotopy equivalence. The reverse direction is obvious. □

13. CELLULAR AND CW APPROXIMATIONS

Cellular Approximation.

Definition 13.1. Let (X, Y) be CW complexes. A map $f : X \rightarrow Y$ is called **cellular** if $f(X^n) \subset Y^n$ for any n . We define the category $\underline{\text{CW}}$ whose objects are CW complexes and morphisms are cellular maps.

Definition 13.2. A **cellular homotopy** between two cellular maps $X \rightarrow Y$ of CW complexes is a homotopy $X \times I \rightarrow Y$ that is itself a cellular map. Here I is naturally a CW complex. We define the quotient category $\underline{\text{hCW}}$ of $\underline{\text{CW}}$ whose morphisms are cellular homotopy class of cellular maps.

Lemma 13.3. *Let X be obtained from A by attaching n -cells ($n \geq 1$), then (X, A) is $(n - 1)$ -connected.*

Proof. Let $r < n$. Consider a diagram

$$\begin{array}{ccc} S^{r-1} & \longrightarrow & A \\ \downarrow & & \downarrow \\ D^r & \xrightarrow{f} & X \end{array}$$

Since D^r is compact, $f(D^r)$ meets only finitely many attached n -cells on X , say e_1, \dots, e_m . Let p_i be the center of e_i . Let $e_i^* = e_i - \{p_i\}$. $Y = X - \{p_1, \dots, p_m\}$. We subdivide D^r into small disks $D^r = \cup_{\alpha} D_{\alpha}^r$ such that $f(D_{\alpha}^r) \subset Y$ or $f(D_{\alpha}^r) \subset e_i^*$. For each D_{α}^r such that $f(D_{\alpha}^r) \subset e_i$ but not in Y , we use the fact that

$(e_i, e_i^*) \simeq (D^n, S^{n-1})$ is $(n - 1)$ -connected to find a homotopy rel ∂D_α^r to adjust mapping D_α^r into e_i^* . It glues together to obtain

$$\begin{array}{ccc} S^{r-1} & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ D^r & \xrightarrow{\quad} & X \\ & \Downarrow & \\ & \xrightarrow{\quad} & \end{array}$$

Then we can further find a homotopy

$$\begin{array}{ccc} S^{r-1} & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow \\ D^r & \xrightarrow{\quad} & Y \\ & \Downarrow & \\ & \xrightarrow{\quad} & \end{array}$$

□

Corollary 13.4. *Let (X, A) be a relative CW complex, then for any $n \geq 0$, the pair (X, X^n) is n -connected.*

Theorem 13.5. *Let $f : (X, A) \rightarrow (\tilde{X}, \tilde{A})$ between relative CW complexes which is cellular on a subcomplex (Y, B) of (X, A) . Then f is homotopic rel Y to a cellular map $g : (X, A) \rightarrow (\tilde{X}, \tilde{A})$.*

Proof. Assume we have constructed $f_{n-1} : (X, A) \rightarrow (\tilde{X}, \tilde{A})$ which is homotopic to f rel Y and cellular on the $(n - 1)$ -skeleton X^{n-1} . Let X^n be obtained from X^{n-1} by attaching n -cells. Consider

$$\begin{array}{ccc} X^{n-1} & \longrightarrow & \tilde{X}^n \\ \downarrow & & \downarrow \\ X^n & \xrightarrow{f_{n-1}} & \tilde{X} \end{array}$$

Since X^n is obtained from X^{n-1} by attaching n -cells and (\tilde{X}, \tilde{X}^n) is n -connected,

$$\begin{array}{ccc} X^{n-1} & \longrightarrow & \tilde{X}^n \\ \downarrow & \nearrow & \downarrow \\ X^n & \xrightarrow{f_{n-1}} & \tilde{X} \\ & \Downarrow & \\ & \xrightarrow{f_{n-1}} & \end{array}$$

we can find a homotopy rel X^{n-1} from $f_{n-1}|_{X^n} : X^n \rightarrow \tilde{X}$ to a map $X^n \rightarrow \tilde{X}^n$. Since f is cellular on Y , we can choose this homotopy rel Y by adjusting only those n -cells not in Y . This homotopy extends to a homotopy rel $X^{n-1} \cup Y$ from f_{n-1} to a map $f_n : X \rightarrow \tilde{X}$ since $X^n \subset X$ is a cofibration. Then f_∞ works. □

Theorem 13.6 (Cellular Approximation Theorem). *Any map between relative CW complexes is homotopic to a cellular map. If two cellular maps between relative CW complexes are homotopic, then they are cellular homotopic.*

Proof. Apply the previous Theorem to (X, \emptyset) and $(X \times I, X \times \partial I)$. □

Remark 13.7. This theorem says that hCW is a full subcategory of hTop.

CW Approximation.

Definition 13.8. A CW approximation of a topological space Y is a CW complex X with a weak homotopy equivalence $f : X \rightarrow Y$.

Theorem 13.9. Any space has a CW approximation.

Proof. We may assume Y is path connected. We construct a CW approximation X of Y by induction on the skeleton X^n . Assume we have constructed $f_n : X^n \rightarrow Y$ which is an n -equivalence. We attach an $(n+1)$ -cell to every generator of $\ker(\pi_n(X^n) \rightarrow \pi_n(Y))$ to obtain \tilde{X}^{n+1} . We can extend f_n to a map $\tilde{f}_{n+1} : \tilde{X}^{n+1} \rightarrow Y$

$$\begin{array}{ccc}
 \coprod S^n & \longrightarrow & X^n \\
 \downarrow & & \downarrow \\
 \coprod D^{n+1} & \longrightarrow & \tilde{X}^{n+1} \\
 & \searrow & \downarrow \\
 & & Y
 \end{array}
 \begin{array}{l}
 \\
 \\
 \\
 \begin{array}{l}
 \xrightarrow{f_n} \\
 \xrightarrow{\tilde{f}_{n+1}}
 \end{array}
 \end{array}$$

Since (\tilde{X}^{n+1}, X^n) is also n -connected, \tilde{f}_{n+1} is an n -equivalence. By construction and the surjectivity of $\pi_n(\tilde{X}^{n+1}) \rightarrow \pi_n(X^n)$, \tilde{f}_{n+1} defines also an isomorphism for $\pi_n(\tilde{X}^{n+1}) \rightarrow \pi_n(Y)$.

Now for every generator S_α^{n+1} of $\text{coker}(\pi_{n+1}(\tilde{X}^{n+1}) \rightarrow \pi_{n+1}(Y))$, we take a wedge sum to obtain

$$X^{n+1} = \tilde{X}^{n+1} \vee (\vee_\alpha S_\alpha^{n+1}).$$

Then the induced map $f_{n+1} : X^{n+1} \rightarrow Y$ extends f_n to an $(n+1)$ -equivalence. Inductively we obtain a weak homotopy equivalence $f_\infty : X = X^\infty \rightarrow Y$. \square

Theorem 13.10. Let $f : X \rightarrow Y$. Let $\Gamma X \rightarrow X$, and $\Gamma Y \rightarrow Y$ be CW approximations. Then there exists a unique map in $[\Gamma X, \Gamma Y]$ making the following diagram commutes in \mathbf{hTop}

$$\begin{array}{ccc}
 \Gamma X & \xrightarrow{\Gamma f} & \Gamma Y \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{f} & Y
 \end{array}$$

Proof. Weak homotopy equivalence of $\Gamma Y \rightarrow Y$ implies the bijection $[\Gamma X, \Gamma Y] \rightarrow [\Gamma X, Y]$. \square

Definition 13.11. Two spaces X_1, X_2 are said to have the same **weak homotopy type** if there exists a space Y and weak homotopy equivalences $f_i : Y \rightarrow X_i, i = 1, 2$.

Proposition 13.12. Weak homotopy type is an equivalence relation.

14. EILENBERG-MACLANE SPACE

Graphs.

Definition 14.1. A **graph** is a one-dimensional CW complex. The points of the 0-skeleton are called **vertices** and the 1-cells are called **edges**.

By definition, a basis for the topology of a graph consists of the open intervals in the edges together with the path-connected neighborhoods of the vertices. A graph is compact if and only if it contains only finitely many vertices and edges.

Definition 14.2. A **subgraph** of a graph is a CW subcomplex. A **tree** is a contractible graph. By a tree in a graph X we mean a subgraph that is a tree. We call a tree in X **maximal** if it contains all the vertices of X .

Proposition 14.3. *Every connected graph contains a maximal tree, and in fact any tree in the graph is contained in a maximal tree.*

Lemma 14.4. *Let $A \subset X$ be a cofibration and A is contractible, then $X \rightarrow X/A$ is a homotopy equivalence.*

Theorem 14.5. *For a connected graph X with maximal tree T , $\pi_1(X)$ is a free group with basis the classes corresponding to the edges e of $X - T$.*

Theorem 14.6 (Nielsen-Schreier theorem). *Every subgroup of a free group is itself free.*

Proof. Let F be a free group with basis indexed by I . Let $X = \bigvee_{I \in B} S^1$. Then $\pi_1(X) = F$. Let $G \subset F$ and $\tilde{X} \rightarrow X$ be the covering such that $\pi_1(\tilde{X}) = G$. Then \tilde{X} is also a CW complex. It follows that G is free. \square

$\pi_n(S^n)$.

We have seen that $\pi_k(S^n) = 1$ for $k < n$. In this subsection we compute

$$\pi_n(S^n) = [S^n, S^n]_0 \cong \mathbb{Z}.$$

Given $f : S^n \rightarrow S^n$, its class $[f] \in \mathbb{Z}$ under the above isomorphism is called the **degree** of f .

Theorem 14.7 (Homotopy Excision Theorem). *Let $(A, C), (B, C)$ be relative CW complex. Let X be the push-out*

$$\begin{array}{ccc} C & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

If (A, C) is m -connected and (B, C) is n -connected, then

$$\pi_i(A, C) \rightarrow \pi_i(X, B)$$

is an isomorphism for $i < m + n$, and a surjection for $i = m + n$.

Corollary 14.8 (Freudenthal Suspension Theorem). *The suspension map*

$$\pi_i(S^n) \rightarrow \pi_{i+1}(S^{n+1})$$

is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$.

Proof. Apply Homotopy Excision to $X = S^{n+1}, C = S^n, A$ the upper half disk, B the lower half disk. \square

Freudenthal Suspension Theorem holds similarly replacing S^n by general $(n - 1)$ -connected space.

Proposition 14.9. $\pi_n(S^n) \cong \mathbb{Z}$ for $n \geq 1$.

Proof. Freudenthal Suspension Theorem reduces to show $\pi_2(S^2) \cong \mathbb{Z}$. This follows from the Hopf fibration

$$S^1 \rightarrow S^3 \rightarrow S^2.$$

\square

Eilenberg-MacLane Space.

Definition 14.10. An **Eilenberg-MacLane Space** is a CW complex $K(G, n)$ such that $\pi_n(K(G, n)) \cong G$ and $\pi_k(K(G, n)) = 0$ for $k \neq n$. Here G is abelian if $n > 1$.

Theorem 14.11. *Eilenberg-MacLane Space $K(G, n)$ exists.*

Proof. We prove the case for $n \geq 2$. There exists an exact sequence

$$0 \rightarrow F_1 \rightarrow F_2 \rightarrow G \rightarrow 0$$

where F_1, F_2 are free abelian groups. Let B_i be a basis of F_i . Let

$$A = \bigvee_{i \in B_1} S^n, \quad B = \bigvee_{j \in B_2} S^n.$$

A, B are $(n-1)$ -connected and $\pi_n(A) = F_1, \pi_n(B) = F_2$. Using the degree map, we can construct

$$f : A \rightarrow B$$

such that $\pi_n(A) \rightarrow \pi_n(B)$ realizes the map $F_1 \rightarrow F_2$. Let X be obtained from B by attaching $(n+1)$ -cells via f . Then X is $(n-1)$ -connected and $\pi_n(X) = G$. Now we proceed as the proof of CW approximation theorem to attach cells of dimension $\geq (n+2)$ to kill all higher homotopy groups of X to get $K(G, n)$. \square

As we will see, $K(G, n)$ is the representing space for cohomology functor with coefficients in G

$$H^n(X; G) \cong [X, K(G, n)] \quad \text{for any CW complex } X.$$

Example 14.12. $S^1 = K(\mathbb{Z}, 1)$. Connected graphs are Eilenberg-MacLane space for free groups at $n = 1$.

Example 14.13. Using the fibration $S^1 \rightarrow S^\infty \rightarrow \mathbb{C}P^\infty$, we find $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$.

Example 14.14. A **knot** is an embedding $K : S^1 \hookrightarrow S^3$. Let $G = \pi_1(S^3 - K)$. Then $S^3 - K = K(G, 1)$.

15. SINGULAR HOMOLOGY

Chain complex.

Definition 15.1. Let R be a commutative ring. A **chain complex** over R is sequence of R -module maps

$$\cdots \rightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \cdots$$

such that $\partial_n \circ \partial_{n+1} = 0 \forall n$. When R is not specified, we mean chain complex of abelian groups (i.e. $R = \mathbb{Z}$).

Sometimes we just write the map by ∂ and the chain complex by (C_\bullet, ∂) . Then $\partial_n = \partial|_{C_n}$ and $\partial^2 = 0$.

Definition 15.2. A **chain map** $f : C_\bullet \rightarrow C'_\bullet$ between two chain complexes over R is a sequence of R -module maps $f_n : C_n \rightarrow C'_n$ such that the following diagram commutes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\ \cdots & \longrightarrow & C'_{n+1} & \xrightarrow{\partial'_{n+1}} & C'_n & \xrightarrow{\partial'_n} & C'_{n-1} & \longrightarrow & \cdots \end{array}$$

We simply write it as

$$\boxed{f \circ \partial = \partial' \circ f}$$

Chain complexes over R together with chain maps form the category $\underline{\mathbf{Ch}}_\bullet(R)$ of chain complexes over R , or simply $\underline{\mathbf{Ch}}_\bullet$ when $R = \mathbb{Z}$.

Definition 15.3. Given a chain complex (C_\bullet, ∂) , its n -cycles Z_n and n -boundaries B_n are

$$Z_n = \text{Ker}(\partial : C_n \rightarrow C_{n-1}), \quad B_n = \text{Im}(\partial : C_{n+1} \rightarrow C_n).$$

$\partial^2 = 0$ implies $B_n \subset Z_n$. We define the n -th homology group by

$$H_n(C_\bullet, \partial) := \frac{Z_n}{B_n} = \frac{\text{ker}(\partial_n)}{\text{im}(\partial_{n+1})}.$$

A chain complex C_\bullet is called **acyclic** or **exact** if $H_n(C_\bullet) = 0, \forall n$.

Proposition 15.4. n -th homology group defines a functor

$$H_n : \underline{\mathbf{Ch}}_\bullet \rightarrow \underline{\mathbf{Ab}}$$

Definition 15.5. A **chain homotopy** $f \stackrel{s}{\simeq} g$ between two chain maps $f, g : C_\bullet \rightarrow C'_\bullet$ is a sequence of homomorphisms $s_n : C_n \rightarrow C'_{n+1}$ such that $f_n - g_n = s_{n-1} \circ \partial_n + \partial'_{n+1} \circ s_n$, or simply

$$f - g = s \circ \partial + \partial' \circ s.$$

Two complexes C_\bullet, C'_\bullet are called **chain homotopy equivalent** if there exists chain maps $f : C_\bullet \rightarrow C'_\bullet$ and $h : C'_\bullet \rightarrow C_\bullet$ such that $f \circ h \simeq 1$ and $g \circ f \simeq 1$.

Proposition 15.6. Chain homotopy defines an equivalence relation on chain maps and compatible with compositions.

In other word, chain homotopy defines an equivalence relation on $\underline{\mathbf{Ch}}_\bullet$. We define the quotient category

$$\underline{\mathbf{hCh}}_\bullet = \underline{\mathbf{Ch}}_\bullet / \simeq.$$

Chain homotopy equivalence becomes an equivalence in $\underline{\mathbf{hCh}}_\bullet$.

Proposition 15.7. Let f, g be chain homotopic chain maps. Then they induce identical map on homology groups

$$H_n(f) = H_n(g) : H_n(C_\bullet) \rightarrow H_n(C'_\bullet).$$

In other words, the functor H_n factor through

$$H_n : \underline{\mathbf{Ch}}_\bullet \rightarrow \underline{\mathbf{hCh}}_\bullet \rightarrow \underline{\mathbf{Ab}}.$$

Singular homology.

Definition 15.8. We define the **standard n -simplex**

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1, t_i \geq 0\}$$

We let $\{v_0, \dots, v_n\}$ denote its vertices. Here $v_i = (0, \dots, 0, 1, 0, \dots, 0)$ where 1 sits at the i -th position.

Definition 15.9. Let X be a topological space. A **singular n -simplex** in X is a continuous map $\sigma : \Delta^n \rightarrow X$. For each $n \geq 0$, we define $S_n(X)$ as the free abelian group with basis all singular n -simplexes in X

$$S_n(X) = \bigoplus_{\sigma \in \text{Hom}(\Delta^n, X)} \mathbb{Z}\sigma.$$

The elements of $S_n(X)$ are called **singular n -chains** in X .

A singular n -chain is given by a finite formal sum

$$\gamma = \sum_{\sigma \in \text{Hom}(\Delta^n, X)} m_\sigma \sigma, \quad m_\sigma \in \mathbb{Z} \text{ and only finitely many } m_\sigma \text{'s are nonzero.}$$

The abelian group structure is: $-\gamma := \sum_\sigma (-m_\sigma) \sigma$ and

$$\left(\sum_\sigma m_\sigma \sigma \right) + \left(\sum_\sigma m'_\sigma \sigma \right) = \sum_\sigma (m_\sigma + m'_\sigma) \sigma.$$

Definition 15.10. Given a n -simplex $\sigma : \Delta^n \rightarrow X$ and $0 \leq i \leq n$, we define

$$\partial^{(i)} \sigma : \Delta^{n-1} \rightarrow X$$

to be the $(n-1)$ -simplex by restricting σ to the i -th face of Δ^n whose vertices are given by $\{v_0, v_1, \dots, \hat{v}_i, \dots, v_n\}$.

We define the **boundary map**

$$\partial : S_n(X) \rightarrow S_{n-1}(X)$$

by the abelian group homomorphism generated by

$$\partial \sigma := \sum_{i=0}^n (-1)^i \partial^{(i)} \sigma.$$

Proposition 15.11. $(S_\bullet(X), \partial)$ defines a chain complex, i.e., $\partial^2 = \partial \circ \partial = 0$.

Definition 15.12. For each $n \geq 0$, we define n -th singular homology group of X by

$$H_n(X) := H_n(S_\bullet(X), \partial).$$

Let $f : X \rightarrow Y$ be a continuous map, it defines a chain map

$$S_\bullet(f) : S_\bullet(X) \rightarrow S_\bullet(Y).$$

This defines the functor of singular chain complex

$$S_\bullet : \text{Top} \rightarrow \text{Ch}_\bullet.$$

Singular homology group can be viewed as the composition of functors

$$\text{Top} \rightarrow \text{Ch}_\bullet \xrightarrow{H_n} \text{Ab}.$$

Proposition 15.13. Let $f, g : X \rightarrow Y$ be homotopic maps. Then $S_\bullet(f), S_\bullet(g) : S_\bullet(X) \rightarrow S_\bullet(Y)$ are chain homotopic. In particular, they induce identical map $H_n(f) = H_n(g) : H_n(X) \rightarrow H_n(Y)$.

Proof. We only need to prove that for $i_0, i_1 : X \rightarrow X \times I$, the induced map

$$S_\bullet(i_0), S_\bullet(i_1) : S_\bullet(X) \rightarrow S_\bullet(X \times I)$$

are chain homotopic. Then their composition with the homotopy $X \times I \rightarrow Y$ gives the proposition.

Let us define a homotopy

$$s : S_n(X) \rightarrow S_{n+1}(X \times I).$$

For $\sigma : \Delta^n \rightarrow X$, we define (topologically)

$$s(\sigma) : \Delta^n \times I \xrightarrow{\sigma \times 1} X \times I$$

Here we treat $\Delta^n \times I$ as a collection of $(n+1)$ -simplexes as follows: let $\{v_0, \dots, v_n\}$ denote the vertices of Δ^n , then the vertices of $\Delta^n \times I$ contain two copies $\{v_0, \dots, v_n\}$ and $\{w_0, \dots, w_n\}$. Then

$$\Delta^n \times I = \sum_{i=0}^n (-1)^i [v_0, v_1, \dots, v_i, w_i, w_{i+1}, \dots, w_n]$$

cuts $\Delta^n \times I$ into $(n + 1)$ -simplexes. Its sum defines $s(\sigma) \in S_{n+1}(X \times I)$. The intuitive formula holds

$$\partial(\Delta^n \times I) = \Delta \times \partial I - (\partial \Delta^n) \times I$$

as an equation for singular chains, leading to

$$S_\bullet(i_1) - S_\bullet(i_0) = \partial \circ s + s \circ \partial.$$

□

Theorem 15.14. *Singular homologies are homotopy invariants. They factor through*

$$H_n : \underline{\mathbf{hTop}} \rightarrow \underline{\mathbf{hCh}}_\bullet \rightarrow \underline{\mathbf{Ab}}.$$

16. EXACT HOMOLOGY SEQUENCE

Exact homology sequence.

Definition 16.1. Chain maps $0 \rightarrow C'_\bullet \xrightarrow{i} C_\bullet \xrightarrow{p} C''_\bullet \rightarrow 0$ is called an **short exact sequence** if for each n

$$0 \rightarrow C'_n \xrightarrow{i} C_n \xrightarrow{p} C''_n \rightarrow 0$$

is an exact sequence of abelian groups.

We have the following commuting diagram

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C'_{n+1} & \xrightarrow{i} & C_{n+1} & \xrightarrow{p} & C''_{n+1} \longrightarrow 0 \\
 & & \downarrow \partial' & & \downarrow \partial & & \downarrow \partial'' \\
 0 & \longrightarrow & C'_n & \xrightarrow{i} & C_n & \xrightarrow{p} & C''_n \longrightarrow 0 \\
 & & \downarrow \partial' & & \downarrow \partial & & \downarrow \partial'' \\
 0 & \longrightarrow & C'_{n-1} & \xrightarrow{i} & C_{n-1} & \xrightarrow{p} & C''_{n-1} \longrightarrow 0 \\
 & & \downarrow \partial' & & \downarrow \partial & & \downarrow \partial''
 \end{array}$$

Lemma/Definition 16.2. *Let $0 \rightarrow C'_\bullet \xrightarrow{i} C_\bullet \xrightarrow{p} C''_\bullet \rightarrow 0$ be a short exact sequence. There is a natural homomorphism*

$$\delta : H_n(C''_\bullet) \rightarrow H_{n-1}(C'_\bullet)$$

*called the **connecting map**. It induces a long exact sequence of abelian groups*

$$\cdots \rightarrow H_n(C'_\bullet) \xrightarrow{i_*} H_n(C_\bullet) \xrightarrow{p_*} H_n(C''_\bullet) \xrightarrow{\delta} H_{n-1}(C'_\bullet) \xrightarrow{i_*} H_{n-1}(C_\bullet) \xrightarrow{p_*} H_{n-1}(C''_\bullet) \rightarrow \cdots$$

The connecting map δ is natural in the sense that a commutative diagram of complexes with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C'_\bullet & \longrightarrow & C_\bullet & \longrightarrow & C''_\bullet \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & D'_\bullet & \longrightarrow & D_\bullet & \longrightarrow & D''_\bullet \longrightarrow 0
 \end{array}$$

induces a commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & H_n(C'') & \longrightarrow & H_n(C_\bullet) & \longrightarrow & H_n(C'') & \xrightarrow{\delta} & H_{n-1}(C'_\bullet) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & H_n(D'') & \longrightarrow & H_n(D_\bullet) & \longrightarrow & H_n(D'') & \xrightarrow{\delta} & H_{n-1}(D'_\bullet) & \longrightarrow & \cdots
 \end{array}$$

Relative homology.

Definition 16.3. Let $A \subset X$ be a subspace. It induces a natural injective chain map $S_\bullet(A) \hookrightarrow S_\bullet(X)$. We define the singular chain complex of X relative to A to be

$$S_n(X, A) := S_n(X) / S_n(A)$$

with the induced differential. Its homology $H_n(X, A) := H_n(S_\bullet(X, A))$ is called the n -th relative homology.

Proposition 16.4. For $A \subset X$, there is a long exact sequence of abelian groups

$$\cdots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \rightarrow \cdots$$

Proof. This follows from the short exact sequence of complexes

$$0 \rightarrow S_\bullet(A) \rightarrow S_\bullet(X) \rightarrow S_\bullet(X, A) \rightarrow 0.$$

□

Let us define relative n -cycles $Z_n(X, A)$ and relative n -boundaries $B_n(X, A)$ to be

$$Z_n(X, A) = \{\gamma \in S_n(X) : \partial\gamma \in S_{n-1}(A)\}$$

$$B_n(X, A) = B_n(X) + S_n(A) \subset S_n(X).$$

Then it is easy to check that $S_n(A) \subset B_n(X, A) \subset Z_n(X, A) \subset S_n(X)$ and

$$H_n(X, A) = Z_n(X, A) / B_n(X, A)$$

Two relative n -cycles γ_1, γ_2 defines the same class $[\gamma_1] = [\gamma_2]$ in $H_n(X, A)$ if and only if $\gamma_1 - \gamma_2$ is homologous to a chain in A . The connecting map

$$\delta : H_n(X, A) \rightarrow H_{n-1}(A)$$

can be understood as follows: a n -cycle in $H_n(X, A)$ is represented by a n -chain $\gamma \in S_n(X)$ such that its boundary $\partial(\gamma)$ lies in A . Viewing $\partial(\gamma)$ as a $(n-1)$ -cycle in A , then

$$\delta[\gamma] = [\partial(\gamma)].$$

Let $f : (X, A) \rightarrow (Y, B)$ be a map of pairs. It naturally induces a commutative diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & S_\bullet(A) & \longrightarrow & S_\bullet(X) & \longrightarrow & S_\bullet(X, A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & S_\bullet(B) & \longrightarrow & S_\bullet(Y) & \longrightarrow & S_\bullet(Y, B) & \longrightarrow & 0
 \end{array}$$

which further induces compatible maps on various homology groups.

Proposition 16.5. Let $\{X_\alpha\}$ be path connected components of X , then

$$H_n(X) = \bigoplus_{\alpha} H_n(X_\alpha).$$

Proposition 16.6. *Let X be path connected. Then $H_0(X) \cong \mathbb{Z}$.*

In general, we have a surjective map

$$\epsilon : H_0(X) \rightarrow \mathbb{Z}, \quad \sum_{p \in X} m_p p \rightarrow \sum_p m_p.$$

Definition 16.7. We define the reduced homology group by

$$\tilde{H}_n(X) = \begin{cases} H_n(X) & n > 0 \\ \ker(H_0(X) \rightarrow \mathbb{Z}) & n = 0 \end{cases}$$

The long exact sequence still holds for the reduced case

$$\cdots \rightarrow \tilde{H}_n(A) \rightarrow \tilde{H}_n(X) \rightarrow H_n(X, A) \xrightarrow{\delta} \tilde{H}_{n-1}(A) \rightarrow \cdots$$

Example 16.8. If X is contractible, then $\tilde{H}_n(X) = 0$ for all n .

Example 16.9. Let $x_0 \in X$ be a point. Using the long exact sequence for $A = \{x_0\} \subset X$, we find

$$H_n(X, x_0) = \tilde{H}_n(X).$$

17. EXCISION

The fundamental property of homology which makes it computable is **excision**.

Barycentric Subdivision.

Definition 17.1. Let Δ^n be the standard n -simplex with vertices v_0, \dots, v_n . We define its **barycenter** to be

$$c(\Delta^n) = \frac{1}{n+1} \sum_{i=0}^n v_i \in \Delta^n.$$

Definition 17.2. We define the **barycentric subdivision** $\mathcal{B}\Delta^n$ of a n -simplex Δ^n as follows:

- (1) $\mathcal{B}\Delta^0 = \Delta^0$.
- (2) Let F_0, \dots, F_n be the n -simplexes of faces of Δ^{n+1} . c be the barycenter of Δ^{n+1} . Then $\mathcal{B}\Delta^{n+1}$ consists of $(n+1)$ -simplexes with ordered vertices $[c, w_0, \dots, w_n]$ where $[w_0, \dots, w_n]$ is a n -simplexes in $\mathcal{B}F_0, \dots, \mathcal{B}F_n$.

Equivalently, a simplex in $\mathcal{B}\Delta^n$ is indexed by a sequence $\{S_0 \subset S_1 \cdots \subset S_n = \Delta^n\}$ where S_i is a face of S_{i+1} . Then its vertices are $[c(S_n), c(S_{n-1}), \dots, c(S_0)]$. It is seen that Δ^n is the union of simplexes in $\mathcal{B}\Delta^n$.

Definition 17.3. We define the n -chain of barycentric subdivision \mathcal{B}_n by

$$\mathcal{B}_n = \sum_{\alpha} \pm \sigma_{\alpha} \in S_n(\Delta^n)$$

where the summation is over all sequence $\alpha = \{S_0 \subset S_1 \cdots \subset S_n = \Delta^n\}$. σ_{α} is the simplex with ordered vertices $[c(S_n), c(S_{n-1}), \dots, c(S_0)]$, viewed as a singular n -chain in Δ^n . The sign \pm is about orientation: if the orientation of $[c(S_n), c(S_{n-1}), \dots, c(S_0)]$ coincides with that of Δ^n , we take $+$; otherwise we take $-$.

Definition 17.4. We define the composition map denoted by

$$S_k(\Delta^m) \times S_n(\Delta^k) \rightarrow S_n(\Delta^m), \quad \sigma \times \eta \rightarrow \sigma \circ \eta.$$

This is defined on generators via the composition $\Delta^n \rightarrow \Delta^k \rightarrow \Delta^m$ and extended linearly on singular chains.

Similarly, there is a natural map denoted by

$$S_n(\Delta^m) : S_m(X) \rightarrow S_n(X), \quad \eta : \sigma \rightarrow \eta^*(\sigma) = \sigma \circ \eta$$

where $\eta^*(\sigma) = \sigma \circ \eta$ is the composition of σ with η .

Example 17.5. Let $\tilde{\partial}_n \in S_{n-1}(\Delta^n)$ be the faces. Then $\tilde{\partial}_n \circ \tilde{\partial}_{n-1} = 0$ and

$$\partial_n = \tilde{\partial}_n^* : S_n(X) \rightarrow S_{n-1}(X)$$

defines the boundary map in singular chains.

Lemma 17.6.

$$\mathcal{B}_n \circ \tilde{\partial}_n = \tilde{\partial}_n \circ \mathcal{B}_{n-1}$$

Proof. The choice of ordering and orientation guarantees that

$$\partial \mathcal{B}_n = \mathcal{B}(\partial \Delta^n)$$

where $\mathcal{B}(\partial \Delta^n)$ is the barycentric subdivision of faces $\partial \Delta^n$ of Δ^n , viewed as a $(n-1)$ -chain in Δ^n . \square

Definition 17.7. We define the **barycentric subdivision** on singular chain complex by

$$\mathcal{B}^* : S_\bullet(X) \rightarrow S_\bullet(X)$$

where $\mathcal{B}^* = \mathcal{B}_n^*$ on $S_n(X)$.

Lemma 17.8. $\mathcal{B} : S_\bullet(X) \rightarrow S_\bullet(X)$ is a chain map. Moreover, it is chain homotopic to the identity map.

Proof. The previous lemma implies

$$\partial_n \circ \mathcal{B}_n^* = \tilde{\partial}_n^* \circ \mathcal{B}_n^* = (\mathcal{B}_n \circ \tilde{\partial}_n)^* = (\tilde{\partial}_n \circ \mathcal{B}_{n-1})^* = \mathcal{B}_{n-1}^* \circ \partial_n.$$

This show that \mathcal{B}^* is a chain map.

To show the chain homotopy, it is enough to construct $T_{n+1} \in S_{n+1}(\Delta^n)$ such that

$$\mathcal{B}_n - 1_{\Delta^n} = T_{n+1} \circ \tilde{\partial}_{n+1} + \tilde{\partial}_n \circ T_n.$$

Here $1_{\Delta^n} : \Delta^n \rightarrow \Delta^n$ is the identity map, viewed as a n -chain. Then T_{n+1}^* gives the required homotopy. T is constructed inductively in n as follows. $T_1 = 0$. Suppose we have constructed T_n . We need to find T_{n+1} such that

$$\partial(T_{n+1}) = \mathcal{B}_n - 1_{\Delta^n} - \tilde{\partial}_n \circ T_n.$$

Observe

$$\partial(\mathcal{B}_n - 1_{\Delta^n} - \tilde{\partial}_n \circ T_n) = (\mathcal{B}_n - 1_{\Delta^n} - \tilde{\partial}_n \circ T_n) \circ \tilde{\partial}_n = \tilde{\partial}_n \circ (\mathcal{B}_{n-1} - 1_{\Delta^{n-1}} - T_n \circ \tilde{\partial}_n) = \tilde{\partial}_n \circ \tilde{\partial}_{n-1} \circ T_{n-1} = 0.$$

Therefore $\mathcal{B}_n - 1_{\Delta^n} - \tilde{\partial}_n \circ T_n$ is a n -cycle. However $H_n(\Delta^n) = 0$ for $n \geq 1$. It follows that T_{n+1} can be constructed. \square

Corollary 17.9. The barycentric subdivision map $\mathcal{B}^* : S_\bullet(X) \rightarrow S_\bullet(X)$ is a quasi-isomorphism.

Excision.

Theorem 17.10 (Excision). *Let $U \subset A \subset X$ be subspaces such that $\bar{U} \subset A^\circ$ (the interior of A). Then the inclusion $i : (X - U, A - U) \hookrightarrow (X, A)$ induces isomorphisms*

$$i_* : H_n(X - U, A - U) \cong H_n(X, A), \quad \forall n.$$

Proof. Let us call $\sigma : \Delta^n \rightarrow X$ *small* if

$$\sigma(\Delta^n) \subset A \quad \text{or} \quad \sigma(\Delta^n) \subset X - U.$$

Let $S'_\bullet(X) \subset S_\bullet(X)$ denote the subcomplex generated by small simplexes, and $S'_\bullet(X, A)$ defined by the exact sequence

$$0 \rightarrow S_\bullet(A) \rightarrow S'_\bullet(X) \rightarrow S'_\bullet(X, A) \rightarrow 0.$$

It is easy to see that

$$S'_\bullet(X, A) \cong S_\bullet(X - U, A - U).$$

There is a natural commutative diagram of chain maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_\bullet(A) & \longrightarrow & S'_\bullet(X) & \longrightarrow & S'_\bullet(X, A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S_\bullet(A) & \longrightarrow & S_\bullet(X) & \longrightarrow & S_\bullet(X, A) & \longrightarrow & 0 \end{array}$$

By the Five Lemma, it is enough to show that

$$S'_\bullet(X) \rightarrow S_\bullet(X)$$

is a quasi-isomorphism.

(1) Injectivity of $H(S'_\bullet(X)) \rightarrow H(S_\bullet(X))$:

Let α be a cycle in $S'_\bullet(X)$ and $\alpha = \partial\beta$ for $\beta \in S_\bullet(X)$. Take k big enough that $(\mathcal{B}^*)^k(\beta) \in S'_\bullet(X)$. Then

$$(\mathcal{B}^*)^k(\alpha) = \partial(\mathcal{B}^*)^k(\beta).$$

Hence $(\mathcal{B}^*)^k(\alpha)$ is zero in $H(S'_\bullet(X))$, so is α which is homologous to $(\mathcal{B}^*)^k(\alpha)$.

(2) Surjectivity of $H(S'_\bullet(X)) \rightarrow H(S_\bullet(X))$:

Let α be a cycle in $S_\bullet(X)$. Take k big enough that $(\mathcal{B}^*)^k(\alpha) \in S'_\bullet(X)$. Then $(\mathcal{B}^*)^k(\alpha)$ is a small cycle which is homologous to α . \square

Theorem 17.11. *Let X_1, X_2 be subspaces of X and $X = X_1^\circ \cup X_2^\circ$. Then*

$$H_\bullet(X_1, X_1 \cap X_2) \rightarrow H_\bullet(X, X_2)$$

is an isomorphism for all n .

Proof. Apply Excision to $U = X - X_1, A = X_2$. \square

Theorem 17.12 (Mayer-Vietoris). *Let X_1, X_2 be subspaces of X and $X = X_1^\circ \cup X_2^\circ$. Then there is an exact sequence*

$$\cdots \rightarrow H_n(X_1 \cap X_2) \xrightarrow{(i_{1*}, i_{2*})} H_n(X_1) \oplus H_n(X_2) \xrightarrow{j_{1*} - j_{2*}} H_n(X) \xrightarrow{\delta} H_{n-1}(X_1 \cap X_2) \rightarrow \cdots$$

It is also true for the reduced homology.

Proof. Let $S_\bullet(X_1) + S_\bullet(X_2) \subset S_\bullet(X)$ be the subspace spanned by $S_\bullet(X_1)$ and $S_\bullet(X_2)$. We have a short exact sequence

$$0 \rightarrow S_\bullet(X_1 \cap X_2) \xrightarrow{(i_1, i_2)} S_\bullet(X_1) \oplus S_\bullet(X_2) \xrightarrow{j_1 - j_2} S_\bullet(X_1) + S_\bullet(X_2) \rightarrow 0.$$

Similar to the proof of Excision via barycentric subdivision, the embedding $S_\bullet(X_1) + S_\bullet(X_2) \subset S_\bullet(X)$ is a quasi-isomorphism. Mayer-Vietoris sequence follows. \square

Theorem 17.13. *Let $A \subset X$ be a closed subspace. Assume A is a strong deformation retract of a neighborhood in X . Then the map $(X, A) \rightarrow (X/A, A/A)$ induces an isomorphism*

$$H_\bullet(X, A) \cong \tilde{H}_\bullet(X/A).$$

Proof. Let U be an open neighborhood of A that deformation retracts to A . Then $H_\bullet(A) \cong H_\bullet(U)$, hence

$$H_\bullet(X, A) \cong H_\bullet(X, U)$$

by Five Lemma. Since A is closed and U is open, we can apply Excision to find

$$H_\bullet(X, A) \cong H_\bullet(X, U) \cong H_\bullet(X - A, U - A).$$

The same consideration applied to $(X/A, A/A)$ and U/A gives

$$H_\bullet(X/A, A/A) \cong H_\bullet(X/A - A/A, U/A - A/A) = H_\bullet(X - A, U - A).$$

\square

This Theorem in particular applies to cofibrations.

18. HOMOLOGY OF SPHERES

Theorem 18.1. *The reduced homology of the sphere S^n is given by*

$$\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$$

Proof. Let $S^n = D_+^n \cup D_-^n$, where D_+^n (D_-^n) is the upper (lower) hemi-sphere and we choose a bit bigger ones to satisfy excision. $D_+^n \cap D_-^n = S^{n-1} \times I \simeq S^{n-1}$. Apply Mayer-Vietoris sequence we find

$$\tilde{H}_k(S^n) = \tilde{H}_{k-1}(S^{n-1}).$$

The theorem follows. \square

Corollary 18.2. *If $m \neq n$, then \mathbb{R}^m and \mathbb{R}^n are not homeomorphic.*

Definition 18.3. A continuous map $f : S^n \rightarrow S^n$ ($n \geq 0$) has **degree** d , denoted by $\deg(f) = d$, if

$$f_* : \tilde{H}_n(S^n) = \mathbb{Z} \rightarrow \tilde{H}_n(S^n) = \mathbb{Z}$$

is multiplication by d .

Lemma 18.4. *Let $f, g : S^n \rightarrow S^n$ be continuous maps.*

- (1) $\deg(f \circ g) = \deg(f) \deg(g)$.
- (2) If $f \simeq g$ are homotopic, then $\deg(f) = \deg(g)$
- (3) If f is a homotopy equivalence, then $\deg(f) = \pm 1$.

Proposition 18.5. *Let $r : S^n \rightarrow S^n, (x_0, \dots, x_n) \rightarrow (-x_0, x_1, \dots, x_n)$ be the reflection. Then*

$$\deg(r) = -1.$$

Proof. Prove by induction on n . This is true for $n = 0$. The induction follows from the commutative diagram

$$\begin{array}{ccc} \tilde{H}_n(S^n) & \xrightarrow{\delta} & \tilde{H}_{n-1}(S^{n-1}) \\ \downarrow r_* & & \downarrow r_* \\ \tilde{H}_n(S^n) & \xrightarrow{\delta} & \tilde{H}_{n-1}(S^{n-1}) \end{array}$$

□

Corollary 18.6. *Let $\sigma : S^n \rightarrow S^n, (x_0, \dots, x_n) \rightarrow (-x_0, \dots, -x_n)$ be the antipodal map. Then*

$$\deg(\sigma) = (-1)^{n+1}.$$

Proof. σ is a composition of $n + 1$ reflections. □

Theorem 18.7 (Hairy Ball Theorem). *S^n has a nowhere vanishing tangent vector field if and only if n is odd.*

Proof. If n is odd, we construct

$$v(x_0, \dots, x_n) = (-x_1, x_0, -x_3, x_2, \dots).$$

Conversely, assume v is no-where vanishing vector field. Let

$$f : S^n \rightarrow S^n, \quad x \rightarrow \frac{v(x)}{|v(x)|}.$$

The map

$$F : S^n \times I \rightarrow S^n, \quad F(x, t) = \cos(\pi t)x + \sin(\pi t)f(x)$$

defines a homotopy between the identity map 1 and the antipodal map σ . It follows that

$$\deg(\sigma) = 1 \implies n = \text{odd}.$$

□

Theorem 18.8 (Brower's Fixed Point Theorem). *Any continuous map $f : D^n \rightarrow D^n$ has a fixed point.*

Proof. Assume f has no fixed point. Define

$$r : D^n \rightarrow S^{n-1}$$

where $r(p)$ is the intersection of ∂D^n with the ray starting from $f(p)$ pointing toward p . Then r defines a retract of $S^{n-1} \hookrightarrow D^n$. This implies $H_\bullet(D^n) = H_\bullet(S^{n-1}) \oplus H_\bullet(D^n, S^{n-1})$, a contradiction. □

We give a geometric interpretation of the degree of $f : S^n \rightarrow S^n$. Let $V \subset S^n$ be a small open ball such that $f^{-1}(V) \rightarrow V$ is a disjoint union of open balls

$$f^{-1}(V) = U_1 \cup \dots \cup U_d.$$

Let $f_i : \bar{U}_i / \partial \bar{U}_i \cong S^n \rightarrow \bar{V} / \partial \bar{V} \cong S^n$. We have the commutative diagram

$$\begin{array}{ccc} H_n(S^n) & \longrightarrow & H_n(S^n/S^n - \cup_i U_i) \cong \bigoplus_i H_n(S^n) \\ \downarrow f_* & & \downarrow \bigoplus_i (f_i)_* \\ H_n(S^n) & \longrightarrow & H_n(S^n/S^n - V) \cong H_n(S^n) \end{array}$$

It is easy to see that first row is $k \rightarrow (k, k, \dots, k)$ and the second row is $k \rightarrow k$. It follows that

$$\deg(f) = \sum_{i=1}^d \deg(f_i).$$

Note that when $f^{-1}(V) \rightarrow V$ is a covering map, then $f : U_i \rightarrow V$ is a homeomorphism. We have $\deg(f_i) = \pm 1$ and $\deg(f)$ is given by a counting with signs.

Example 18.9. Identify $S^2 = \mathbb{C} \cup \{\infty\}$. Consider the map $f : S^2 \rightarrow S^2, z \rightarrow z^k$. Then $\deg(f) = k$.

19. CELLULAR HOMOLOGY

Cellular homology.

Definition 19.1. Let (X, A) be a relative CW complex with skeletons: $A = X^{-1} \subset X^0 \subset \dots \subset X^n \subset \dots$. We define the relative **cellular chain complex** $(C_\bullet^{cell}(X, A), \partial)$

$$\dots \rightarrow C_n^{cell}(X, A) \xrightarrow{\partial} C_{n-1}^{cell}(X, A) \xrightarrow{\partial} \dots \rightarrow C_0^{cell}(X, A) \rightarrow 0$$

where $C_n^{cell}(X, A) := H_n(X^n, X^{n-1})$ and the boundary map ∂ is defined by the commutative diagram

$$\begin{array}{ccc} H_n(X^n, X^{n-1}) & \xrightarrow{\quad \partial \quad} & H_{n-1}(X^{n-1}, X^{n-2}) \\ & \searrow \delta & \nearrow j \\ & & H_{n-1}(X^{n-1}, A) \end{array}$$

Here δ is the connecting map of relative homology for $A \subset X^{n-1} \subset X^n$ and j is the natural map.

Assume X^n is obtained from X^{n-1} by attaching n -cells indexed by J_n

$$\begin{array}{ccc} \coprod_{\alpha \in J_n} S^{n-1} & \xrightarrow{f} & X^{n-1} \\ \downarrow & & \downarrow \\ \coprod_{\alpha \in J_n} D^n & \xrightarrow{\Phi_f} & X^n \end{array}$$

Since $X^{n-1} \hookrightarrow X^n$ is a cofibration,

$$C_n^{cell}(X, A) \cong \tilde{H}_n(X^n / X^{n-1}) \cong \bigoplus_{J_n} \tilde{H}_n(S^n) \cong \bigoplus_{J_n} \mathbb{Z}$$

is the free abelian group generated by each attached $H_n(D^n, S^{n-1})$. Using the diagram

$$\begin{array}{ccccc}
 H_n(X^n, X^{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(X^{n-1}, X^{n-2}) & \xrightarrow{\partial_{n-1}} & H_{n-2}(X^{n-2}, X^{n-3}) \\
 \searrow \delta_n & & \nearrow j_n & & \searrow \delta_{n-1} \\
 & & H_{n-1}(X^{n-1}, A) & & H_{n-1}(X^{n-2}, A) \\
 & & & & \nearrow j_{n-1}
 \end{array}$$

and $\delta_{n-1} \circ j_n = 0$, we see that

$$\partial_{n-1} \circ \partial_n = j_{n-1} \circ \delta_{n-1} \circ j_n \circ \delta_n = 0.$$

Therefore $(C_\bullet^{cell}(X, A), \partial)$ indeed defines a chain complex.

Definition 19.2. Let (X, A) be a relative CW complex. We define its **n-th relative cellular homology** by

$$H_n^{cell}(X, A) := H_n(C_\bullet^{cell}(X, A), \partial).$$

When $A = \emptyset$, we simply denote it by $H_n^{cell}(X)$ called the **n-th cellular homology**.

Lemma 19.3. Let (X, A) be a relative CW complex. Let $0 \leq q < p \leq \infty$. Then

$$H_n(X^p, X^q) = 0, \quad n \leq q \quad \text{or} \quad n > p.$$

Theorem 19.4. Let (X, A) be a relative CW complex. Then cellular homology coincides with singular homology

$$H_n^{cell}(X, A) \cong H_n(X, A).$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc}
 & & H_{n+1}(X^{n+1}, X^n) & & & & H_n(X^{n-2}, A)(=0) \\
 & & \downarrow & \searrow \partial_{n+1} & & & \downarrow \\
 & & H_n(X^n, A) & \longrightarrow & H_n(X^n, X^{n-1}) & \longrightarrow & H_{n-1}(X^{n-1}, A) \\
 H_n(X^{n-1}, A)(=0) & \longrightarrow & & & & & \downarrow \\
 & & \downarrow & & & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & & H_n(X^{n+1}, A) & & & & \downarrow \\
 & & \downarrow & & & & H_{n-1}(X^{n-2}, A)(=0) \\
 & & H_n(X^{n+1}, X^n)(=0) & & & &
 \end{array}$$

Diagram chasing implies $H_n(X^{n+1}, A) \cong H_n^{cell}(X, A)$. Theorem follows from the exact sequence

$$H_{n+1}(X, X^{n+1})(=0) \rightarrow H_n(X^{n+1}, A) \rightarrow H_n(X, A) \rightarrow H_n(X, X^{n+1})(=0)$$

□

Let $f : (X, A) \rightarrow (Y, B)$ be a cellular map. It induces a map on cellular homology

$$f_* : H_\bullet^{cell}(X, A) \rightarrow H_\bullet^{cell}(Y, B).$$

Therefore in the category of CW complexes, we can work entirely with cellular homology which is combinatorially easier to compute by the next formula.

Cellular Boundary Formula.

Let us now analyze cellular differential

$$\partial_n : H_n(X^n, X^{n-1}) \rightarrow H_{n-1}(X^{n-1}, X^{n-2}).$$

For each n -cell e_α^n , we have the gluing map

$$f_{e_\alpha^n} : S^{n-1} \rightarrow X^{n-1}.$$

This defines a map

$$\tilde{f}_{e_\alpha^n} : S^{n-1} \rightarrow X^{n-1}/X^{n-2} = \bigvee_{I_{n-1}} S^{n-1}$$

which induces a degree map

$$(\tilde{f}_{e_\alpha^n})_* : \tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z} \rightarrow \bigoplus_{I_{n-1}} \tilde{H}_{n-1}(S^{n-1}) \cong \bigoplus_{I_{n-1}} \mathbb{Z}.$$

Collecting all n -cells, this generates the degree map

$$d_n : \bigoplus_{I_n} \mathbb{Z} \rightarrow \bigoplus_{I_{n-1}} \mathbb{Z}.$$

Theorem 19.5. Under the identification $C_n^{\text{cell}}(X^n, X^{n-1}) \cong \bigoplus_{I_n} \mathbb{Z}$, cellular differential coincides with the degree map

$$\partial_n \cong d_n.$$

Example 19.6. $\mathbb{C}P^n$ has a CW structure with a single $2m$ -cell for each $m \leq n$. Since there is no odd dim cells, the degree map $d = 0$. We find

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & k = 0, 2, \dots, 2n \\ 0 & \text{otherwise} \end{cases}$$

Example 19.7. A closed oriented surface Σ_g of genus g has a CW structure with a 0-cell, $2g$ 1-cells, and a 2-cell. It is easy to see that the degree map is zero. We find

$$H_k(\Sigma_g) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^{2g} & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & k > 2. \end{cases}$$

Example 19.8. $\mathbb{R}P^n$ has a CW structure with a k -cell for each $0 \leq k \leq n$. The degree map is $d_k = 1 + (-1)^k$.

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 < k < n, k \text{ odd} \\ \mathbb{Z} & k = n = \text{odd} \\ 0 & k = n = \text{even} \\ 0 & k > n \end{cases}$$

Euler characteristic.

Definition 19.9. Let X be a finite CW complex of dimension n and denote by c_i the number of i -cells of X . The Euler characteristic of X is defined as:

$$\chi(X) := \sum_i (-1)^i c_i.$$

Recall that any finitely generated abelian group G is decomposed into a free part and a torsion part

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_k\mathbb{Z}.$$

The integer $r := rk(G)$ is called the **rank** of G .

Theorem 19.10. Let X be a finite CW complex. Then

$$\chi(X) = \sum_i (-1)^i b_i(X)$$

where $b_i(X) := rk(H_i(X))$ is called the i -th **Betti number** of X . In particular, $\chi(X)$ is independent of the chosen CW structure on X and only depend on the cellular homotopy class of X .

20. COHOMOLOGY AND UNIVERSAL COEFFICIENT THEOREM

Cohomology.

Definition 20.1. Let R be a commutative ring. A **cochain complex** over R is sequence of R -module maps

$$\cdots \rightarrow C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \rightarrow \cdots$$

such that $d_n \circ d_{n-1} = 0 \forall n$. When R is not specified, we mean cochain complex of abelian groups (i.e. $R = \mathbb{Z}$).

Sometimes we just write the map by d and the cochain complex by (C^\bullet, d) . Then $d_n = d|_{C^n}$ and $d^2 = 0$.

Definition 20.2. Given a cochain complex (C^\bullet, d) , its n -cocycles Z^n and n -coboundaries B^n are

$$Z^n = \text{Ker}(d : C^n \rightarrow C^{n+1}), \quad B^n = \text{Im}(d : C^{n-1} \rightarrow C^n).$$

$d^2 = 0$ implies $B^n \subset Z^n$. We define the n -th **cohomology group** by

$$H^n(C^\bullet, d) := \frac{Z^n}{B^n} = \frac{\text{ker}(d_n)}{\text{im}(d_{n-1})}.$$

A cochain complex C^\bullet is called **acyclic** or **exact** if $H^n(C^\bullet) = 0, \forall n$.

We are interested in the following relation between cochain and chain complex.

Definition 20.3. Let (C_\bullet, ∂) be a chain complex over R , and G be a R -module. We define its dual cochain complex $(C^\bullet, d) = \text{Hom}_R(C_\bullet, G)$ by

$$\cdots \text{Hom}_R(C_{n-1}, G) \rightarrow \text{Hom}_R(C_n, G) \rightarrow \text{Hom}_R(C_{n+1}, G) \rightarrow \cdots$$

Here given $f \in \text{Hom}_R(C_n, G)$, we define $d_n f \in \text{Hom}_R(C_{n+1}, G)$ by

$$d_n f(c) := f(\partial_{n+1}(c)), \quad \forall c \in C_{n+1}.$$

Definition 20.4. Let G be an abelian group and X be a topological space. For $n \geq 0$, we define the group of **singular n -cochains** in X with coefficient in G to be

$$S^n(X; G) := \text{Hom}(S_n(X), G).$$

The dual cochain complex $S^\bullet(X; G) = \text{Hom}(S_\bullet(X), G)$ is called the **singular cochain complex** with coefficient in G . Its cohomology is called the **singular cohomology** with coefficient in G , denoted by

$$H^n(X; G) := H^n(S^\bullet(X; G)).$$

When $G = \mathbb{Z}$, we simply write it as $H^n(X)$.

Theorem 20.5. $H^n(-; G)$ defines a contra-variant functor

$$H^n(-; G) : \underline{\text{hTop}} \rightarrow \underline{\text{Ab}}.$$

Example 20.6 (Dimension Axiom). Let X be a point. Then

$$H^n(X; G) = \begin{cases} G & k = 0 \\ 0 & k > 0 \end{cases}$$

Lemma 20.7. Let G be a R -module. If $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ is an exact sequence of R -modules, then applying $\text{Hom}_R(-, G)$ gives an exact sequence

$$0 \rightarrow \text{Hom}_R(A_3, G) \rightarrow \text{Hom}_R(A_2, G) \rightarrow \text{Hom}_R(A_1, G).$$

If A_3 is a free R -module (or more generally projective R -module), then the last morphism is also surjective.

Definition 20.8. Let G be an abelian group. Let $A \subset X$ be a subspace. We define the **relative singular cochain complex** with coefficient in G by

$$S^\bullet(X, A; G) := \text{Hom}(S_\bullet(X)/S_\bullet(A), G).$$

Its cohomology is called the **relative singular cohomology**, denoted by $H^\bullet(X, A; G)$.

Since $S_\bullet(X)/S_\bullet(A)$ is a free abelian group, we have a short exact sequence of cochain complex

$$0 \rightarrow S^\bullet(X, A; G) \rightarrow S^\bullet(X; G) \rightarrow S^\bullet(A; G) \rightarrow 0$$

which induces a long exact sequence of cohomology groups

$$0 \rightarrow H^0(X, A; G) \rightarrow H^0(X; G) \rightarrow H^0(A; G) \rightarrow H^1(X, A; G) \rightarrow \cdots$$

Moreover, the connecting maps

$$\delta : H^n(A, G) \rightarrow H^{n+1}(X, A; G)$$

is natural in the same sense as homology case.

Theorem 20.9 (Excision). Let $U \subset A \subset X$ be subspaces such that $\bar{U} \subset A^\circ$ (the interior of A). Then the inclusion $i : (X - U, A - U) \hookrightarrow (X, A)$ induces isomorphisms

$$i^* : H^n(X, A; G) \cong H^n(X - U, A - U; G), \quad \forall n.$$

Universal Coefficient Theorem for Cohomology.

Definition 20.10. Let M, N be two R -modules. Let $P_\bullet \rightarrow M$ be a free R -module resolution of M :

$$\cdots P_n \rightarrow P_{n-1} \rightarrow \cdots P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

is an exact sequence of R -modules and P^i 's are free. We define the Ext group

$$\text{Ext}_R^k(M, N) = H^k(\text{Hom}(P_\bullet, N))$$

and the Tor group

$$\text{Tor}_k^R(M, N) = H_k(P_\bullet \otimes_R N).$$

Note that

$$\text{Ext}_R^0(M, N) = \text{Hom}_R(M, N), \quad \text{Tor}_0^R(M, N) = M \otimes_R N.$$

Ext and Tor are called the derived functors of Hom and \otimes . It is a classical result in homological algebra that $\text{Ext}_R^k(M, N)$ and $\text{Tor}_k^R(M, N)$ don't depend on the choice of resolution of M . They are functorial with respect to both variables and Tor_k^R is symmetric in two variables

$$\text{Tor}_k^R(M, N) = \text{Tor}_k^R(N, M).$$

Moreover, for any short exact sequence of R -modules

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

there associate long exact sequences

$$\begin{aligned} 0 \rightarrow \text{Hom}_R(M_3, N) &\rightarrow \text{Hom}_R(M_2, N) \rightarrow \text{Hom}_R(M_1, N) \\ &\rightarrow \text{Ext}_R^1(M_3, N) \rightarrow \text{Ext}_R^1(M_2, N) \rightarrow \text{Ext}_R^1(M_1, N) \\ &\rightarrow \text{Ext}_R^2(M_3, N) \rightarrow \text{Ext}_R^2(M_2, N) \rightarrow \text{Ext}_R^2(M_1, N) \rightarrow \cdots \end{aligned}$$

and

$$\begin{aligned} 0 \rightarrow \text{Hom}_R(N, M_1) &\rightarrow \text{Hom}_R(N, M_2) \rightarrow \text{Hom}_R(N, M_3) \\ &\rightarrow \text{Ext}_R^1(N, M_1) \rightarrow \text{Ext}_R^1(N, M_2) \rightarrow \text{Ext}_R^1(N, M_3) \\ &\rightarrow \text{Ext}_R^2(N, M_1) \rightarrow \text{Ext}_R^2(N, M_2) \rightarrow \text{Ext}_R^2(N, M_3) \rightarrow \cdots \end{aligned}$$

and

$$\begin{aligned} \cdots \rightarrow \text{Tor}_2^R(M_1, N) &\rightarrow \text{Tor}_2^R(M_2, N) \rightarrow \text{Tor}_3^R(M_3, N) \\ &\rightarrow \text{Tor}_1^R(M_1, N) \rightarrow \text{Tor}_1^R(M_2, N) \rightarrow \text{Tor}_1^R(M_3, N) \\ &\rightarrow M_1 \otimes_R N \rightarrow M_2 \otimes_R N \rightarrow M_3 \otimes_R N \rightarrow 0 \end{aligned}$$

Now we focus on the case of abelian groups $R = \mathbb{Z}$. For any abelian group M , let P_0 be a free abelian group such that $P_0 \rightarrow M$ is surjective. Let P_1 be its kernel. Then P_1 is also free and

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

defines a free resolution of abelian groups. This implies that

$$\text{Ext}^k(M, N) = 0, \quad \text{Tor}_k(M, N) = 0 \quad \text{for } k \geq 2.$$

In the case of abelian groups we will simply denote

$$\boxed{\text{Ext}(M, N) := \text{Ext}_{\mathbb{Z}}^1(M, N), \quad \text{Tor}(M, N) := \text{Tor}_{\mathbb{Z}}^1(M, N)}.$$

Lemma 20.11. *If either M is free or N is divisible, then $\text{Ext}(M, N) = 0$.*

Proposition 20.12. *Let (C_\bullet, ∂) be a chain complex of free abelian groups, then*

$$H^n(\text{Hom}(C_\bullet, G)) \cong \text{Hom}(H_n(C_\bullet), G) \oplus \text{Ext}(H_{n-1}(C_\bullet), G)$$

Proof. Let B_n be n -boundaries and Z_n be n -cycles, which are both free. We have exact sequences

$$0 \rightarrow B_n \rightarrow Z_n \rightarrow H_n \rightarrow 0, \quad 0 \rightarrow Z_n \rightarrow C_n \rightarrow B_{n-1} \rightarrow 0.$$

This implies exact sequences

$$0 \rightarrow \text{Hom}(H_n, G) \rightarrow \text{Hom}(Z_n, G) \rightarrow \text{Hom}(B_n, G) \rightarrow \text{Ext}(H_n, G) \rightarrow 0$$

and the split exact sequence

$$0 \rightarrow \text{Hom}(B_{n-1}, G) \rightarrow \text{Hom}(C_n, G) \rightarrow \text{Hom}(Z_n, G) \rightarrow 0.$$

Consider the commutative diagram with exact columns

$$\begin{array}{ccccc}
 & 0 & & 0 & \\
 & \uparrow & & \downarrow & \\
 & \text{Hom}(Z_{n-1}, G) & \longrightarrow & \text{Hom}(B_{n-1}, G) & \\
 & \uparrow & & \downarrow & \\
 & \text{Hom}(C_{n-1}, G) & \longrightarrow & \text{Hom}(C_n, G) & \longrightarrow & \text{Hom}(C_{n+1}, G) \\
 & & & \downarrow & & \uparrow \\
 & & & \text{Hom}(Z_n, G) & \longrightarrow & \text{Hom}(B_n, G) \\
 & & & \downarrow & & \uparrow \\
 & & & 0 & & 0
 \end{array}$$

This implies a short exact sequence

$$0 \rightarrow \text{Ext}(H_{n-1}, G) \rightarrow H^n(\text{Hom}(C_\bullet, G)) \rightarrow \text{Hom}(H_n, G) \rightarrow 0$$

which is also split due to the split of the middle column in the above diagram. \square

Theorem 20.13 (Universal Coefficient Theorem for Cohomology). *Let G be an abelian group and X be a topological space. Then for any $n \geq 0$, there exists a split exact sequence*

$$0 \rightarrow \text{Ext}(H_{n-1}(X), G) \rightarrow H^n(X; G) \rightarrow \text{Hom}(H_n(X), G) \rightarrow 0$$

which induces isomorphisms

$$H^n(X; G) \cong \text{Hom}(H_n(X), G) \oplus \text{Ext}(H_{n-1}(X), G).$$

Proof. Apply the previous Lemma to $C_\bullet = S_\bullet(X)$. \square

Universal Coefficient Theorem for Homology.

Definition 20.14. Let G be an abelian group. Let $A \subset X$ be a subspace. We define the **relative singular chain complex** with coefficient in G by

$$S_{\bullet}(X, A; G) := S_{\bullet}(X, A) \otimes_{\mathbb{Z}} G$$

Its cohomology is called the **relative singular homology** with coefficient in G , denoted by $H_{\bullet}(X, A; G)$. When $A = \emptyset$, we simply get the singular homology $H_{\bullet}(X; G)$.

Similar long exact sequence for relative singular homologies follows from the exact sequence

$$0 \rightarrow S_{\bullet}(A; G) \rightarrow S_{\bullet}(X; G) \rightarrow S_{\bullet}(X, A; G) \rightarrow 0.$$

Theorem 20.15 (Universal Coefficient Theorem for homology). *Let G be an abelian group and X be a topological space. Then for any $n \geq 0$, there exists a split exact sequence*

$$0 \rightarrow H_n(X) \otimes G \rightarrow H_n(X; G) \rightarrow \text{Tor}(H_{n-1}(X), G) \rightarrow 0$$

which induces isomorphisms

$$H_n(X; G) \cong H_n(X) \otimes G \oplus \text{Tor}(H_{n-1}(X), G).$$

The proof is similar to the cohomology case.

21. EILENBERG-ZILBER THEOREM AND KÜNNETH FORMULA

Eilenberg-Zilber Theorem.

Definition 21.1. Let $(C_{\bullet}, \partial_C)$ and $(D_{\bullet}, \partial_D)$ be two chain complexes. We define their tensor product $C_{\bullet} \otimes D_{\bullet}$ as the chain complex

$$(C_{\bullet} \otimes D_{\bullet})_k := \sum_{p+q=k} C_p \otimes D_q$$

with the boundary map

$$\partial(c_p \otimes d_q) := \partial_C(c_p) \otimes d_q + (-1)^p c_p \otimes \partial_D(d_q), \quad c_p \in C_p, d_q \in D_q.$$

Proposition 21.2. *Assume C_{\bullet} is chain homotopy equivalent to C'_{\bullet} . Then $C_{\bullet} \otimes D_{\bullet}$ is chain homotopy equivalent to $C'_{\bullet} \otimes D_{\bullet}$.*

We would like to compare two functors from $\underline{\text{Top}} \times \underline{\text{Top}} \rightarrow \underline{\text{Ch}}_{\bullet}$:

$$S_{\bullet}(X \times Y), \quad S_{\bullet}(X) \otimes S_{\bullet}(Y).$$

We first observe that there exists a canonical isomorphism

$$H_0(X \times Y) \cong H_0(X) \otimes H_0(Y).$$

The following theorem of Eilenberg-Zilber says that such initial condition determines a natural homotopy equivalent between the above two functors which is unique up to chain homotopy.

Theorem 21.3 (Eilenberg-Zilber). *Let X, Y be two topological spaces. Then there exists a chain equivalence*

$$\boxed{S_{\bullet}(X \times Y) \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} S_{\bullet}(X) \otimes S_{\bullet}(Y)}$$

which is natural with respect to X, Y and induce the canonical isomorphism $H_0(X \times Y) \cong H_0(X) \otimes H_0(Y)$. Such chain equivalence is unique up to chain homotopy. In particular, there are canonical isomorphisms

$$H_n(X \times Y) = H_n(S_\bullet(X) \otimes S_\bullet(Y)), \quad \forall n \geq 0.$$

F, G will be called **Eilenberg-Zilber maps**.

Proof. Observe that any map $\Delta^p \xrightarrow{(\sigma_x, \sigma_y)} X \times Y$ factors through

$$\Delta^p \xrightarrow{\delta_p} \Delta^p \times \Delta^p \xrightarrow{\sigma_x \times \sigma_y} X \times Y$$

where $\Delta^p \xrightarrow{\delta_p} \Delta^p \times \Delta^p$ is the diagonal map. This implies that a natural transformation of the functor $S_\bullet(-, -)$ is determined by its value on $\{\delta_p\}_{p \geq 0}$. For example,

$$F((\sigma_x, \sigma_y)) = (\sigma_x \otimes \sigma_y)_* F(\delta_p).$$

Similarly, a natural transformation of the functor $S_\bullet(-) \otimes S_\bullet(-)$ is determined by its value on $1_p \otimes 1_q$ where $1_p : \Delta^p \rightarrow \Delta^p$ is the identity map. For example, for any $\sigma_x : \Delta^p \rightarrow X, \sigma_y : \Delta^q \rightarrow Y$, we have

$$G(\sigma_x \otimes \sigma_y) = (\sigma_x \times \sigma_y)_* G(1_p \otimes 1_q).$$

Therefore F and G are completely determined by

$$f_n := F(\delta_n) \in \bigoplus_{p+q=n} S_p(\Delta^n) \otimes S_q(\Delta^n), \quad g_n := G(1_p \otimes 1_q) \in \bigoplus_{p+q=n} S_n(\Delta^p \times \Delta^q).$$

We will use the same notations as in the discussion of Barycentric decomposition. Then

$$f_n \circ g_n \simeq \delta_n \in S_n(\Delta^n \times \Delta^n), \quad g_n \circ f_n \in \bigoplus_{p+q=n} (S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q))_n.$$

Let us denote the following complexes

$$C_n = \prod_{k \geq 0} (S_\bullet(\Delta^k) \otimes S_\bullet(\Delta^k))_{n+k}, \quad D_n = \prod_{m \geq 0} \left(\bigoplus_{p+q=m} S_{n+p+q}(\Delta^p \times \Delta^q) \right)$$

with boundary map

$$\partial + \tilde{\partial} : C_n \rightarrow C_{n-1}, \quad \partial + \tilde{\partial} : D_n \rightarrow D_{n-1}$$

as follows. ∂ is the usual boundary map of singular chain complexes

$$\partial : (S_\bullet(\Delta^k) \otimes S_\bullet(\Delta^k))_n \rightarrow (S_\bullet(\Delta^k) \otimes S_\bullet(\Delta^k))_{n-1}, \quad \partial : S_n(\Delta^p \times \Delta^q) \rightarrow S_{n-1}(\Delta^p \times \Delta^q).$$

$\tilde{\partial}$ is the map induced by composing with the face singular chain $\tilde{\partial} \in \bigoplus_k S_{k-1}(\Delta^k)$

$$\tilde{\partial} : S_p(\Delta^{k-1}) \otimes S_q(\Delta^{k-1}) \rightarrow S_p(\Delta^k) \otimes S_q(\Delta^k), \quad \sigma_p \otimes \sigma_q \rightarrow \tilde{\partial} \circ \sigma_p \otimes \tilde{\partial} \circ \sigma_q$$

and

$$\tilde{\partial} : S_n(\Delta^p \times \Delta^q) \rightarrow S_n(\Delta^{p+1} \times \Delta^q) \oplus S_n(\Delta^p \times \Delta^{q+1}), \quad \sigma_p \times \sigma_q \rightarrow (\tilde{\partial} \circ \sigma_p) \times \sigma_q + (-1)^{n-p} \sigma_p \times (\tilde{\partial} \circ \sigma_q).$$

Then $f = (f_n) \in C_0$ and $g = (g_n) \in D_0$.

$$F, G \text{ are chain maps} \iff f, g \text{ are 0-cycles in } C_\bullet, D_\bullet$$

and natural chain homotopy of F, G are given by 0-boundaries. We claim that

$$H_n(C_\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}, \quad H_n(D_\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}.$$

This follows from a spectral sequence computation by first computing ∂ -homology and then computing $\tilde{\partial}$ -homology. For example the first page $(H_\bullet(C_\bullet, \partial), \tilde{\partial})$ is

$$0 \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \dots$$

whose $\tilde{\partial}$ -homology is now \mathbb{Z} at degree 0. The case of D is similar. This implies that the initial condition completely determines chain maps F, G up to chain homotopy.

Let us now analyze the composition $F \circ G$ and $G \circ F$. We similarly form the chain complexes

$$C'_n = \prod_{k \geq 0} S_{n+k}(\Delta^k \times \Delta^k), \quad D'_n := \prod_{m \geq 0} \bigoplus_{p+q=m} (S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q))_{n+p+q}$$

with boundary map $\partial + \tilde{\partial}$ defined similarly. Homology of C'_\bullet controls natural chain maps of $S_\bullet(X \times Y)$ to itself up to chain homotopy, and similarly for D'_\bullet . We still have

$$H_n(C'_\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}, \quad H_n(D'_\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}.$$

It follows that $F \circ G$ and $G \circ F$ are both naturally chain homotopic to the identity map. The theorem follows. \square

An explicit construction of G can be described as follows: given $\sigma_p : \Delta^p \rightarrow X, \sigma_q : \Delta^q \rightarrow Y$,

$$G(\sigma_p \otimes \sigma_q) : \Delta^p \times \Delta^q \rightarrow X \times Y$$

where we have to chop $\Delta^p \times \Delta^q$ into $p + q$ -simplexes. This is the **shuffle product**.

An explicit construction of F can be given by **Alexander-Whitney map** described as follows.

Definition 21.4. Given a singular n -simplex $\sigma : \Delta^n \rightarrow X$ and $0 \leq p, q \leq n$, we define

- the **front p -face** of σ to be the singular p -simplex

$${}_p\sigma : \Delta^p \rightarrow X, \quad {}_p\sigma(t_0, \dots, t_p) := \sigma(t_0, \dots, t_p, 0, \dots, 0)$$

- the **back q -face** of σ to be the singular q -simplex

$$\sigma_q : \Delta^q \rightarrow X, \quad \sigma_q(t_0, \dots, t_q) := \sigma(0, \dots, 0, t_0, \dots, t_q).$$

Definition 21.5. Let X, Y be topological spaces. Let $\pi_X : X \times Y \rightarrow X, \pi_Y : X \times Y \rightarrow Y$ be the projections. We define the **Alexander-Whitney map**

$$AW : S_\bullet(X \times Y) \rightarrow S_\bullet(X) \otimes S_\bullet(Y)$$

by the natural transformation given by the formula

$$AW(\sigma) := \sum_{p+q=n} p(\pi_X \circ \sigma) \otimes (\pi_Y \circ \sigma)_q.$$

Theorem 21.6. *The Alexander-Whitney map is a chain homotopy equivalence.*

Proof. It is easy to see that AW is a natural chain map which induces the canonical isomorphism

$$H_0(X \times Y) \rightarrow H_0(X) \otimes H_0(Y).$$

Therefore AW is a chain homotopy equivalence by Eilenberg-Zilber Theorem. \square

Künneth formula.

Theorem 21.7 (Algebraic Künneth formula). *Let C_\bullet and D_\bullet be chain complex of free abelian groups. Then there is a split exact sequence*

$$0 \rightarrow (H_\bullet(C) \otimes H_\bullet(D))_n \rightarrow H_n(C_\bullet \otimes D_\bullet) \rightarrow \text{Tor}(H_\bullet(C), H_\bullet(D))_{n-1} \rightarrow 0.$$

Here $\text{Tor}(H_\bullet(C), H_\bullet(D))_k = \bigoplus_{p+q=k} \text{Tor}(H_p(C), H_q(D))$.

Proof. Using the freeness of C_\bullet we can show that

$$H_\bullet(C_\bullet \otimes D_\bullet) = H_\bullet(C_\bullet \otimes H_\bullet(D)).$$

Applying Universal Coefficient Theorem for Homology, we find

$$0 \rightarrow H_p(C) \otimes H_q(D) \rightarrow H_{p+q}(C_{\bullet-q} \otimes H_q(D)) \rightarrow \text{Tor}(H_{p-1}(C), H_q(D)) \rightarrow 0.$$

Summing over p, q gives the theorem. □

Theorem 21.8 (Künneth formula). *For any topological spaces X, Y and $n \geq 0$, there is a split exact sequence*

$$0 \rightarrow \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \rightarrow H_n(X \times Y) \rightarrow \bigoplus_{p+q=n-1} \text{Tor}(H_p(X), H_q(Y)) \rightarrow 0.$$

Proof. This follows from Eilenberg-Zilber Theorem and algebraic Künneth formula. □

22. CUP AND CAP PRODUCT

Let R be a commutative ring with unit. We have natural cochain maps

$$S^\bullet(X; R) \otimes_R S^\bullet(Y; R) \rightarrow \text{Hom}(S^\bullet(X) \otimes S^\bullet(Y), R) \rightarrow S^\bullet(X \times Y; R)$$

where the first map maps $\varphi_p \in S^p(X; R), \eta_q \in S^q(Y; R)$ to $\varphi_p \otimes \eta_q$ where

$$\varphi_p \otimes \eta_q : \sigma_p \otimes \sigma_q \rightarrow \varphi_p(\sigma_p) \cdot \eta_q(\sigma_q), \quad \sigma_p \in S_p(X), \quad \sigma_q \in S_q(Y).$$

Here \cdot is the product in R . This leads to a cochain map

$$S^\bullet(X; R) \otimes_R S^\bullet(Y; R) \rightarrow S^\bullet(X \times Y; R)$$

which further induces

$$H^\bullet(X; R) \otimes_R H^\bullet(Y; R) \rightarrow H^\bullet(X \times Y; R)$$

Cup product.

Definition 22.1. Let R be a commutative ring with unit. We define the **cup product** on cohomology groups

$$\cup : H^p(X; R) \otimes_R H^q(X; R) \rightarrow H^{p+q}(X; R)$$

by the composition

$$\begin{array}{ccc} H^\bullet(X; R) \otimes_R H^\bullet(X; R) & \longrightarrow & H^\bullet(X \times X; R) \\ & \searrow \cup & \downarrow \Delta^* \\ & & H^\bullet(X; R) \end{array}$$

Here $\Delta : X \rightarrow X \times X$ is the diagonal map.

Alexander-Whitney map gives a specific product formula

$$(\alpha \cup \beta)(\sigma) = \alpha(\rho\sigma) \cdot \beta(\sigma_q), \quad \alpha \in S^p(X; R), \beta \in S^q(X; R), \sigma : \Delta^{p+q} \rightarrow X.$$

Theorem 22.2. $H^\bullet(X; R)$ is a graded commutative ring with unit:

(1) **Unit:** let $1 \in H^0(X; R)$ be represented by the cocycle which takes every singular 0-simplex to $1 \in R$. Then

$$1 \cup \alpha = \alpha \cup 1 = \alpha, \quad \forall \alpha \in H^\bullet(X; R).$$

(2) **Associativity:**

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma).$$

(3) **Graded commutativity:**

$$\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha, \quad \forall \alpha \in H^p(X; R), \beta \in H^q(X; R).$$

Proof. Unit of 1 is checked easily. Observe that the following two compositions of Eilenberg-Zilber maps are chain homotopic (similar to Eilenberg-Zilber Theorem)

$$\begin{aligned} S_\bullet(X \times Y \times Z) &\rightarrow S_\bullet(X \times Y) \otimes S_\bullet(Z) \rightarrow S_\bullet(X) \otimes S_\bullet(Y) \otimes S_\bullet(Z) \\ S_\bullet(X \times Y \times Z) &\rightarrow S_\bullet(X) \otimes S_\bullet(Y \times Z) \rightarrow S_\bullet(X) \otimes S_\bullet(Y) \otimes S_\bullet(Z). \end{aligned}$$

Associativity follows from the commutative diagram (R is hidden for simplicity)

$$\begin{array}{ccccc} H^\bullet(X) \otimes H^\bullet(X) \otimes H^\bullet(X) & \longrightarrow & H^\bullet(X \times X) \otimes H^\bullet(X) & \xrightarrow{(\Delta \times 1)^*} & H^\bullet(X) \otimes H^\bullet(X) \\ \downarrow & & \downarrow & & \downarrow \\ H^\bullet(X) \otimes H^\bullet(X \times X) & \longrightarrow & H^\bullet(X \times X \times X) & \xrightarrow{(\Delta \times 1)^*} & H^\bullet(X \times X) \\ \downarrow (1 \times \Delta)^* & & \downarrow (1 \times \Delta)^* & & \downarrow \Delta^* \\ H^\bullet(X) \otimes H^\bullet(X) & \longrightarrow & H^\bullet(X \times X) & \xrightarrow{\Delta^*} & H^\bullet(X) \end{array}$$

Graded commutativity follows from the fact that the interchange map of tensor product of chain complexes

$$\begin{aligned} T : C_\bullet \otimes D_\bullet &\rightarrow D_\bullet \otimes C_\bullet \\ c_p \otimes d_q &\rightarrow (-1)^{pq} d_q \otimes c_p \end{aligned}$$

is a chain isomorphism. Therefore the two chain maps

$$\begin{aligned} S_\bullet(X \times Y) &\rightarrow S_\bullet(Y \times X) \rightarrow S_\bullet(Y) \otimes S_\bullet(X) \\ S_\bullet(X \times Y) &\rightarrow S_\bullet(X) \otimes S_\bullet(Y) \xrightarrow{T} S_\bullet(Y) \otimes S_\bullet(X) \end{aligned}$$

are chain homotopic, again by the uniqueness in Eilenberg-Zilber Theorem.

Set $Y = X$ we find the following commutative diagram

$$\begin{array}{ccc} H^\bullet(X) \otimes H^\bullet(X) & \longrightarrow & H^\bullet(X \times X) \\ \downarrow T & & \downarrow = \\ H^\bullet(X) \otimes H^\bullet(X) & \longrightarrow & H^\bullet(X \times X). \end{array}$$

which gives graded commutativity.

Alternately, all the above can be checked explicitly using Alexander-Whitney map □

Theorem 22.3. Let $f : X \rightarrow Y$ be a continuous map. Then

$$f^* : H^\bullet(Y; R) \rightarrow H^\bullet(X; R)$$

is a morphism of graded commutative rings, i.e. $f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$. In other words, $H^\bullet(-)$ defines a functor from the category of topological spaces to the category of graded commutative rings.

Proof. The theorem follows from the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta \downarrow & & \downarrow \Delta \\ X \times X & \xrightarrow{f \times f} & Y \times Y. \end{array}$$

□

Theorem 22.4 (Künneth formula). Assume R is a PID, and $H_i(X; R)$ are finitely generated R -module, then there exists a split exact sequence of R -modules

$$0 \rightarrow \bigoplus_{p+q=n} H^p(X; R) \otimes H^q(Y; R) \rightarrow H^n(X \times Y; R) \rightarrow \bigoplus_{p+q=n+1} \text{Tor}_1^R(H^p(X; R), H^q(Y; R)).$$

In particular, if $H^\bullet(X; R)$ or $H^\bullet(Y; R)$ are free R -modules, we have an isomorphism of graded commutative rings

$$H^\bullet(X \times Y; R) \cong H^\bullet(X; R) \otimes_R H^\bullet(Y; R).$$

Example 22.5. $H^\bullet(S^n) = \mathbb{Z}[\eta]/\eta^2$ where $\eta \in H^n(S^n)$ is a generator.

Example 22.6. Let $T^n = S^1 \times \cdots \times S^1$ be the n -torus. Then

$$H^\bullet(T^n) \cong \mathbb{Z}[\eta_1, \dots, \eta_n], \quad \eta_i \eta_j = -\eta_j \eta_i$$

is the exterior algebra with n generators. Each η_i corresponds a generator of $H^1(S^1)$.

Proposition 22.7. $H^\bullet(\mathbb{C}P^n) = \mathbb{Z}[x]/x^{n+1}$, where $x \in H^2(\mathbb{C}P^n)$ is a generator.

Proof. We prove by induction n . We know that

$$H^k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & k = 2m \leq 2n \\ 0 & \text{otherwise} \end{cases}$$

Let x be a generator of $H^2(\mathbb{C}P^n)$. We only need to show that x^k is a generator of $H^{2k}(\mathbb{C}P^n)$ for each $k \leq n$. Using cellular chain complex, we know that for $k < n$

$$H^{2k}(\mathbb{C}P^n) \rightarrow H^{2k}(\mathbb{C}P^k)$$

is an isomorphism. By induction, this implies that x^k is a generator of $H^{2k}(\mathbb{C}P^n)$ for $k < n$. Poincare duality theorem (which will be proved in the next section) implies that

$$H^2(\mathbb{C}P^n) \otimes H^{2n-2}(\mathbb{C}P^n) \xrightarrow{\cup} H^{2n}(\mathbb{C}P^n)$$

is an isomorphism. This says that x^n is a generator of $H^{2n}(\mathbb{C}P^n)$. This proves the proposition.

□

Cap product.

Definition 22.8. We define the evaluation map

$$\langle -, - \rangle : S^\bullet(X; R) \times_R S_\bullet(X; R) \rightarrow R$$

as follows: for $\alpha \in S^p(X; R), \sigma \in S_p(X), r \in R$,

$$\langle \alpha, \sigma \otimes r \rangle := \alpha(\sigma) \cdot r.$$

The evaluation map is compatible with boundary map and induces an evaluation map

$$\langle -, - \rangle : H^p(X; R) \otimes_R H_p(X; R) \rightarrow R.$$

This generalized to

$$S^\bullet(X; R) \otimes_R S_\bullet(X \times Y; R) \rightarrow S^\bullet(X; R) \otimes_R S_\bullet(X; R) \otimes_R S_\bullet(Y; R) \xrightarrow{\langle -, - \rangle^{\otimes 1}} S_\bullet(Y; R)$$

which induces

$$H^p(X; R) \otimes_R H_{p+q}(X \times Y; R) \rightarrow H_q(Y; R).$$

Definition 22.9. We define the **cap product**

$$\boxed{\cap : H^p(X; R) \otimes H_{p+q}(X; R) \rightarrow H_q(X; R)}$$

by the composition

$$\begin{array}{ccc} H^p(X; R) \otimes H_{p+q}(X; R) & \xrightarrow{1 \otimes \Delta} & H^p(X; R) \otimes H_{p+q}(X \times X; R) \\ & \searrow \cap & \downarrow \\ & & H_q(X; R) \end{array}$$

Theorem 22.10. The cap product gives $H_\bullet(X; R)$ a structure of $H^\bullet(X; R)$ -module.

Theorem 22.11. The cap product extends naturally to the relative case: for any pair $A \subset X$

$$\cap : H^p(X, A) \otimes H_{p+q}(X, A) \rightarrow H_q(X)$$

$$\cap : H^p(X) \otimes H_{p+q}(X, A) \rightarrow H_q(X, A)$$

Proof. Since $S^\bullet(X, A) \subset S^\bullet(X)$, we have

$$\cap : S^\bullet(X, A) \times S_\bullet(X) \rightarrow S_\bullet(X).$$

We model the cap product on chains via the Alexander-Whitney map. Then

$$\cap : S^\bullet(X, A) \times S_\bullet(A) \rightarrow 0.$$

Therefore \cap factors through

$$\cap : S^\bullet(X, A) \times \frac{S_\bullet(X)}{S_\bullet(A)} \rightarrow S_\bullet(X).$$

Passing to homology (cohomology) we find the first cap product. The second one is proved similarly using

$$\cap : S^\bullet(X) \times \frac{S_\bullet(X)}{S_\bullet(A)} \rightarrow \frac{S_\bullet(X)}{S_\bullet(A)}.$$

□

23. POINCARÉ DUALITY

Definition 23.1. A **topological manifold** of dimension n , or a topological n -manifold, is a Hausdorff space in which each point has an open neighborhood homeomorphic to \mathbb{R}^n .

In this section, a manifold always means a topological manifold. For any point $x \in X$, there exists an open neighborhood U and a homeomorphism $\phi : U \rightarrow \mathbb{R}^n$. (U, ϕ) is called a **chart** around x .

Orientation.

Definition 23.2. Let X be a n -manifold. $x \in X$ be a point. A generator of

$$H_n(X, X - x) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong \mathbb{Z}$$

is called a **local orientation** of X at x .

For any $x \in X$, there are two choices of local orientation at x . We obtain a two-sheet cover

$$\pi : \tilde{X} \rightarrow X, \quad \text{where} \quad \tilde{X} = \{(x, \mu_x) \mid \mu_x \text{ is a local orientation of } X \text{ at } x\}$$

Here π is the natural projection $(x, \mu_x) \rightarrow x$. \tilde{X} is topologized as follows. Let U be a small open ball in X . Then for any $x \in U$, we have an isomorphism

$$H_n(X, X - U) \cong H_n(X, X - x)$$

which induces a set theoretical identification

$$\pi^{-1}(U) \cong U \times \mathbb{Z}_2.$$

Then we give a topology on \tilde{X} by requiring all such identifications being homeomorphisms. In particular, $\pi : \tilde{X} \rightarrow X$ is a \mathbb{Z}_2 -covering map.

Definition 23.3. A (global) **orientation** of X is a section of $\pi : \tilde{X} \rightarrow X$, i.e., a continuous map $s : X \rightarrow \tilde{X}$ such that $\pi \circ s = 1_X$. If an orientation exists, we say X is **orientable**.

Theorem 23.4. Let X be a connected manifold. Then X is orientable if and only if \tilde{X} has two connected components. In particular, a connected orientable manifold has precisely two orientations.

Example 23.5. A simply connected manifold is orientable.

Example 23.6. Let X be connected non-orientable manifold. Then \tilde{X} is connected orientable.

Lemma 23.7. Let $U \subset \mathbb{R}^n$ be open. Then the natural map

$$H_n(\mathbb{R}^n, U) \rightarrow \prod_{x \in \mathbb{R}^n - U} H_n(\mathbb{R}^n, \mathbb{R}^n - x)$$

is injective.

Proof. This is equivalent to the injectivity of

$$\tilde{H}_{n-1}(U) \rightarrow \prod_{x \in \mathbb{R}^n - U} H_{n-1}(\mathbb{R}^n - x).$$

Let α be a singular $(n-1)$ -chain representing a class $[\alpha]_U$ in $\tilde{H}_{n-1}(U)$. We can choose a big ball B containing U and finite small cubes D_1, \dots, D_N such that D_i is not a subset of U but

$$\text{Supp}(\alpha) \subset B - D_1 \cup \dots \cup D_N \subset U.$$

Then α represents a class in $\tilde{H}_{n-1}(D_1 \cup \cdots \cup D_N) \cong H_n(B, B - D_1 \cup \cdots \cup D_N)$ which maps to zero in each $H_n(B, B - D_i) \cong H_n(B, B - x_i)$ where $x_i \in D_i - U$. It follows via Mayer-Vietoris argument that α is the zero class in $\tilde{H}_{n-1}(D_1 \cup \cdots \cup D_N)$, hence zero in $\tilde{H}_{n-1}(U)$. \square

Fundamental class.

Theorem 23.8. *Let X be a connected n -manifold. For any abelian group G , we have the following vanishing statement*

$$\begin{cases} H_i(X; G) = 0 & i > n \\ H_n(X; G) = 0 & \text{if } X \text{ is noncompact.} \end{cases}$$

Proof. We prove the case for $G = \mathbb{Z}$. General G is similar. We assume X is connected.

Step 1: $X = U \subset \mathbb{R}^n$ is an open subset.

Let $\alpha \in S_i(U)$ represent an element of $[\alpha] \in H_i(U)$. Let $K \subset U$ be a compact subset such that $\text{Supp}(\alpha) \in K$. Equip \mathbb{R}^n with a CW structure in terms of small enough cubes such that

$$K \subset L \subset U$$

where L is a finite CW subcomplex. We have a commutative diagram

$$\begin{array}{ccc} H_{i+1}(\mathbb{R}^n, L) & \longrightarrow & H_{i+1}(\mathbb{R}^n, U) \\ \downarrow & & \downarrow \\ H_i(L) & \longrightarrow & H_i(U) \end{array}$$

By construction, $[\alpha] \in H_i(U)$ lies in the image of $H_i(L)$. But $H_{i+1}(\mathbb{R}^n, L) \cong H_{i+1}^{cell}(\mathbb{R}^n, L) = 0$ for $i \geq n$.

Step 2: $X = U \cup V$ where U open is homeomorphic to \mathbb{R}^n and V open satisfies the vanishing condition.

Consider the Mayer-Vietoris sequence

$$\tilde{H}_i(U) \oplus \tilde{H}_i(V) \rightarrow \tilde{H}_i(U \cup V) \rightarrow \tilde{H}_{i-1}(U \cap V) \rightarrow \tilde{H}_{i-1}(U) \oplus \tilde{H}_{i-1}(V)$$

For $i > n$, we find $H_i(U \cup V) = 0$ by Step 1. Assume that $X = U \cup V$ is not compact. We need to prove

$$\tilde{H}_{n-1}(U \cap V) \rightarrow \tilde{H}_{n-1}(V)$$

is injective. The noncompactness and connectedness of X implies that

$$H_n(U \cup V) \rightarrow H_n(U \cup V, U \cup V - x)$$

is zero map for any $x \in X$. Consider the commutative diagram, where $x \in U - U \cap V$

$$\begin{array}{ccccccc} & H_n(U \cup V) & & & & & \\ & \downarrow & \searrow & & & & \\ H_n(U \cup V, U \cup V - x) & & H_n(U \cup V, U \cap V) & \longleftrightarrow & H_n(V, U \cap V) & & \\ & \downarrow \cong & \uparrow & \searrow & \downarrow & & \\ H_n(U, U - x) & \longleftarrow & H_n(U, U \cap V) & \longrightarrow & \tilde{H}_{n-1}(U \cap V) & \longrightarrow & 0 \\ & & & & \downarrow & & \\ & & & & \tilde{H}_{n-1}(V) & & \end{array}$$

Let $\alpha \in H_n(U, U \cap V)$ maps to $\ker(\tilde{H}_{n-1}(U \cap V) \rightarrow \tilde{H}_{n-1}(V))$. Diagram chasing implies that α maps to $H_n(U, U - x)$ for any $x \in U - U \cap V$. Since x is arbitrary, this implies $\alpha = 0$ by the previous lemma.

Step 3: General case. Let $\alpha \in \mathcal{S}_i(X)$ representing a class in $H_i(X)$. We can choose finite coordinate charts U_1, \dots, U_N such that $\text{Supp}(\alpha) \subset U_1 \cup \dots \cup U_N$. Then the class of α lies in the image of the map

$$H_i(U_1 \cup \dots \cup U_N) \rightarrow H_i(X).$$

We only need to prove the theorem for $U_1 \cup \dots \cup U_N$. This follows from Step 2 and induction on N . \square

Definition 23.9. Let X be an n -manifold. A **fundamental class** of X at a subspace $A \subset X$ is an element $s \in H_n(X, X - A)$ whose image

$$H_n(X, X - A) \rightarrow H_n(X, X - x)$$

defines a local orientation for each $x \in A$. When $A = X$, $s \in H_n(X)$ is called a fundamental class of X .

Theorem 23.10. Let X be an oriented n -manifold, $K \subset X$ be compact subspace. Then

- (1) $H_i(X, X - K) = 0$ for any $i > n$.
- (2) The orientation of X defines a unique fundamental class of X at K .

In particular, if X is compact, then there exists a unique fundamental class of X associated to the orientation.

Proof.

Step 1: K is a compact subset inside a coordinate chart $U \cong \mathbb{R}^n$. Then

$$H_i(X, X - K) \cong H_i(U, U - K) \cong \tilde{H}_{i-1}(U - K) = 0 \quad i > n.$$

Take a big enough ball B such that $K \subset B \subset U$. The orientation of X at the local chart U determines an element of $H_n(X, X - U)$ which maps to the required fundamental class of X at K .

Step 2: $K = K_1 \cup K_2$ where $K_1, K_2, K_1 \cap K_2$ satisfy (1)(2). Using Mayer-Vietoris sequence

$$\dots H_{i+1}(X, X - K_1 \cap K_2) \rightarrow H_i(X, X - K_1 \cup K_2) \rightarrow H_i(X, X - K_1) \oplus H_i(X, X - K_2) \rightarrow H_i(X, X - K_1 \cap K_2) \rightarrow \dots$$

we see K satisfies (1). The unique fundamental classes at K_1 and K_2 map to the unique fundamental class at $K_1 \cap K_2$, giving rise to a unique fundamental class at $K_1 \cup K_2$ by the exact sequence

$$0 \rightarrow H_n(X, X - K_1 \cup K_2) \rightarrow H_n(X, X - K_1) \oplus H_n(X, X - K_2) \rightarrow H_n(X, X - K_1 \cap K_2)$$

Step 3: For arbitrary K , it is covered by a finite number of coordinates charts $\{U_i\}_{1 \leq i \leq N}$. Let $K_i = K \cap U_i$. Then $K = K_1 \cup \dots \cup K_N$. The theorem holds for K by induction on N and Step 1, 2. \square

Poincaré duality.

Definition 23.11. Let \mathcal{K} denote the set of compact subspaces of X . We define **compactly supported cohomology** of X by

$$H_c^k(X) := \text{colim}_{K \in \mathcal{K}} H^k(X, X - K)$$

where the colimit is taken with respect to the homomorphisms

$$H^k(X, X - K_1) \rightarrow H^k(X, X - K_2)$$

for $K_1 \subset K_2$ compact. In particular, if X is compact, then $H_c^k(X) = H^k(X)$.

The functorial structure is with respect to the proper maps: let $f : X \rightarrow Y$ be proper, then

$$f^* : H_c^k(Y) \rightarrow H_c^k(X).$$

Example 23.12. Let $X = \mathbb{R}^n$. Consider the sequence of compact subspaces $B_1 \subset B_2 \subset B_3 \subset \dots$, where B_k is the closed ball of radius k . Any compact subspace is contained in some ball. Therefore

$$H_c^i(\mathbb{R}^n) = \operatorname{colim}_k H^i(\mathbb{R}^n, \mathbb{R}^n - B_k) = \tilde{H}^i(S^{n-1}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}.$$

Theorem 23.13. Let $X = U \cup V$ where U, V open. Then we have the Mayer-Vietoris exact sequence

$$\dots \rightarrow H_c^k(U \cap V) \rightarrow H_c^k(U) \oplus H_c^k(V) \rightarrow H_c^k(X) \rightarrow H_c^{k+1}(U \cap V) \rightarrow \dots$$

Let X be an oriented n -manifold. For each compact K , let $\xi_K \in H_n(X, X - K)$ be the fundamental class determined by the orientation. Taking the cap product we find

$$D_K : H^p(X, X - K) \xrightarrow{\cap \xi_K} H_{n-p}(X).$$

This passes to the colimit and induces a map

$$D : H_c^p(X) \rightarrow H_{n-p}(X).$$

Theorem 23.14 (Poincaré Duality). Let X be an oriented n -manifold. Then for any p ,

$$D : H_c^p(X) \rightarrow H_{n-p}(X)$$

is an isomorphism. In particular, if X is compact then $H^p(X) \cong H_{n-p}(X)$.

Proof. We prove the theorem for all open subset U of X .

Step 1: If the theorem holds for open U, V and $U \cap V$, then the theorem holds for $U \cup V$.

This follows from Mayer-Vietoris sequence and the commutative diagram

$$\begin{array}{ccccccccc} \longrightarrow & H_c^k(U \cap V) & \longrightarrow & H_c^k(U) \oplus H_c^k(V) & \longrightarrow & H_c^k(U \cup V) & \longrightarrow & H_c^{k+1}(U \cap V) & \longrightarrow & \dots \\ & \downarrow D & & \downarrow D \oplus D & & \downarrow D & & \downarrow D & & \\ \longrightarrow & H_{n-k}(U \cap V) & \longrightarrow & H_{n-k}(U) \oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(U \cup V) & \longrightarrow & H_{n-k-1}(U \cap V) & \longrightarrow & \dots \end{array}$$

Step 2: Let $U_1 \subset U_2 \subset \dots$ and $U = \cup_i U_i$. Assume the theorem holds for U_i , then it holds for U .

This follows from the isomorphism

$$H_c^k(U) = \operatorname{colim}_i H_c^k(U_i), \quad H_{n-k}(U) = \operatorname{colim}_i H_{n-k}(U_i).$$

Step 3: The theorem holds for an open U contained in a coordinate chart.

This follows by expressing U as a countable union of convex subsets of \mathbb{R}^n .

Step 4: For any open U .

By Step 2, 3 and Zorn's lemma, there is a maximal open subset U of X for which the theorem is true. By Step 1, U must be the same as X . \square

24. INTERSECTION AND LEFSCHETZ FIXED POINT THEOREM

In this section X will be an oriented connected closed n -dim manifold. $[X]$ its fundamental class.

Intersection form. Poincaré duality gives an isomorphism

$$H^i(X) \cong H_{n-i}(X).$$

The cup product on cohomology has a geometric meaning under Poincaré duality as follows. Let Y, Z be two oriented closed submanifold of X . Assume $\dim(Y) = i, \dim(Z) = j$, and Y intersects Z transversely so that their intersection $Y \cap Z$ is manifold of dimension $i + j - n$. $Y \cap Z$ has an induced orientation. Let $[Y]^* \in H^{n-i}(X)$ be the Poincaré dual of the fundamental class $[Y] \in H_i(X)$. Then

$$[Y]^* \cup [Z]^* = [Y \cap Z]^*.$$

Therefore the cup product is interpreted as intersection under Poincaré duality.

An important case is when Y and Z have complementary dimension, i.e. $i + j = n$ so that $Y \cap Z$ is a finite set of points, whose signed sum gives the intersection number of Y and Z .

Definition 24.1. We define the intersection pairing

$$\langle -, - \rangle : H_i(X) \times H_{n-i}(X) \rightarrow H_0(X) \cong \mathbb{Z}.$$

Equivalently, we have the pairing on cohomology

$$\langle -, - \rangle : H^i(X) \times H^{n-i}(X) \rightarrow H^n(X) \cong \mathbb{Z}.$$

The intersection pairing is non-degenerate when torsion elements are factored out. In particular

$$H^i(X; \mathbb{Q}) \times H^{n-i}(X; \mathbb{Q}) \rightarrow \mathbb{Q}$$

is a non-degenerate pairing.

Example 24.2. $T^2 = S^1 \times S^1$. $Y_1 = S^1 \times \{1\}, Y_2 = \{1\} \times S^1$. $Y_1 \cap Y_2$ is a point. This is dual to the ring structure $H^\bullet(T^2) = \mathbb{Z}[\eta_1, \eta_2]$, where η_i is dual to Y_i .

Example 24.3. Let $f : \Sigma_g \rightarrow \Sigma_h$.

Lefschetz Fixed Point Theorem. Let us consider the diagonal $\Delta \subset X \times X$. Let $\{e_i\}$ be a basis of $H_\bullet(X; \mathbb{R})$, consisting of elements of pure degree. Let e^i be its dual basis of $H_\bullet(X; \mathbb{Q})$ such that

$$\langle e^j, e_i \rangle = \delta_i^j.$$

First we observe that

$$[\Delta] \in H_n(X \times X; \mathbb{Q}) \cong \bigoplus_p H_p(X; \mathbb{Q}) \otimes H_{n-p}(X; \mathbb{Q})$$

is given by

$$[\Delta] = \sum_i e_i \otimes e^i.$$

This can be checked by intersecting with a basis of $H_\bullet(X \times X; \mathbb{Q})$.

Let $f : X \rightarrow X$ be a smooth map. Let

$$\Gamma_f := \{(x, f(x)) | x \in X\} \subset X \times X$$

be the graph of f . Let $\alpha \in H_p(X), \beta \in H_{n-p}(X)$. From the geometry of graph, we find

$$[\Gamma_f] \cdot \alpha \times \beta = (-1)^p f_* \alpha \cdot \beta.$$

Applying this to $[\Delta]$, we find

$$[\Gamma_f] \cdot [\Delta] = \sum_i (-1)^{|e_i|} f_* e_i \cdot e^i = \sum_p (-1)^p \text{Tr}(f_* : H_p(X; \mathbb{Q}) \rightarrow H_p(X; \mathbb{Q})).$$

Definition 24.4. We define the Lefschetz number of f by

$$L(f) := \sum_p (-1)^p \operatorname{Tr}(f_* : H_p(X; \mathbb{Q}) \rightarrow H_p(X; \mathbb{Q})).$$

When Γ_f and Δ intersects transversely,

$$\#\operatorname{Fix}(f) = [\Gamma_f] \cdot [\Delta]$$

gives a signed count of fixed points of the map f . This gives the Lefschetz Fixed Point Theorem

$$\#\operatorname{Fix}(f) = L(f).$$

In particular, if the right hand side is not zero, there must exist a fixed point of f .

Example 24.5. Let n be even. Then any map $f : \mathbb{C}P^n \rightarrow \mathbb{C}P^n$ has a fixed point. In fact,

$$f^* : H^*(\mathbb{C}P^n; \mathbb{Q}) \rightarrow H^*(\mathbb{C}P^n; \mathbb{Q})$$

is a ring map. Let $x \in H^2(\mathbb{C}P^n)$ be a generator, let $f^*(x) = kx$ for some $k \in \mathbb{Z}$. Then

$$\sum_p (-1)^p \operatorname{Tr}(f_* |_{H_p(\mathbb{C}P^n; \mathbb{Q})}) = \sum_{i=0}^n k^n$$

is an odd number, hence not zero. By Lefschetz Fixed Point Theorem, f must have a fixed point.

Example 24.6. The Lefschetz number of the identity map $\operatorname{id} : X \rightarrow X$ is precisely the Euler characteristic

$$L(\operatorname{id}) = \chi(X).$$

Consider the sphere S^2 , and the map

$$f : S^2 \rightarrow S^2, \quad x \rightarrow \frac{x+v}{|x+v|}, \quad v = (0, 0, 1/2).$$

f has two fixed points: north and south pole, and f is homotopy to the identity. We find

$$\chi(S^2) = L(\operatorname{id}) = L(f) = 2.$$

For another example, consider a compact connected Lie group G . Let $g \in G$ which is not identity but close to identity. Then multiplication by g has no fixed point, and it is homotopic to the identity map. We find

$$\chi(G) = 0.$$

25. SPECTRAL SEQUENCE

Spectral sequences usually arise in two situations

- (1) A \mathbb{Z} -filtration of a chain complex: a sequence of subcomplexes $\cdots \subset F_p \subset F_{p+1} \subset \cdots$.
- (2) A \mathbb{Z} -filtration of a topological space: a family of subspaces $\cdots \subset X_p \subset X_{p+1} \subset \cdots$.

Definition 25.1. A **filtered R -module** is an R -module A with an increasing sequence of submodules

$$\cdots \subset F_p A \subset F_{p+1} A \subset \cdots$$

indexed by $p \in \mathbb{Z}$. We always assume that it is exhaustive and Hausdorff

$$\bigcup_p F_p A = A \quad (\text{exhaustive}), \quad \bigcap_p F_p A = 0 \quad (\text{Hausdorff}).$$

The filtration is bounded if $F_p A = 0$ for p sufficiently small and $F_p A$ for p sufficiently large. The **associated graded module** $G_{\bullet}^F A$ is defined by

$$G_{\bullet}^F(A) := \bigoplus_{p \in \mathbb{Z}} G_p^F A, \quad G_p^F A := F_p A / F_{p-1} A.$$

A **filtered chain complex** is a chain complex (C_{\bullet}, ∂) together with a filtration $F_p C_i$ of each C_i such that the differential preserves the filtration

$$\partial(F_p C_i) \subset F_p C_{i-1}.$$

In other words, we have an increasing sequence of subcomplexes $F_p C_{\bullet}$ of C_{\bullet} .

A filtered chain complex induces a filtration on its homology

$$F_p H_i(C_{\bullet}) = \text{Im}(H_i(F_p C_{\bullet}) \rightarrow H_i(C_{\bullet})).$$

In other words, an element $[\alpha] \in H_i(C_{\bullet})$ lies in $F_p H_i(C_{\bullet})$ if and only if there exists a representative $x \in F_p C_i$ such that $[\alpha] = [x]$. Its graded piece is given by

$$G_p^F H_i(C_{\bullet}) = \frac{\text{Ker}(\partial : F_p C_i \rightarrow F_p C_{i-1})}{F_{p-1} C_i + \partial C_{i+1}}.$$

Notation 25.2. In this section, our notation of quotient means the quotient of the numerator by its intersection with the denominator, i.e., $\frac{A}{B} := \frac{A}{A \cap B}$.

Given a filtered R -module A , we define its Rees module as a submodule of $A[z, z^{-1}]$ defined by

$$A_F := \bigoplus_{p \in \mathbb{Z}} F_p A z^p \subset A[z, z^{-1}].$$

Our conditions for the filtration can be interpreted as follows

- (1) increasing filtration: A_F is a $R[z]$ -submodule of $A[z, z^{-1}]$ and $z : A_F \rightarrow A_F$ is injective.
- (2) exhaustive: $A_F[z^{-1}] := A_F \otimes_{R[z]} R[z, z^{-1}]$ equals $A[z, z^{-1}]$.
- (3) Hausdorff: $\bigcap_{p \geq 0} z^{-p} A_F = 0$ in $A[z, z^{-1}]$.

We have

$$G_{\bullet}^F(A) := A_F / z A_F.$$

26. OBSTRUCTION THEORY

27. THE THEOREM OF HUREWICZ

Definition 27.1. We define the Hurewicz map $\rho : \pi_n(X, A) \rightarrow H_n(X, A)$ by

$$\rho([f]) := f_*(\eta)$$

where $f : (D^n, S^{n-1}) \rightarrow (X, A)$ represents an element of $\pi_n(X, A)$ and η is a generator of $H_n(D^n, S^{n-1})$.

The following diagram commutes

$$\begin{array}{cccccccc} \cdots & \longrightarrow & \pi_n(A) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_n(X, A) & \longrightarrow & \pi_{n-1}(A) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & \cdots \end{array}$$

Theorem 27.2 (Hurewicz).

- (1) If a space X is $(n-1)$ -connected, $n \geq 2$, then $H_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$.
- (2) If a pair (X, A) is $(n-1)$ -connected, $n \geq 2$, with A simply-connected and nonempty, then $H_i(X, A) = 0$ for $i < n$ and $\pi_n(X, A) \cong H_n(X, A)$.

28. EILENBERG-STEENROD AXIOMS

In this section we discuss Eilenberg-Steenrod's axiomatic approach to homology theory.

Eilenberg-Steenrod Axioms.

Definition 28.1. A homology theory consists of a sequence of functors H_n ($n \in \mathbb{Z}$) from the category of pairs (X, A) of topological spaces to the category of abelian groups, together with a natural transformation

$$\partial : H_i(X, A) \rightarrow H_{i-1}(A) \quad (:= H_{i-1}(A, \emptyset))$$

called the *connecting map*. They satisfy the following properties

- (1) **Exactness.** For any pair (X, A) with inclusions $i : A \subset X, j : (X, \emptyset) \subset (X, A)$, there is an exact sequence

$$\cdots \rightarrow H_q(A) \xrightarrow{H_q(i)} H_q(X) \xrightarrow{H_q(j)} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \rightarrow$$

- (2) **Homotopy.** If $f_0, f_1 : (X, A) \rightarrow (Y, B)$ are homotopic, then

$$H(f_0) = H(f_1) : H_\bullet(X, A) \rightarrow H_\bullet(Y, B).$$

- (3) **Excision.** For any pair (X, A) , if U is a subset of X such that the closure of U is contained in the interior of A , then the inclusion $j : (X - U, A - U) \subset (X, A)$ induces isomorphisms

$$H(j) : H_\bullet(X - U, A - U) \cong H_\bullet(X, A).$$

- (4) **Dimension.** If \star is a point, then $H_i(\star) = 0$ for any $i \neq 0$.

We also add two additional axioms

- (5) **Additivity.** If $X = \coprod_\alpha X_\alpha$ is a disjoint union, then

$$H_\bullet(X) = \bigoplus_\alpha H_\bullet(X_\alpha).$$

- (6) **Weak equivalence.** If $f : (X, A) \rightarrow (Y, B)$ is a weak equivalence, then $H_\bullet(f)$ are isomorphism.

For a homology theory, $H_0(\star) = G$ is called the *coefficient group* of the theory.

Remark 28.2. The weak equivalence axiom ensures that a homology theory is uniquely determined by the subcategory of CW complexes.

Definition 28.3. Let H, H' be two homology theories. A natural transformation $\Phi : H \rightarrow H'$ is a sequence of natural transformations $\Phi_i : H_i \rightarrow H'_i$ such that the following diagram commutes for any pair (X, A)

$$\begin{array}{ccc} H_i(X, A) & \xrightarrow{\partial} & H_{i-1}(A) \\ \Phi_i \downarrow & & \downarrow \Phi_{i-1} \\ H'_i(X, A) & \xrightarrow{\partial'} & H'_{i-1}(A) \end{array}$$

If Φ_i is a natural isomorphism for each i , then we say H, H' are naturally isomorphic.

Example 28.4. Singularity homology $H(X, A; G)$ is a homology theory with coefficient G .

Given a homology theory, we can similarly define its reduced homology by

$$\tilde{H}_i(X, A) = \begin{cases} \ker(H_0(X) \rightarrow H_0(\star)) & i = 0, A = \emptyset \\ H_i(X, A) & \text{otherwise} \end{cases}$$

The reduced homology sequence is also exact.

Proposition 28.5. *Let H be a homology theory with coefficient G , Then*

$$\tilde{H}_i(S^n) = \begin{cases} G & i = n \\ 0 & i \neq n \end{cases}$$

Hurewicz Theorem gives a natural isomorphism

$$H_n(X; \mathbb{Z}) \rightarrow H_n(X)$$

from the singular homology to our given homology theory, for $(n - 1)$ -connected space X . This is the key to prove the following uniqueness theorem.

Theorem 28.6. *Any homology theory is naturally isomorphic to the singular homology.*

Proof. We only need to prove for CW complex. We can use the axioms to construct the cellular chain complex $C_n(X) = H_n(X^{(n)}/X^{(n-1)})$ for any homology theory, and show by the same method that the homology of

$$\cdots \rightarrow C_n(X) \xrightarrow{d} C_{n-1}(X) \rightarrow \cdots$$

is isomorphic to $H_\bullet(X)$ of our given homology theory. Hurewicz Theorem will imply that this chain complex is isomorphic to the cellular chain complex associated to the singular homology. This proves the theorem. \square

Generalized homology theory.

Definition 28.7. A (co)-homology functor H that satisfies Eilenberg-Steenrod Axioms except the Dimension axiom is called a *generalized (co)-homology theory*

Example 28.8 (K-theory).

Example 28.9 (Bordism).