# INTRODUCTION TO PERTURBATIVE QUANTUM FIELD THEORY AND GEOMETRIC APPLICATIONS

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ABSTRACT. This note is the part I of author's course on perturbative quantum field theory at Tsinghua university during spring 2015.

CONTENTS



## 1. INTEGRATION AND BV

A physics system is always described by a function

$$S : \mathcal{E} \to \mathbb{C}.$$

$\mathcal{E}$ is called the *space of fields*, and $S$ is called the *action functional*. Usually $\mathcal{E}$ is infinite dimensional, introducing many fundamental difficulties.

The classical physics is described by the critical locus

$$\mathrm{Crit}(S) = \{\delta S = 0\}.$$

$\delta S = 0$ is called the *equation of motion*. The quantum physics is related to the integration

$$\int_{\mathcal{E}} \mathcal{O} e^{iS/\hbar}.$$

$\mathcal{O}$ is a function on $\mathcal{E}$ called the *observable*. When $\hbar$ is very small, the above integral is asymptotically approximated around the critical locus of $S$, a method called *stationary phase approximation*. The classical limit is obtained by $\hbar \to 0$.

When $\mathcal{E}$ is infinite dimensional, the above integral is called the *path integral*, which is mostly not rigorously defined. This causes a big trouble in mathematics to understand quantum physics. However, the asymptotic expansion as $\hbar \to 0$ around a critical point of $S$ can usually be understood rigorously. Understanding the asymptotic $\hbar$-expansion is called *perturbative theory*. Such asymptotic expansion around critical points of $S$ is far from understanding the full path integral (the latter is therefore called *nonperturbative*). However, this is the first approximations, and sometimes the approximation becomes exact like in certain supersymmetric theories. Moreover, some recent development of *resurgent method* tells that perturbative theory gives some nonperturbative information as well. In this lecture, we will focus on the perturbative study.

Here are some basic examples of quantum field theory that we will discuss through this lecture.

(1) Scalar field theory. $\mathcal{E} = C^\infty(X)$ smooth functions on a manifold $X$.
(2) Gauge theory. $\mathcal{E} = \{\text{connections on a bundle over } X\}$.
(3) $\sigma$-model. $\mathcal{E} = \mathrm{map}(\Sigma, X)$ maps between two manifolds.
(4) *Gravity*. $\mathcal{E} =$ metrics on $X$.

We will mainly use Batalin-Vilkovisky (BV) formalism to understand the path integral. Let us first explain the basic idea in finite dimensional situation. Let

$$\mathcal{E} = X$$

be a smooth oriented manifold of dimension $n$. Let $\Omega$ be a fixed volume form on $X$, which gives the integration

$$\int_X : C_c^\infty(X) \to \mathbb{R}, \quad g \to \int_X g\Omega.$$

This has a homological interpretation as follows. Let

$$\Omega_c(X) = \oplus_p \Omega_c^p(X)$$

be differential forms with compact support. There is a de Rham differential

$$d : \Omega_c^p(X) \to \Omega_c^{p+1}(X).$$

We know that

$$H^n(\Omega_c(X), d) \cong \mathbb{R}.$$

Therefore the integration $\int_X$ has a homological interpretation

$$g \to [g\Omega] \in H^n(\Omega_c(X), d) \cong \mathbb{R}.$$

Let us do in a bit different way. Let us define *polyvector fields* by

$$\mathrm{PV}(X) = \oplus_p \mathrm{PV}^p(X) = \oplus_p \Gamma(X, \wedge^p T_X).$$

We also denote $\mathrm{PV}_c(X)$ by the subspace of compactly supported polyvector fields. We can naturally identify

$$\Gamma_\Omega : \mathrm{PV}(X) \to \Omega(X), \quad \wedge^p T_X \to \Omega^{n-p}(X),$$

by contracting with $\Omega$. Pulling back the de Rham $d$, we get a differential

$$\Delta_\Omega : \mathrm{PV}(X) \to \mathrm{PV}(X)$$

such that

$$\Gamma_\Omega(\Delta_\Omega \mu) = d\Gamma_\Omega(\mu).$$

We associate a grading such that $\mathrm{PV}^p(X)$ has degree $-p$. Then

$$\Delta_\Omega : \mathrm{PV}^p(X) \to \mathrm{PV}^{p-1}(X)$$

has degree 1. Under the above identification, we find that the integration $\int_X$ can be equivalently described by

$$g \to [g] \in H^0(\mathrm{PV}_c(X), \Delta_\Omega) \cong \mathbb{R}.$$

We have replaced differential forms by polyvector fields, and transform the integration by

$$H^n \to H^0.$$

The upshot is that in the infinite dimensional case, when $n \to \infty$, $H^n$ is difficult to handle, while $H^0$ looks good as long as we can make sense of $\Delta_\Omega$ accordingly. As we will see, this can be really done, which is the main point of view in [1].

$\Delta_\Omega$ will be called the BV-operator. To get a feeling on how $\Delta_\Omega$ looks like. Let us work locally on $X$ and choose local coordinates $\{x^i\}$ on an open subset $U$. Let

$$\Omega = e^{f(x)} dx^1 \wedge \cdots \wedge dx^n.$$

Locally,

$$\mathrm{PV}(U) = C^\infty(U)[\partial_1, \cdots, \partial_n],$$

where $\partial_i$'s are anti-commuting

$$\partial_i \partial_j = -\partial_j \partial_i.$$

Let us introduce Grassmann variable $\theta_i$ to represent

$$\theta_i \equiv \partial_i. \quad \theta_i\theta_j = -\theta_j\theta_i.$$

Then a local section $\mu \in \mathrm{PV}(X)$ can be written as a function of $x^i, \theta_i$

$$\mu = \mu(x^i, \theta_i).$$

$\theta_i$'s behave like fermions. Let $\frac{\partial}{\partial\theta_i}$ be the derivative with respect to $\theta_i$. This can be defined similarly as that of the ordinary derivative with extra care about the signs.

**Lemma 1.1.** *The BV operator $\Delta_\Omega$ is given locally by*

$$\Delta_\Omega = \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial\theta_i} + \sum_i \partial_i f \frac{\partial}{\partial\theta_i}.$$

Note that the first term looks like a Laplacian, but it it NOT. $\theta_i$ is an odd variable. $\Delta_\Omega$ is also sometimes called BV Laplacian, or odd Laplacian.

In contrast to the de Rham differential $d$, which is a first order differential operator hence a derivation, $\Delta_\Omega$ looks like a second order operator. We can measure how far it is from a derivation.

**Definition 1.1.** We define a bracket

$$\{-,-\} : \mathrm{PV}(X) \times \mathrm{PV}(X) \to \mathrm{PV}(X)$$

by

$$\{\alpha, \beta\} = \Delta_\Omega(\alpha\beta) - (\Delta_\alpha)\beta - (-1)^{|\alpha|}\alpha\Delta_\Omega\beta.$$

Here $|\alpha|$ is the degree of $\alpha$.

$\{,\}$ is essentially the Schouten-Nijenhuis bracket. As we will explore, the quantization process can be understood as deforming

$$\{,\} \to \Delta_\Omega.$$

*Exercise* 1.

(1) Prove Lemma 1.1.
(2) Show that $\{-,-\}$ doesn't depend on the choice of $\Omega$. Therefore it is an intrinsic structure of the polyvector fields.
(3) Let $\mathrm{Vect}_X \subset \mathrm{PV}(X)$ be the subspace given by vector fields. Show that

$$\{,\} : \mathrm{Vect}_X \times \mathrm{Vect}_X \to \mathrm{Vect}_X$$

coincides with the usual Lie bracket.

## 2. Feynman diagram

Let us explore the idea of BV for the simplest example $X = \mathbb{R}$. Let us choose the volume form

$$\Omega = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

and try to under the integration

$$\int : \mathbb{C}[x] \to \mathbb{C}, \quad g \to \int_{\mathbb{R}} g\Omega.$$

The BV operator is given by

$$\Delta_\Omega = \frac{\partial}{\partial x}\frac{\partial}{\partial \theta} - x\frac{\partial}{\partial \theta}.$$

For any given polynomial $g(x)$, we have

$$\Delta_\Omega g = 0$$

since $g(x)$ doesn't depend on $\theta$. We let $[g]$ represent its class in $\Delta_\Omega$-cohomology. Since

$$\Delta_\Omega(x^{m-1}\theta) = (m-1)x^{m-2} - x^m,$$

we find that

$$[x^m] = (m-1)[x^{m-2}].$$

It follows that $[x^m] = 0$ if $m$ is odd, and

$$[x^{2k}] = (2k-1)!![1].$$

Therefore

$$\int_{\mathbb{R}} x^{2k}\Omega = (2k-1)!! \int_{\mathbb{R}} \Omega = (2k-1)!!,$$

which is nothing but a successive applications of integration by part.

Let us organize the structure a bit. Introduce the operator

$$U = e^{\frac{1}{2}\frac{\partial}{\partial x}\frac{\partial}{\partial x}} : \mathbb{C}[x] \to \mathbb{C}[x],$$

which extends to $\mathbb{C}[x, \theta]$ naturally.

**Lemma 2.1.** *$\Delta_\Omega = U^{-1}(-x\frac{\partial}{\partial \theta})U$.*

It follows that we have a cochain isomorphism of complexes

$$U : (\mathbb{C}[x,\theta], \Delta_\Omega) \to (\mathbb{C}[x,\theta], -x\frac{\partial}{\partial \theta})$$

with $U(1) = 1$. The cohomology of the operator $-x\frac{\partial}{\partial\theta}$ is easy to describe. Any function $h(x)$ is cohomolgical equivalent to $h(0)$. Since $U$ induces an isomorphism on cohomologies, we find that

$$U([g(x)]) = [U(g)(x)] = [U(g)(0)].$$

Equivalently,

$$\int_{\mathbb{R}} g(x)\Omega = e^{\frac{1}{2}\partial_x^2} g(x)|_{x=0}[1] = e^{\frac{1}{2}\partial_x^2} g(x)|_{x=0}.$$

More generally, let us introduce an auxiliary parameter $a$. Then

$$\int_{\mathbb{R}} g(x+a)\Omega = e^{\frac{1}{2}\partial_a^2} g(a).$$

The upshot is that the operator $U$ describes the full information of integration against the volume form $\Omega$.

Now let us consider a toy model describing an interacting system

$$\int_{\mathbb{R}} e^{(-\frac{1}{2}x^2+\frac{1}{3!}x^3)/\hbar} \frac{dx}{\sqrt{2\pi}}.$$

The quadratic term $\frac{1}{2}x^2$ we have studied so far corresponds to the *free part*, while the cubic term represents the *interaction*. This integration is of course not convergent. There are several ways to render this. The first is to deform the integrated domain $\mathbb{R}$ into a differential cycle in $\mathbb{C}$ to make the integration convergent. This highlights some method in nonperturbative theory. Another thing is that we can view the interaction term as a small perturbation. For example, if we rescale $x \to \sqrt{\hbar}x$, the integral becomes

$$\sqrt{\hbar}\int_{\mathbb{R}} e^{(-\frac{1}{2}x^2+\frac{\sqrt{\hbar}}{3!}x^3)} \frac{dx}{\sqrt{2\pi}}.$$

We can Taylor expand $e^{\frac{\sqrt{\hbar}}{3!}x^3}$ and study its power series expansion in terms of $\hbar$. This gives some asymptotic behavior as $\hbar \to 0$. Let us take this viewpoint.

Let us bring back our $\hbar$ and our $U$ operator is given by

$$U_\hbar = e^{\frac{\hbar}{2}\partial_x^2}.$$

Then it is not hard to see that we can formally write

$$\int_{\mathbb{R}} e^{(-\frac{1}{2}x^2+\frac{1}{3!}(x+a)^3)/\hbar} \frac{dx}{\sqrt{2\pi\hbar}} \text{``}=\text{''} e^{\frac{\hbar}{2}\partial_a^2} e^{\frac{1}{3!}a^3/\hbar} = \sum_{k,m\geq 0} \frac{(\frac{\hbar}{2}\partial_a^2)^k}{k!} \frac{(\frac{1}{3!\hbar}a^3)^m}{m!}.$$

This infinite series can be grouped in terms of graphs. We introduce an edge called *propagator* and a cubic vertex. The propagator represents $\frac{\hbar}{2}\partial_a^2$, which carries a power of $\hbar$. The vertex represents $\frac{1}{3!\hbar}a^3$ which carries a power of $\hbar^{-1}$. For each $\partial_a$ acting on some $a$, we attach a edge connecting that vertex. Therefore the above sum is decomposed into types of trivalent graphs. We put an $a$ on each external edge. For each graph $\Gamma$, we follow the above rule and write down a polynomial of $a$ denoted by

$$W_\Gamma(a) = a^D \hbar^{E-V}$$

where $D$ is the number of external edges, $E$ is the number of internal edges, $V$ is the number of vertices. Let $Aut(\Gamma)$ be the permutations of vertices, internal edges and external edges which map $\Gamma$ to itself.

**Proposition 2.1** (Feynman graph formula)**.** *We have the graph expression of the formal sum*

$$\sum_{k,m\geq 0} \frac{(\frac{\hbar}{2}\partial_a^2)^k}{k!}\frac{(\frac{1}{3!\hbar}a^3)^m}{m!} = \sum_{\Gamma:trivalent\ graphs} \frac{W_\Gamma(a)}{|Aut(\Gamma)|} = \exp\left(\sum_{\Gamma:connected} \frac{W_\Gamma(a)}{|Aut(\Gamma)|}\right)$$

*where* $\sum\limits_{\Gamma:connected}$ *is the summation over all connected graphs.*

Let us formally write

$$e^{W(a)/\hbar}" = " \int_{\mathbb{R}} e^{(-\frac{1}{2}x^2+\frac{1}{3!}(x+a)^3)/\hbar} \frac{dx}{\sqrt{2\pi}}.$$

Feynman graph formula suggests that we *define* $W(a)$ by

$$W(a) := \hbar \sum_{\Gamma:\text{connected}} \frac{W_\Gamma(a)}{|Aut(\Gamma)|}.$$

We can also expand $W(a)$ in terms of $\hbar$-orders

$$W(a) = \sum_{g\geq 0} W_g(a)\hbar^g.$$

Note that $\hbar W_\Gamma(a)$ contributes to $\hbar^{E-V+1} = \hbar^{g(\Gamma)}$, where $g(\Gamma)$ is the number of loops in $\Gamma$. If follows that $W_g$ is contributed by $g$-loop connected graphs. Graphs with $g = 0$ are called *trees*. Graphs with $g = 1$ are called 1-loop diagrams, etc. Tree diagrams correspond to classical information, while graphs starting with 1-loop can be viewed as *quantum corrections*.

The above graph formulation can be generalized. Let us consider

$$\int_{\mathbb{R}} e^{(-\frac{1}{2}x^2+I(x+a))/\hbar} \frac{dx}{\sqrt{2\pi}}" = " e^{\frac{\hbar}{2}\partial_a^2} e^{I(a)/\hbar}.$$

The interaction $I$ can contain several terms, each contributing to a type of vertex. Terms in $I$ can also contain powers in $\hbar$. In that case we put a loop order of $k$ to the vertex containing $\hbar^k$. For example, a term $\hbar^2 x^4$ could be viewed as giving a vertex of valency 4 and loop 2. When we compute the loop of a graph $\Gamma$, we should add the extra loops coming from each vertex. Let us write

$$e^{W(P,I)/\hbar} = e^{\frac{\hbar}{2}\partial_a^2} e^{I(a)/\hbar}.$$

$P = \frac{1}{2}\partial_x^2$ represent the propagator, $I$ represent the vertex. Then Feynman graph formula suggests that we define $W(P, I)$ by

$$W(P,I) = \hbar \sum_{\Gamma:\text{connected}} \frac{W_\Gamma(a)}{|Aut(\Gamma)|}. \tag{2.1}$$

where $\Gamma$ runs over all possible connected graphs with vertices represented by terms in $I$. It defines a transformation

$$I \to W(P, I)$$

such that

$$e^{W(P,I)/\hbar} = e^{\hbar P} e^{I/\hbar}.$$

**Proposition 2.2.** *$W(P, -)$ defines a well-defined transformation on*

$$W(P, -) : \mathbb{C}[[x, \hbar]]^+ \to \mathbb{C}[[x, \hbar]]^+,$$

*where $\mathbb{C}[[x, \hbar]]^+$ is the subspace of $\mathbb{C}[[x, h]]$ which is at least cubic modulo $\hbar$, i.e.,*

$$\mathbb{C}[[x, \hbar]]^+ = x^3\mathbb{C}[[x]] \oplus \hbar\mathbb{C}[[x, \hbar]].$$

*Proof.* For $I \in \mathbb{C}[[x, \hbar]]^+$, we need to prove that given a fixed number of external edges and loop number, there exists only finitely number graphs contributing. Let $\Gamma$ be a $G$-loop graph with $D$ external edges, $V$ vertices, $E$ edges. Assume $\Gamma$ contains $n_{g,k}$ vertices of valency $k$ and loop number $g$. Then

$$\begin{aligned} G &= E - V + 1 + \sum_{g,k} g n_{g,k} \\ V &= \sum_{g,k} n_{g,k} \\ \sum_{g,k} k n_{g,k} &= 2E + D. \end{aligned}$$

It follows that

$$(2G + D - 2) = \sum_{g,k} (2g + k - 2) n_{g,k}.$$

Since $I \in \mathbb{C}[[x, k]]^+$, we know that $2g + k - 2 \geq 0$, with equality holds only when $g = 1, k = 0$. It follows that only finitely $n_{g,k}$'s are nonzero. Moreover,

$$2G + D - 2 \geq 0$$

with equality holds only when $G = 1, D = 0$. It follows that $W(P, I) \in \mathbb{C}[[x, \hbar]]^+$. □

$W(P, -)$ will be called the renormalization group flow with respect to $P$

*Exercise* 2. Give a precise definition of $Aut(\Gamma)$ and prove the Feynman graph formula.

## 3. Stokes phenomenon

Before we move on to the issue of infinite dimension, let us first take a look at some nonperturbative aspects of the integral

$$\int_\Gamma e^{(-\frac{1}{2}x^2 + \frac{1}{3!}x^3)/\hbar} \frac{dx}{\sqrt{2\pi}}$$

where $\Gamma$ is a contour in the complex plane $\mathbb{C}$. We know that if $C$ is the real line, then the above integral is in fact divergent. We have pretended to ignore this issue and study some formal aspect of the "integral" and find the expression of Feynman diagram. Let us do some analysis in the real situation which highlights a difference between perturbative and nonperturbative aspects. We will follow the presentation [2].

Up to a change of variable and rescale, our integral can be changed to

$$\int e^{(x^3 - x)/\hbar} dx.$$

To make a convergent integral, we need to change the domain $\mathbb{R}$ to another contour $\Gamma \subset \mathbb{C}$. Let us assume first that $\hbar > 0$. In this case, it turns out that the imaginary line

$$\Gamma = i\mathbb{R}, \quad \hbar > 0,$$

works, and the integral

$$\int_{-i\infty}^{i\infty} e^{(x^3 - x)/\hbar} dx = 2\int_0^\infty \cos\left((x^3 - x)/\hbar\right) dx$$

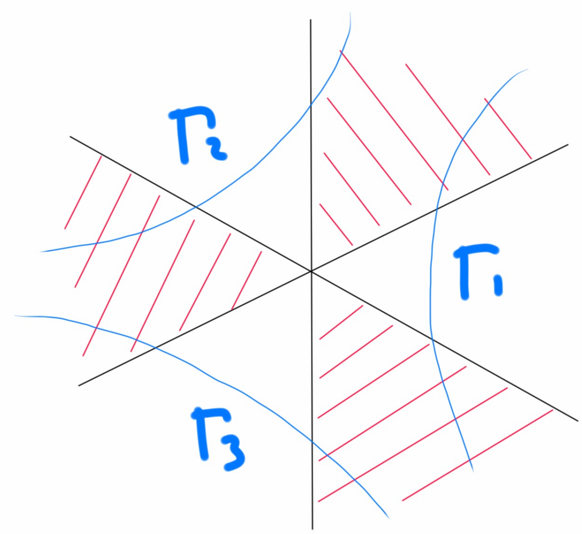
is called *Airy integral*.

*Exercise* 3. Show that the above Airy integral is convergent.

More generally, we can choose $\Gamma$ such that $(x^3 - x)/\hbar$ decays around the boundary of $\Gamma$, i.e., along the asymptotic region

$$\mathrm{Re}(x^3/\hbar) < 0.$$

If $\hbar > 0$, then we find the following region




and curves $\Gamma_1, \Gamma_2, \Gamma_3$ as above. Then

$$\int_{\Gamma_i} e^{(x^3-x)/\hbar} dx, \quad i = 1, 2, 3,$$

are convergent. From the topology, we find the relation

$$\sum_{i=1}^{3} \int_{\Gamma_i} e^{(x^3-x)/\hbar} dx = 0.$$

*Exercise* 4. Show that the integral $\int_{\Gamma_i} e^{(x^3-x)/\hbar} dx$ satisfies a second order differential equation in terms of $\hbar$. In particular, there are only two linearly independent solutions.

Now here comes the subtle point. Let

$$Z_i(\hbar) = \int_{\Gamma_i} e^{(x^3-x)/\hbar} dx.$$

What's the analyticity of $Z_i(\hbar)$ as a function of $\hbar$? If we do analytic continuation starting from a point $\hbar > 0$ by rotation

$$\hbar \to e^{i\theta} \hbar,$$

and deform the curves $\Gamma_i$'s accordingly. Since the integral doesn't change value under small deformation of the curve, after we go back by a circle

$$\hbar \to e^{2\pi i}\hbar,$$

the three curves $\Gamma_i$'s are permuted. We find

$$Z_i(e^{2\pi i}\hbar) = Z_{i+1}(\hbar).$$

If we let $Z_1(\hbar), Z_2(\hbar)$ represent the two linearly independent integrals, then

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \quad \text{as } \hbar \to e^{2\pi i}\hbar.$$

The matrix $M = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ is called the *monodromy*. We have $M^3 = 1$.

To get some intrinsic geometric picture, we can look at the so-called *Lefschetz thimbles*. The fundamental reason for two independent cycles is that for $I = (x^3 - x)/\hbar$, there are two critical points:

$$\mathrm{Crit}(I) = \{\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\} = \{p_+, p_-\}.$$

The critical values are given by

$$I(p_\pm) = \mp\frac{2}{3\sqrt{3}\hbar}.$$

Starting from the critical points $p_\pm$, we can construct two cycles $\Gamma_\pm$ as follows. Consider the flow equation

$$\frac{dx}{dt} = -\overline{\frac{\partial I}{\partial x}}, \quad t \in \mathbb{R}, \quad x \in \mathbb{C}.$$

On each flow line, we have

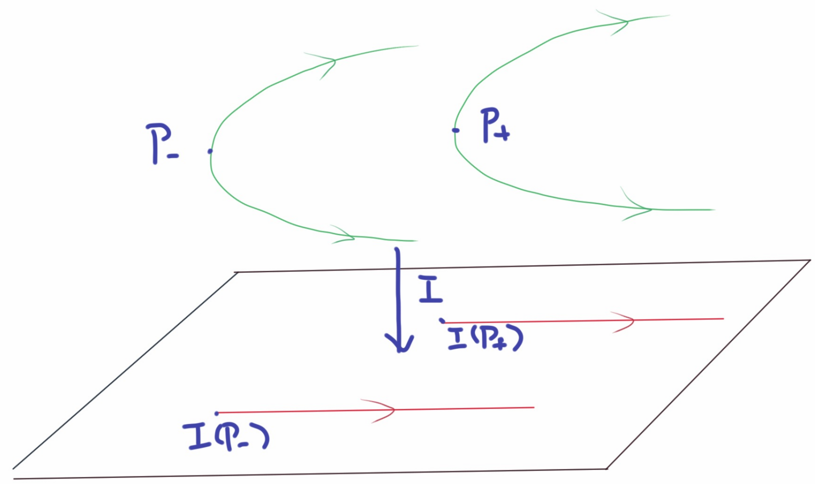
$$\frac{dI}{dt} = \frac{dx}{dt}\frac{\partial I}{\partial x} = -\left|\frac{\partial I}{\partial x}\right|^2.$$

This implies the following properties along each flow line

(1) $\mathrm{Re}(I)$ is decreasing.
(2) $\mathrm{Im}(I)$ is constant.
(3) As $t \to \pm\infty$, $x(t)$ approaches $\infty$ or $p_\pm$.

Let

$$\Gamma_\pm = \cup\{\text{flows} |\ x(-\infty) = p_\pm\}.$$




The cycles $\Gamma_\pm$ are called Lefschetz thimbles associated to the critical points $p_\pm$. There is a very subtle point here. On each flow line, $\mathrm{Im}(I)$ is constant. Since

$$\mathrm{Im}(I)(\Gamma_+) = \mathrm{Im}(I)(p_+) = \mathrm{Im}(-\frac{2}{3\sqrt{3}\hbar}), \quad \mathrm{Im}(I)(\Gamma_-) = \mathrm{Im}(\frac{2}{3\sqrt{3}\hbar}).$$

If $\hbar$ is NOT real, then $\Gamma_+$ does not intersect $\Gamma_-$, and we find two independent integrals

$$\int_{\Gamma_+} e^I dx, \quad \int_{\Gamma_-} e^I dx.$$

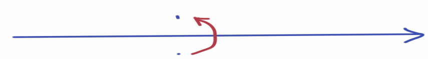
However, if $\hbar \neq 0$ is real, then there exists a flow line starting from one critical point to another. Therefore it is ambiguous to talk about $\Gamma_\pm$. The two rays $\{\hbar > 0\}$ and $\{\hbar < 0\}$ are called *Stokes rays*.

What happens if we cross a Stokes ray? Consider $\{\hbar > 0\}$. On this ray,

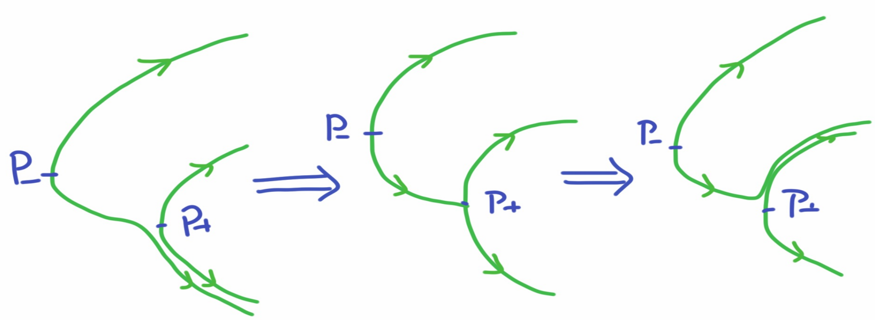
$$I(p_-) > I(p_+).$$

Since $\mathrm{Re}(I)$ decays along flow lines, there exists a flow with $x(-\infty) = p_-, x(+\infty) = p_+$.


When we cross the Stokes ray,



the following picture explains the situation




So we have the folioing transformation

$$\begin{pmatrix} \Gamma_+ \\ \Gamma_- \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_+ \\ \Gamma_- \end{pmatrix}.$$

In real life, if we want a meaningful integral

$$Z(\hbar) = \int e^I dx := n_+ \int_{\Gamma_+} e^I dx + n_- \int_{\Gamma_-} e^I dx$$

and expect $Z(\hbar)$ be an analytic function of $\hbar$, then the numbers $(n_+, n_-)$ are locally constant in the $\hbar$-plane, but jumps across the Stokes rays. This is also called *wall-crossing phenomenon*.

How is this related to the power series expansions in terms of Feynman diagrams? In fact, the power series we get is basically an asymptotic series of the above integral around one of the critical point. Recall our previous situation $I = -\frac{1}{2}x^2 + \frac{1}{6}x^3$, we have two critical points $\{0, 2\}$. There we have studied the asymptotic expansion around one of them, $\{0\}$, and view $x^3$ as perturbation. There is an another critical point that we have ignored.

Those two crucial points are the basically the same as $\{p_\pm\}$. Let us analyze a bit on the asymptotic structures around these two points. Our Feynman diagram gives the following answer

$$p_+ : \int_{\Gamma_+} e^I dx \sim \frac{e^{I(p_+)}}{\sqrt{\hbar}}(a_0 + a_1\hbar + \cdots).$$

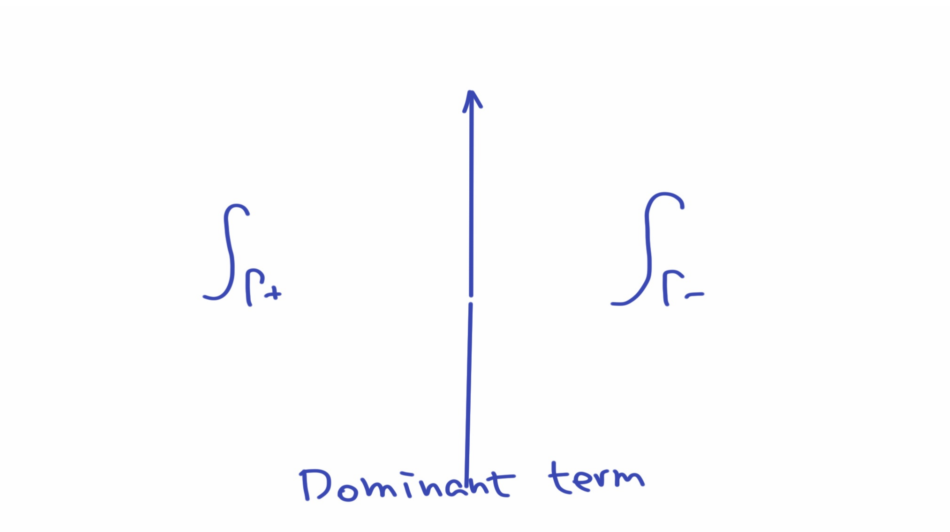
and

$$p_- : \int_{\Gamma_-} e^I dx \sim \frac{e^{I(p_-)}}{\sqrt{\hbar}}(b_0 + b_1\hbar + \cdots).$$

We find

- If $\mathrm{Re}\,\hbar > 0$, then $e^{I(p_-)}$ dominates.
- If $\mathrm{Re}\,\hbar < 0$, then $e^{I(p_+)}$ dominates.

The rays $i\mathbb{R}^+$ and $i\mathbb{R}^-$ on the imaginary line separating the dominate asymptotic behaviors are sometimes called *anti-Stokes rays*.




This also explains why the jumping phenomenon looks as it is. When we cross the Stokes ray $\mathbb{R}^+$, $\int_{\Gamma_+}$ can not jump, but

$$\int_{\Gamma_-} \to \pm \int_{\Gamma_+} + \int_{\Gamma_-}$$

is possible and doesn't change its total asymptotic structure since $\int_{\Gamma_-}$ dominates.

Here we see that there is a much more delicate information in non-perturbative aspects of the integral, and our perturbation treatment in terms of Feynman diagrams zooms only into one critical point of it.

## 4. Determinant

Now we come back to perturbative theory. Let us first generalize our discussion of Feynman diagrams to higher dimensions. Consider the integral

$$\int_{\mathbb{R}^n} \prod_{i=1}^{n} \frac{dx^i}{\sqrt{2\pi\hbar}} e^{(-\frac{1}{2}Q(x)+I(x))/\hbar}.$$

where $Q(x) = \sum_{i,j=1}^n Q_{ij}x^i x^j$, $Q_{ij}$ is a positive matrix, called the *free part*. $I(x)$ is at least cubic, called the *interaction*. The volume form

$$e^{-\frac{1}{2\hbar}Q(x)} \frac{dx^i}{\sqrt{2\pi\hbar}}$$

gives rise to the BV operator

$$\hbar \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \theta_i} - \sum_{i,j} Q_{ij}x^i \frac{\partial}{\partial \theta_j} = U^{-1}(-\sum_{i,j} Q_{ij}x^i \frac{\partial}{\partial \theta_j})U,$$

where

$$U = e^{\hbar P}, \quad P = \frac{1}{2}\sum_{ij} Q^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}.$$

Here $\{Q^{ij}\}$ is the inverse matrix of $\{Q_{ij}\}$. Similar to the one-dim case, the integral is represented by

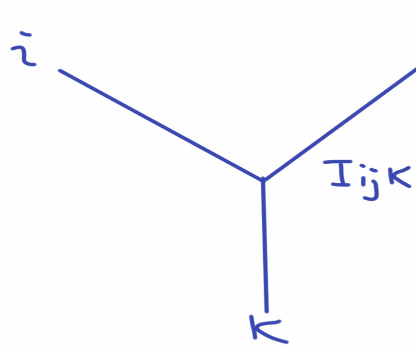
$$e^{\hbar P}e^{I/\hbar},$$

and we get the power series in terms of Feynman diagrams

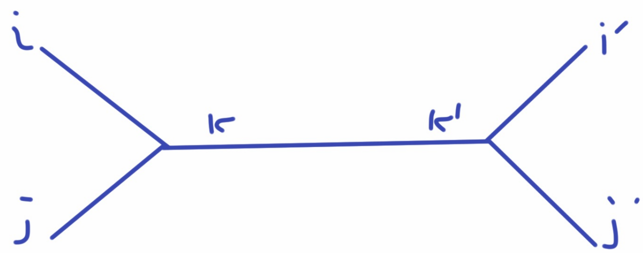
$$\int_{\mathbb{R}^n} \prod_{i=1}^n \frac{dx^i}{\sqrt{2\pi\hbar}} e^{(-\frac{1}{2}Q(x)+I(x+a))/\hbar} \sim \frac{1}{\sqrt{\det(Q)}} \exp\left(\sum_{\Gamma:\text{conn}} \frac{W_\Gamma(a)}{|Aut(\Gamma)|}\right)$$

Here the propagator is represented by

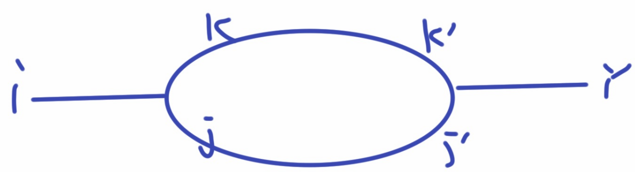

Note that we have an additional index for the variables. Similarly for vertices. For example, if $I = \frac{1}{3!}\sum_{ink} \lambda_{ijk}x^ix^jx^k$, then



And we give some examples of terms in the Feynman diagrams.




$$= \frac{1}{8} \sum a^i a^j Q^{kk'} a^{i'} a^{j'} I_{ijk} I_{i'j'k'}$$




$$= \frac{\hbar}{4} \sum a^i a^{i'} Q^k Q^{k'} Q^j Q^{j'} I_{ijk} I_{i'j'k'}$$

As a result, we can similarly write down a flow operator

$$W(P, -) : \mathbb{C}[[x^i, \hbar]]^+ \to \mathbb{C}[[x^i, \hbar]]^+, \quad W(P, I) = \hbar \sum_{\Gamma:\text{conn}} \frac{W_\Gamma}{|Aut(\Gamma)|}.$$

Formally

$$e^{W(P,I)/\hbar} = e^{\hbar P} e^{I/\hbar}.$$

Now let us come to the infinite dimensional situation. Let us start with the *scalar field theory*. The space of fields is given by

$$\mathcal{E} = C^\infty(X)$$

smooth functions on a manifold $X$. This is an infinite dimensional space. We can put $N$ points on $X$ to have a discrete approximation for $X$, and think about $X$ as the continuous limit $N \to \infty$. Naively,

$$\mathbb{R}^N = \mathrm{Map}(\text{N-points}, \mathbb{R}) \overset{N\to\infty}{\to} \mathrm{Map}(X, \mathbb{R}).$$

Comparing with the finite dimensional case,

$$i \leftrightarrow \text{a point of } \; X.$$

Now we consider the action functional. Let us start with free theory, i.e., without interaction, and consider the following action

$$S = \frac{1}{2}\int_X \phi D\phi, \quad \phi \in C^\infty(X).$$

Here $D$ is a positive Laplacian. A typical example on $X = \mathbb{R}^d$ is

$$D = -\sum_{i=1}^{d} \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i} + m^2.$$

$m^2$ is called the *mass*. Let us try to understand the integral of the free theory

$$\int_{\mathcal{E}} [D\phi] e^{-S/\hbar}.$$

Such integral over the infinite dimensional space is called *path integral*. The symbol $[D\phi]$ represents the measure, which in most cases are not well-defined. In any sense, if we try to mimic the finite dimensional situation, we expect naively

$$\int_{\mathcal{E}} [D\phi] e^{-S/\hbar} \text{“} = \text{”} \frac{1}{\sqrt{\det(D)}}.$$

How to define $\det D$? Let us explain by an example a heuristic physics method, and then explain the mathematical construction.

Consider $X = S^1_\beta$, a circle of length $\beta$. Let

$$D = -\frac{d}{dt}\frac{d}{dt} + 1.$$

The eigenvectors are given by $e^{2\pi i n t/\beta}$, $n \in \mathbb{Z}$, hence the eigenvalues

$$\frac{4\pi^2 n^2}{\beta^2} + 1, \quad n \in \mathbb{Z}.$$

Naively,

$$\sqrt{\det(D)} \overset{?}{=} \text{“} \prod_{n=1}^{\infty} \left(1 + \frac{4\pi^2 n^2}{\beta^2}\right) \text{”}.$$

The right hand side is of course meaningless in the naive sense. Let us do some formal manipulation

$$"\prod_{n=1}^{\infty}\left(\frac{1+4\pi^2n^2}{\beta^2}\right)" = "\prod_{n=1}^{\infty}\left(\frac{4\pi^2n^2}{\beta^2}\right)""\prod_{n=1}^{\infty}\left(1+\frac{\beta^2}{4\pi^2n^2}\right)" = "e^{\sum_{n\geq 1}\log\frac{4\pi^2n^2}{\beta^2}}""\prod_{n=1}^{\infty}\left(1+\frac{\beta^2}{4\pi^2n^2}\right)"$$

The first term is treated as follows. Define

$$\eta(s)=\sum_{n=1}^{\infty}\left(\frac{\beta}{2\pi n}\right)^{2s}=\left(\frac{\beta}{2\pi}\right)^{s}\zeta(s),$$

where

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

is the Riemann zeta function. Symbolically,

$$-\sum_{n=1}^{\infty}\log\frac{4\pi^2n^2}{\beta^2}=\eta'(0).$$

This is understood in the sense of analytic continuation. $\zeta(s)$ is defined for $\mathrm{Re}(s)>1$, and has an analytic continuation at $s=0$, with

$$\zeta(0)=-\frac{1}{2},\quad \zeta'(0)=-\frac{1}{2}\log(2\pi).$$

We find the following heuristic answer

$$"e^{\sum_{n\geq 1}\log\frac{4\pi^2n^2}{\beta^2}}"=e^{-\eta'(0)}=\beta.$$

The second term is convergent

$$\prod_{n=1}^{\infty}\left(1+\frac{\beta^2}{4\pi^2n^2}\right)=\frac{\sinh(\beta/2)}{\beta/2}.$$

Put the above two contributions together, we get

$$\sqrt{\det(D)}=2\sinh(\beta/2).$$

Therefore a reasonable evaluation of the path integral would be

$$\int_{\mathcal{E}}[D\phi]e^{-S/\hbar}"="\frac{1}{2\sinh(\beta/2)}.$$

In the next lecture, we will give a physics explanation of this result.

Let us take more serious look at $\det(D)$ following the above idea. Let $D$ be a positive Laplacian. Let $\lambda_i$ be its eigenvalues. We define an analogue of Riemann zeta function for $D$ by

$$\zeta_D(s)=\sum_{i=1}^{\infty}\frac{1}{\lambda_i^s}.$$

One way to get $\zeta_D(s)$ is by using the heat operator $e^{-tD}$. This is represented by a kernel function $h_t(x,y)$ such that

$$(e^{-tD}\phi)(x) = \int h_t(x,y)\phi(y)dy.$$

$h_t$ is the fundamental solution of the heat equation

$$(\frac{d}{dt} + D_x)h_t(x,y) = 0$$

and $\lim_{t\to 0} h_t(x,y) = \delta(x,y)$ is the $\delta$-function. Using

$$\frac{1}{\lambda^s} = \frac{1}{\Gamma(s)}\int_0^\infty e^{-\lambda t}t^{s-1}dt,$$

we find

$$\zeta_D(s) = \frac{1}{\Gamma(s)}\int_0^\infty \left(\operatorname{Tr} e^{-tD}\right) t^{s-1}dt.$$

Note $\operatorname{Tr} e^{-tD} = \int h_t(x,x)dx$, which has a well-behaved asymptotic structure as $t \to 0$ and implies that $\zeta_D(s)$ is analytic continued to $s = 0$. Then the determinant is defined by

$$\det(D) := e^{-\zeta_D'(0)}.$$

## 5. Quantum mechanics

In this lecture, we give a crash course on quantum mechanics and its relation with path integral to explain the factor $\frac{1}{2\sinh(\beta/2)}$.

Quantum mechanics is basically one-dimensional quantum field theory. Let

$$q : I, \mathbb{R}, \text{or } S^1 \to M$$

be a a curve on a manifold $M$. If you like, you can take $M \cong \mathbb{R}^d$ and think about $q = q(t)$ as $d$ functions on 1-dim space $I, \mathbb{R}$, or $S^1$. Consider th action functional

$$S[q] = \int \mathcal{L}(q,\dot{q})dt = \int (\frac{1}{2}\dot{q}^2 - V(q,\dot{q}))dt,$$

where $\dot{q} = \frac{dq}{dt}$. $V$ is called the *potential*. $\mathcal{L}$ is the *lagrangian*, which can be viewed as a function on $T_*M$

$$\mathcal{L} : T_*M \to \mathbb{R}.$$

The classical physics is characterized by the critical point of $S$. Since

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\delta q,$$

the critical point is described by the following *equation of motion*

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0.$$

Consider the Legendre transformation

$$\lambda : T_* M \to T^* M, \quad (q, \dot{q}) \to (q, p = \frac{\partial \mathcal{L}}{\partial \dot{q}}).$$

Assume $\lambda$ is invertible that we can solve $\dot{q} = \dot{q}(q, p)$. Then we obtain a function

$$H : T^* M \to \mathbb{R}, \quad \mathcal{H}(q, p) = p\dot{q} - \mathcal{L}(q, \dot{q}),$$

called *Hamiltonian*.

**Example 5.1.** Assume $V = V(q)$ only depends on $q$. Then $p = \dot{q}$, and

$$H = \frac{1}{2}p^2 + V(q).$$

The advantage of working with $T^* M$ is that it is a symplectic manfiod with a canonical Poisson bracket

$$\{-, -\} : C^\infty(T^* M) \times C^\infty(T^* M) \to C^\infty(T^* M)$$

such that

$$\{p, q\} = -\{q, p\} = 1.$$

Then the classical equation of motion can be described as Hamiltonian flow

$$\begin{cases} \frac{dq}{dt} = \{H, q\} = \frac{\partial H}{\partial p} \\ \frac{dp}{dt} = \{H, p\} = -\frac{\partial H}{\partial q} \end{cases}.$$

Classically, an *observable* is represented by a function $f \in C^\infty(T^* M)$, whose evolution equation with respect to time is

$$\frac{df}{dt} = \{H, f\}.$$

To get the quantum theory, let us introduce the so-called *canonical quantization*. It simply quantizes

$$(p, q) \to (\hat{p}, \hat{q})$$

where $\hat{p}, \hat{q}$ are operators on certain Hilbert space $\mathcal{H}$ such that

$$[\hat{q}, \hat{p}] = i\hbar, \quad [\hat{q}, \hat{q}] = [\hat{p}, \hat{p}] = 0.$$

For example, we can represent the Hilbert space by $L^2(M)$. Then

$$\hat{q} = q\cdot, \quad \hat{p} = -i\hbar\frac{\partial}{\partial q}.$$

The quantization gives

$$T^*M \to D_M$$
$$\{-,-\} \to i[-,-]$$

where $D_M$ is the space of differential operators on $M$. Similarly, the Hamiltonian $H$ is also quantized to be an operator

$$H \to \hat{H}.$$

The evolution equation for *quantum observable* becomes

$$\frac{d\hat{f}}{dt} = i[\hat{H}, \hat{f}],$$

which can be formally solved by

$$\hat{f}(t) = e^{i\hat{H}t/\hbar}\hat{f}(0)e^{-i\hat{H}t/\hbar}.$$

The time evolution operator $e^{i\hat{H}t/\hbar}$ defines a unitary operator on the Hilbert space $\mathcal{H}$. Let us do some physics. We introduce Dirac's Braket notation to formally write a vector $|q>$ for the eigenvector of $\hat{q}$

$$\hat{q}|q>= q|q> .$$

Any element in $\mathcal{H}$ can be formally represented as a linear combination

$$|\phi>= \sum_q \phi(q)|q> .$$

$\phi(q)$ is called the wave-function representation of $|\phi>$. We also write

$$<\phi|$$

as the conjugation of $|\phi>$. Then another way to represent wave function is

$$\phi(q) =< q|\phi> .$$

Here $<-|->$ represents the Hermitian pairing on $\mathcal{H}$.

As time goes, the position operator evolves as

$$\hat{q}(t) = e^{i\hat{H}t/\hbar}\hat{q}e^{-i\hat{H}t/\hbar}.$$

In particular, its eigenvector evolves as

$$|q,t>= e^{i\hat{H}t/\hbar}|q> .$$

The wave function at time $t$ is represented by

$$\phi(q,t) =< q,t|\phi>=< q|e^{-i\hat{H}t/\hbar}|\phi>,$$

from here it is easy to derive

$$i\hbar\frac{\partial}{\partial t}\phi = \hat{H}\phi,$$

which is nothing but the celebrated Schrödinger equation.

Now here comes the key point. Let us consider the physics amplitude for an initial state $|\phi_i >$ to evolve after time $t$ becoming a final state $|\phi_f >$. That is, we are interested in

$$< \phi_f, t|\phi_i >=< \phi_f|e^{-i\hat{H}t/\hbar}|\phi_i > .$$

The fundamental physics meaning of path integral is that it represents the kernel of the time evolution operator $e^{-i\hat{H}t/\hbar}$!. Precisely, let us formally write

$$K_t(q_f, q_i) = \int_{q(0)=q_i, q(t)=q_f} [Dq]e^{iS[q]},$$

that is the integral over the space of paths with fixed boundary condition. Then

$$< \phi_f, t|\phi_i >= \int dq_f dq_i \overline{\phi_f(q_f)} K_t(q_f, q_i)\phi(q_i).$$

In particular, if we consider the path integral over circles of lenght $t$, then

$$\int_{q(0)=q(t)} [Dq]e^{iS} = \int dq K_t(q, q) = \operatorname{Tr} e^{-i\hat{H}t/\hbar}.$$

I should warn you that the measure for path integrals are in general not rigorously defined. Nevertherless, this above relation gives many insight into physics system. Similar, if we insert some operators

$$< \phi_f, t|\hat{A}_n(q(t_n)) \cdots \hat{A}_1(q(t_1))|\phi_i >$$

where $0 < t_1 < \cdots < t_n < t$. Then it is represented by the kernel

$$\int_{q(0)=q_i, q(t)=q_f} [Dq] A_n(q(t_n)) \cdots A_1(q(t_1))e^{iS[q]}.$$

**Example 5.2.** Now we can explain the physics meanig of the result $\frac{1}{2\sinh(\beta/2)}$. Consider the simple Harmonic oscillator

$$S = \int_0^t \left(\frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2\right) dt.$$

In practice, to get some better behaved path integral, physicists usually do the *Wick rotation*

$$t \to -i\tau,$$

such that

$$e^{iS} \to e^{-S_E}.$$

$S_E$ is called the Euclidean action, which is usually bounded from below. In our case, we get

$$S_E = \int_0^\beta \left(\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 - \frac{1}{2}q^2\right) d\tau = \frac{1}{2}\int_0^\beta qDq, \quad \beta = it.$$

Here

$$D = -\left(\frac{d}{d\tau}\right)^2 + 1.$$

For simplicity, we set $\hbar = 1$. In the previous lecture, we have computed

$$\int_{q(0)=q(\beta)} [Dq]^{-\frac{1}{2}\int_0^\beta qDq} = \frac{1}{\sqrt{\det D}} = \frac{1}{2\sinh(\beta/2)}.$$

Now we come to the Hilbert space picture. The Hamiltonian is

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2.$$

After quantization, we get the Hamiltonian operator

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{q}}{2} = -\frac{1}{2}\left(\frac{d}{dq}\right)^2 + \frac{1}{2}q^2.$$

Let us find the eigenvalues of $\hat{H}$. This can be solved as follows. Consider

$$a^\dagger = \frac{1}{\sqrt{2}}(\hat{p} + i\hat{q}), \quad a = \frac{1}{\sqrt{2}}(\hat{p} - i\hat{q}).$$

$a^\dagger$ is called the *creation operator*, and $a$ is called the *annihilation operator*. Then

$$\hat{H} = a^\dagger a + \frac{1}{2}, \quad [a, a^\dagger] = 1, \quad [\hat{H}, a] = -a, \quad [\hat{H}, a^\dagger] = a^\dagger.$$

In particular, if $|\phi>$ is a eigenstate with eigenvalue $E$

$$\hat{H}|\phi >= E|\phi >,$$

then

$$\hat{H}a^\dagger|\phi >= (E+1)a^\dagger|\phi >, \quad \hat{H}a|\phi >= (E-1)a|\phi >,$$

i.e., $a^\dagger$ raises the eigenvalue by 1, and $a$ lowers the eigenvalue by 1. Since $\hat{H}$ is bounded $\hat{H} \geq \frac{1}{2}$, the lowest eigenstate is represented by

$$a|0 >= 0, \quad \hat{H}|0 >= \frac{1}{2}|0> .$$

$|0>$ is called the *vanuum*. The equation $a|0 >= 0$ is easily solved, and we have the wave function for the vacuum state by

$$\phi_0 = e^{-\frac{1}{2}q^2}.$$

All the other eigenstate can be obtained by

$$|n>= (a^\dagger)^n|0>, \quad \hat{H}|n>= (n+\frac{1}{2})|n> .$$

It can be proved that we have found a complete basis of the $L^2$-space, hence all the eigenvalues. Therefore

$$\operatorname{Tr} e^{-i\hat{H}t} = \operatorname{Tr} e^{-\beta\hat{H}} = \sum_{n\geq 0} e^{-\beta(n+\frac{1}{2})} = \frac{1}{2\sinh(\beta/2)}.$$

Physics works!

## 6. Ultraviolet divergence

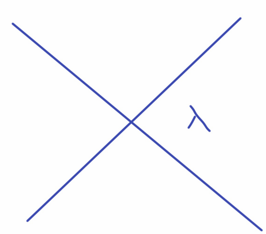
Let us now consider an interacting scalar field theory. To illustrate the issue, let us consider the so-called $\phi^4$-theory on $\mathbb{R}^4$.

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} \phi D\phi + \frac{\lambda}{4!}\int_{\mathbb{R}^4} \phi^4, \quad \phi \in C_c^\infty(\mathbb{R}^4).$$

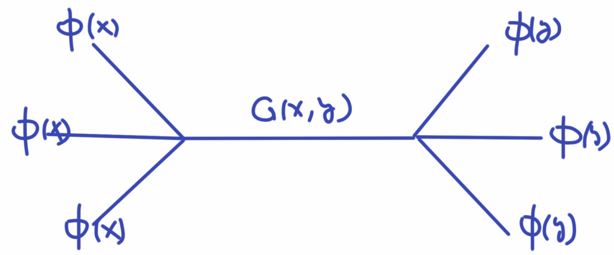
Here $D = -\sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}$ is the standard Laplacian. The propagator is given by $D^{-1}$, or the Green's function

$$G(x,y) = \frac{1}{|x-y|^2}.$$

The vertex is given by the $\phi^4$-interaction



We would like to understand the corresponding Feynman diagrams. Let us start with a tree diagram:

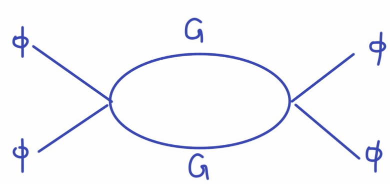



It gives rise to

$$\lambda^2 \int_{\mathbb{R}^4 \times \mathbb{R}^4} \phi(x)^3 \phi(y)^3 G(x,y) = \lambda^2 \int d^4x d^4y \phi(x)^3 \phi(y)^3 \frac{1}{|x-y|^2}.$$

This is convergent for any compactly supported input $\phi$. In fact, you can show that all tree diagrams are convergent. There exists a problem if you allow arbitrary $\phi$ without any control on its decay property at $\infty$. This potential divergence is due to the non-compactness of our working manifold, called *IR divergence*. This is not a big bother for us. We can restrict ourselves to inputs with certain locality property, or we can work with compact manifold.

Now let us consider a one-loop diagram as follows:




Its Feynman diagram gives

$$\hbar\lambda^2 \int d^4x d^4y \phi(x)^2 \phi(y)^2 G(x,y)^2 = \hbar\lambda^2 \int d^4x d^4y \phi(x)^2 \phi(y)^2 \frac{1}{|x-y|^4}.$$

This is in general divergent around the locus $x = y$! If we shift our variable $y$ and change to redial coordinate, then the above integral becomes

$$\int d^4x \int_{S^3} d\Omega \int_0^\infty \phi(x)^2 \phi(x + y(r,\Omega))^2 \frac{dr}{r}$$

that we expect certain logarithmic divergence. This divergence comes from very small distance, i.e., when $r \to 0$ in the above expression. Small distance corresponds to high energy, therefore such divergence is called *UV divergence*, which is a much more serious problem in quantum field theory.

The solution to this problem leads to the celebrated physics idea of *renormalization*. Let us illustrate by this example on what is happening. The problem is about the singularity of the Green's function $G(x,y)$. We can regularize it to make a better behaved Feynman diagram. What physicists usually do is to go to the Fourier modes and do cut-off there. Precisely, we can write

$$G(x,y) = \int d^4k \frac{e^{ik(x-y)}}{k^2}$$

and then cut it to a better behaved one

$$\int_{\Lambda_0 \le |k| \le \Lambda_1} d^4k \frac{e^{ik(x-y)}}{k^2}.$$

That is, we only integrate part of the data, and then do it step by step. It is not quite easy to work with this on arbitrary manifold. Instead, we can use the *heat kernel cut-off*. Let us introduce the regularized propagator

$$P_\epsilon^L = \int_\epsilon^L e^{-tD} dt, \quad 0 < \epsilon < L$$

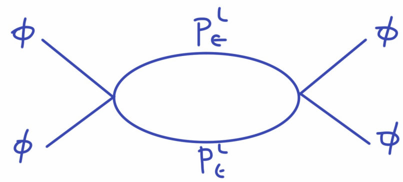
where $e^{-tD}$ is the heat kernel. When we set $\epsilon \to 0$ and $L \to \infty$, we recover

$$G = P_0^\infty.$$

The Feynman diagram with the regularized propagator $P_\epsilon^L$ is obviously convergent, but exhibiting certain singular behavior as $\epsilon \to 0$ (the UV divergence). Let us figure this out in our example. The regularized propagator is given by

$$P_\epsilon^L = \int_\epsilon^L \frac{dt}{(2\pi t)^2} e^{-|x-y|^2/4t}.$$

The regularized one-loop diagram as before becomes




$$\begin{aligned}
W_\Gamma(P_\epsilon^L, I) &= \int d^4x d^4y \phi(x)^2 \phi(y)^2 P_\epsilon^L(x,y)^2 \\
&= \int d^4x d^4y \phi(x)^2 \phi(x+y)^2 \int_\epsilon^L \frac{dt_1}{(2\pi t_1)^2} \frac{dt_2}{(2\pi t_2)^2} e^{-|y|^2/4t_1 - |y|^2/4t_2} \\
&= \int d^4x \int_\epsilon^L \frac{dt_1}{(2\pi t_1)^2} \frac{dt_2}{(2\pi t_2)^2} \int d^4y \phi(x)^2 \phi(x+y)^2 e^{-|y|^2/4t_1 - |y|^2/4t_2}
\end{aligned}$$

Let us write

$$\phi(x)^2 \phi(x+y)^2 = \phi(x)^4 + R(x,y).$$

Since we have assume $\phi$ being compactly supported, $R(x,y)$ has also compact support and bounded by

$$|R(x,y)| \leq C|y|$$

where $C$ is a constant depending on $\phi$. We separate the above integral into two parts by this decomposition. The first one is

$$\begin{aligned}
&\int d^4x \phi(x)^4 \int_\epsilon^L \frac{dt_1}{(2\pi t_1)^2} \frac{dt_2}{(2\pi t_2)^2} \int d^4y e^{-|y|^2/4t_1 - |y|^2/4t_2} \\
=& \int d^4x \phi(x)^4 \int_\epsilon^L \frac{dt_1}{(2\pi t_1)^2} \frac{dt_2}{(2\pi t_2)^2} \frac{(2\pi)^2}{(\frac{1}{2t_1} + \frac{1}{2t_2})^2} \\
=& \frac{1}{\pi^2} \int d^4x \phi(x)^4 \int_\epsilon^L \frac{dt_1 dt_2}{(t_1+t_2)^2} \\
=& \frac{1}{\pi^2} \left( \log \frac{L+\epsilon}{2L} - \log \frac{2\epsilon}{L+\epsilon} \right) \int d^4x \phi(x)^4 \\
=& -\frac{\log \epsilon}{\pi^2} \int d^4x \phi(x)^4 + \text{terms smooth as } \epsilon \to 0.
\end{aligned}$$

In particular, we find this log-divergence in an explicit form as $\epsilon \to 0$.

The second term by $R(x,y)$ is absolutely bounded by

$$C \int_\epsilon^L \frac{dt_1}{(2\pi t_1)^2} \frac{dt_2}{(2\pi t_2)^2} \int d^4y |y| e^{-|y|^2/4t_1 - |y|^2/4t_2}$$

$$=C' \int_{\epsilon}^{L} \frac{dt_1 dt_2}{(t_1 t_2)^2} \left( \frac{1}{t_1} + \frac{1}{t_2} \right)^{-5/2} = C' \int_{\epsilon}^{L} \frac{dt_1 dt_2 \sqrt{t_1 t_2}}{(t_1 + t_2)^{5/2}}$$

It is not hard to prove that

$$\int_{0}^{L} \frac{dt_1 dt_2 \sqrt{t_1 t_2}}{(t_1 + t_2)^{5/2}} < \infty.$$

This implies that the second term has a smooth limit as $\epsilon \to 0$. Put the above information together, we find

$$W_{\Gamma}(P_{\epsilon}^{L}, I) = -\frac{\log \epsilon}{\pi^2} \hbar \lambda^2 \int d^4 x \phi(x)^4 + \text{terms smooth as } \epsilon \to 0.$$

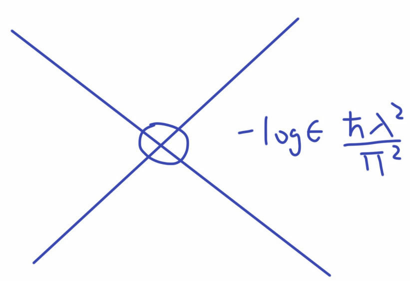
To deal with such divergence, let us correct our action functional by adding a new $\epsilon$-dependent term

$$I^{CT}(\epsilon) = \frac{\log \epsilon}{\pi^2} \int d^4 x \phi(x)^4.$$

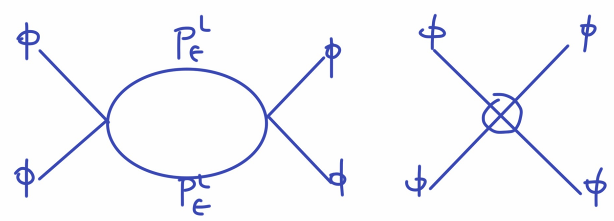
The corrected action functional becomes

$$S + I^{CT}(\epsilon) = \frac{1}{2} \int_{\mathbb{R}^4} \phi D \phi + \frac{\lambda}{4!} \int_{\mathbb{R}^4} \phi^4 + \frac{\log \epsilon}{\pi^2} \hbar \lambda^2 \int d^4 x \phi(x)^4.$$

Therefore we have added a new 1-loop vertex for our Feynman diagram



Now let us evaluate the 1-loop Feynman diagram with 4 external inputs. There are two contributions $\Gamma$




As we have proved above, the following limit

$$\lim_{\epsilon\to 0} W_\Gamma(P_\epsilon^L, I + I^{CT}(\epsilon))$$

exists! This gives a well-defined Feynman integral, or *renormlized*.

Now we can use this new action functional to do other 1-loops and higher loops. As we will prove later, you can always find counter terms, recursively order by order in $\hbar$, such that

$$\lim_{\epsilon\to 0} W_\Gamma(P_\epsilon^L, I + I^{CT}(\epsilon))$$

exists for sum of graphs of any fixed loops and number of inputs. We define the effective functional at scale $L$

$$I[L] = \lim_{\epsilon\to 0} \sum_{\Gamma:conn} W_\Gamma(P_\epsilon^L, I + I^{CT}(\epsilon)),$$

then $I[\infty]$ gives our desired candidate for the perturbative series of the path integral

$$\text{"}\int [D\phi] e^{-S/\hbar}\text{"} \longrightarrow e^{I[\infty]/\hbar}.$$

In general, you may need to add many many terms in $I^{(CT)}(\epsilon)$ to make this limit work at all loops, even infinite many different types. In our $\phi^4$-example, the counter-term you add is still of the form of the original $\phi^4$-interaction, therefore the coupling constant is corrected. If this kind of situation happens, we say the theory is *renormalizable*. If we have to add into infinite many different counter terms, we say the theory is *non-renormalizable*. Physics prefer renormalizable theory because it has only finite types of interactions, and therefore can be tested by experiments in finite steps and do prediction. Non-renormalizable theory loses this prediction power, but otherwise still gives interesting well-defined theory.

To be continued....

## REFERENCES

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