CALABI-YAU GEOMETRY, PRIMITIVE FORM AND MIRROR SYMMETRY

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ABSTRACT. This note comes out of the author's lecture presented at the workshop *Primitive forms and related subjects*, Feb 10-14 2014.

CONTENTS

1. INTRODUCTION

Primitive forms are introduced by K.Saito [40, 41] around early 1980's as a generalization of the elliptic period integral theory associated to an isolated singularity. It leads to systematic examples of Frobenius manifold structure on the universal unfoldings of isolated singularities, characterizing the mathematical structure of topological Landau-Ginzburg B-model. Motivated by mirror symmetry, there is a vast interest in understanding the structure of primitive forms nowadays.

One of the recent development is the mathematical theory of Landau-Ginzburg A-models constructed by Fan, Jarvis and Ruan [17], popularized as the FJRW theory (see also [8, 39] for several purely algebraic constructions of Landau-Ginzburg A-model and their relationships with FJRW theory). With the huge success toward understanding the mirror symmetry between Calabi-Yau manifolds and Calabi-Yau/Landau-Ginzburg correspondence, it is desirable to see whether mirror symmetry between Landau-Ginzburg models hold directly. A first step toward establishing such Landau-Ginzburg mirror symmetry is to understanding the relation between FJRW theory in the A-model and Saito's primitive form in the B-model. The purpose of this lecture is to present some recent progress in this direction related to the author's works [13, 23, 28, 29].

There are two goals in this lecture. The first is to give a unified presentation for both Calabi-Yau and Landau-Ginzburg B-models via the geometry of polyvector fields. On compact Calabi-Yau manifolds, Barannikov and Kontsevich [5] construct a large class of Frobenius manifold structures on their extended moduli spaces of complex structures, which is closely related to the Kodaira-Spencer gauge theory for polyvector fields introduced in [7]. With deformation theory, Barannikov developed the generalized period map via variation of semi-infintie Hodge structures [3] to describe the mirror of Gromov-Witten invariants. The prototype of semi-finite Hodge structures and generalized period maps can be traced back to Landau-Gzinburg models as Saito's higher residue theory and primitive forms [40, 41]. The first part of the lecture is to explain how the geometry of polyvector fields in Calabi-Yau geometry is related to Landau-Ginzburg models [28].

The second part is to present the recent development of perburbative method for primitive forms [28, 29] and mirror symmetry [23]. For weighted homogenous cases, explicit expressions of primitive forms are only known for ADE and simple elliptic singularities. This becomes one of the main obstacles to test mirror symmetry between Landau-Ginzburg models. However, WDVV equation is a powerful integrable equation, which usually reduces the computation of generating functions to a finite few orders. A recursive algorithm is developed in [28, 29] which allows us to compute any primitive forms up to arbitrary finite order. This solves the computation difficulty in Landau-Ginzburg B-model. As an application, a version of Landau-Ginzburg mirror symmetry conjecture for a large class of singularities is established [23, 29].

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2. CALABI-YAU GEOMETRY

2.1. **Polyvector fields.** Let X be a compact Calabi-Yau manifold of dimension d with holomorphic volume form Ω_X.Let

$$\mathrm{PV}(X) = \bigoplus_{0\leq i,j\leq d} \mathrm{PV}^{i,j}(X), \quad \mathrm{PV}^{i,j}(X) = \mathcal{A}^{0,j}(X, \wedge^i T_X)$$

be the space of polyvector fields on X. Here T_X is the holomorphic tangent bundle, and $\mathcal{A}^{0,j}(X, \wedge^i T_X)$ is the space of smooth $(0, j)$-forms valued in $\wedge^i T_X$. $\mathrm{PV}(X)$ is a differential bi-graded commutative algebra: the differential is

$$\bar{\partial} : \mathrm{PV}^{i,j}(X) \to \mathrm{PV}^{i,j+1}(X),$$

and the algebra structure arises from wedge product. Our degree convention is that elements of $\mathrm{PV}^{i,j}(X)$ are of degree $j - i$. The graded-commutativity says

$$\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha$$

where $|\alpha|, |\beta|$ denote the degree of α, β respectively. Ω_X induces an identification between the space of polyvector fields and differential forms

$$\mathrm{PV}^{i,j}(X) \overset{\lrcorner\Omega_X}{\cong} \mathcal{A}^{d-i,j}(X)$$
$$\alpha \to \alpha\lrcorner\Omega_X$$

where $\lrcorner$ is the contraction, and $\mathcal{A}^{i,j}(X)$ denotes smooth differential forms of type (i, j). The holomorphic de Rham differential ∂ on forms defines an operator on

PV(X) via the above isomorphism, which we still denote by

$$\partial : \mathrm{PV}^{i,j}(X) \to \mathrm{PV}^{i-1,j}(X)$$

i.e.

$$(\partial\alpha)\lrcorner\Omega_X \equiv \partial(\alpha\lrcorner\Omega_X), \ \ \alpha \in \mathrm{PV}(X).$$

The definition of ∂ doesn't depend on the choice of Ω_X on compact Calabi-Yau manifolds. It induces a bracket on polyvector fields (Tian-Todorov lemma)

$$\{\alpha, \beta\} = \partial\left(\alpha\beta\right) - \left(\partial\alpha\right)\beta - (-1)^{|\alpha|}\alpha(\partial\beta)$$

which coincides with the Schouten-Nijenhuis bracket (up to a sign).

We can integrate polyvector fields by the *trace map* $\mathrm{Tr} : \mathrm{PV}(X) \to \mathbb{C}$

$$\mathrm{Tr}(\alpha) := \int_X (\alpha\lrcorner\Omega_X) \wedge \Omega_X.$$

Let $\langle -, - \rangle$ be the induced pairing $\mathrm{PV}(X) \otimes \mathrm{PV}(X) \to \mathbb{C}$

$$\alpha \otimes \beta \to \langle \alpha, \beta \rangle \equiv \mathrm{Tr}\left(\alpha\beta\right).$$

It is easy to see that $\bar\partial$ is (graded) skew symmetric for this pairing and ∂ is (graded) symmetric.

2.2. **Symplectic structure.** Following Givental's symplectic formalism, let us add a formal variable z representing the "gravitational descendant" and introduce the following spaces

$$S(X) := \mathrm{PV}(X)((z)), \quad S_+(X) := \mathrm{PV}(X)[[z]], \quad S_-(X) := z^{-1}\,\mathrm{PV}(X)[z^{-1}].$$

There exists a natural symplectic pairing on $S(X)$ given by

$$\omega(f(z)\alpha, g(z)\beta) := \mathrm{Res}_{z=0}\left(f(z)g(-z)dz\right)\mathrm{Tr}(\alpha\beta).$$

It is direct to check that the differential

$$Q = \bar\partial + z\partial$$

is (graded) skew-symmetric with respect to the symplectic pairing ω. Let us denote by

$$\mathcal{H} = H^*(S(X), Q), \quad \mathcal{H}_+ = H^*(S_+(X), Q).$$

ω descends to define a pairing on the cohomology $\mathcal{H}$, still denoted by

$$\omega : \mathcal{H} \otimes_{\mathbb{C}} \mathcal{H} \to \mathbb{C}.$$

$\mathcal{H}_+$ becomes an isotropic subspace. Let us define the isomorphism

$$\Gamma_\Omega : \mathrm{PV}(X)((z)) \to \mathcal{A}(X)((z)), \quad z^k\alpha \to z^{k+i-1}\alpha\lrcorner\Omega_X, \quad \alpha \in \mathrm{PV}^{i,j}(X).$$

It transfers Q to the de Rham differential

$$\Gamma_\Omega \circ Q = d \circ \Gamma.$$

Under the isomorphism Γ_Ω, we have

$$\Gamma_\Omega(S_+(X)) = \prod_{p\in\mathbb{Z}} z^{d-p+1} F^p \mathcal{A}(X),$$

where $F^p\mathcal{A}(X) = \mathcal{A}^{\geq p,*}(X)$. At the cohomology level we have isomorphisms

$$\Gamma_\Omega : \mathcal{H} \xrightarrow{\sim} H^*(X,\mathbb{C})((z))$$

and

$$\Gamma_\Omega : \mathcal{H}_+ \xrightarrow{\sim} \prod_{p\in\mathbb{Z}} z^{d-p+1} F^p H^*(X,\mathbb{C}).$$

In particular, we see that the isotropic embedding $S_+(X) \subset S(X)$ (or $\mathcal{H}_+ \subset \mathcal{H}$) plays the role of Hodge filtration.

The symplectic space $(\mathcal{H}, \omega)$ is basically Givental's loop space formalism in the B-model, where $(S(X), \omega)$ can be viewed as the lifting to the chain level. Givental has formulated the generating function of Gromov-Witten invariants as the geometry of Lagrangian cones inside the symplectic space. The B-model aspect is established via Barannikov's generalized period map [2,3] using deformation theory [5].

In fact, the Lagrangian cone construction can be lifted to the chain level in the B-model, which gives rise to a gauge theory of polyvector fields on Calabi-Yau manifolds. This is developed in [13], which we call BCOV theory, as a generalization of the Kodaira-Spencer gravity theory on Calabi-Yau three-folds discovered by [7]. It turns out that the tree level BCOV theory is equivalent to Givental's formalism and Barannikov's generalized period map at the cohomology level [33], which we now describe.

2.3. **BCOV theory.** Define a formal graded submanifold $\mathcal{L}_X$ of $S(X)$ based at 0

$$\mathcal{L}_X = \{z - ze^{\mu/z} | \mu \in S_+(X)\}.$$

By "formal" we mean it is defined via its functor of points. More precisely, this is a functor from nilpotent Artinian graded algebras R (with maximal ideal $m \subset R$) to sets. If R is an Artinian graded algebra, then the R-points of $S(X)$ is the set of degree 0 elements of $S(X) \otimes m$. We define $\mathcal{L}_X(R)$ to be the set of those $a \in S(X) \otimes m$ which are of degree 0, and which can be expressed (necessarily in a unique way) in the form

$$a = z - ze^{\mu/z}$$

for some $\mu \in S_+(X) \otimes m$. This expression makes sense because the maximal ideal $m \subset R$ is nilpotent.

The fundamental property of $\mathcal{L}_X$ is that it is a formal lagrangian submanifold of $S(X)$ which is preserved by the differential Q [13]. If we formally identify

$$S(X) = T^*(S_+(X))$$

via the natural splitting

$$S(X) = S_+(X) \oplus S_-(X),$$

then there exists a formal functional I^{BCOV} on $S_+(X)$ such that

$$\mathcal{L}_X = \mathrm{Graph}(dI^{BCOV}).$$

We have the following explicit formula [13]

$$I^{BCOV}(\mu) = \mathrm{Tr}\,\langle e^{\mu}\rangle_0\,,$$

where

$$\langle - \rangle_0 : \mathrm{Sym}(S_+(X)) \to \mathrm{PV}(X)$$

is the map via intersection on $\overline{M}_{0,n}$

$$\left\langle z^{k_1}\mu_1, \cdots, z^{k_n}\mu_n \right\rangle_0 = \mu_1 \wedge \cdots \wedge \mu_n \int_{\overline{M}_{0,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} = \binom{n-3}{k_1 \cdots k_n} \mu_1 \wedge \cdots \wedge \mu_n.$$

I^{BCOV} is called the BCOV interaction in [13], which also appears in a finite dimensional toy model in [36]. Note that if we "turn off" gravitational descendants by setting $z = 0$ for the inputs, then I^{BCOV} is reduced to the cubic interaction of Kodaira-Spencer gauge theory [7]. The geometric fact that $\mathcal{L}_X$ is preserved by the differential Q is translated to a classical master equation for I^{BCOV}, which describes the infinitesimal gauge transformation in the BV formalism [13].

2.4. **Generating function.** The genus g generating function of B-model invariants could be obtained via g-loop Feynman diagrams with vertices being I^{BCOV}. For $g > 0$, this requires suitable renormalizations (see [13]). For $g = 0$, however, renormalization is not needed and we obtain the genus zero generating function as a sum of tree diagrams in BCOV theory. This is basically equivalent to a version of homological perturbation lemma, which amounts to go to Q-cohomology.

The Q-fixed point of $\mathcal{L}_X$ can be described as follows. Observe that

$$Q(ze^{\mu/z} - z) = (Q\mu + \frac{1}{2}\{\mu, \mu\})e^{\mu/z}.$$

The Q-fixed point of $\mathcal{L}_X$ are described by solutions of the Maurer-Cartan equation

$$Q\mu + \frac{1}{2}\{\mu, \mu\} = 0,$$

which can be viewed as an extended version of deforming a pair of complex structure on X together with a holomorphic volume form [3].

Let $\mathcal{M}_X$ denote the formal moduli of gauge equivalent solutions

$$\mathcal{M}_X = \left\{\mu \in S_+(X) | Q\mu + \frac{1}{2}\{\mu, \mu\} = 0\right\} / \sim,$$

which again is defined via its functor of points on nilpotent Artinian graded algebras. It follows from Calabi-Yau geometry that the moduli $\mathcal{M}_X$ is smooth [3,5]. In the same fashion, we can view it as a formal lagrangian submanifold

$$\mathcal{M}_X \hookrightarrow \mathcal{H}, \quad \mu \to z - ze^{\mu/z}.$$

To obtain the generating function at the cohomology level, we need a choice of splitting since $S_-(X)$ is not preserved by Q.

Let $\mathcal{L} \subset \mathcal{H}$ a linear isotropic subspace of $\mathcal{H}$ such that

(1) $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{L}$,
(2) $\mathcal{L}$ is preserved by the operator $z^{-1} : \mathcal{H} \to \mathcal{H}$.

The splitting $\mathcal{L}$ allows us to identify

$$H^*(X) = \mathcal{H}_+/z\mathcal{H}_+ \cong \mathcal{H}_+ \cap z\mathcal{L},$$

hence

$$\mathcal{H}_+(X) \cong H^*(X)[[z]], \quad \mathcal{L} \cong z^{-1}H^*(X)[z], \quad \mathcal{H} \cong H^*(X)((z)).$$

In particular, we can formally identify

$$\mathcal{H} \cong T^*(H^*(X)[[z]])$$

from which we obtain a formal generating function $F_0^{\mathcal{L}}$ on $H^*(X)[[z]]$ such that

$$\mathcal{M}_X \cong \mathrm{Graph}(d\mathbf{F}_0^{\mathcal{L}}).$$

We can make this construction explicitly. Let $\{\phi_\alpha\}$ be a basis of $H^*(X)$, and $\{\phi^\alpha\}$ be a dual basis of $H^*(X)$ such that

$$\int \phi_\alpha \wedge \phi^\beta = \delta_\alpha^\beta.$$

Let $\mathbf{s}$ be a coordinate on $\mathcal{M}_X$ and $\mu(\mathbf{s})$ a universal solution of the Maurer-Cantan equation. We can always expand in $\mathcal{H}$

$$ze^{\mu(\mathbf{s})/z} = z + \sum_{k\geq 0}\sum_{\alpha} \tau_k^{\alpha}(\mathbf{s})\phi_\alpha(-1)^k z^k + \sum_{k\geq 0}\sum_{\alpha} p_{k,\alpha}(\mathbf{s})\phi_\alpha z^{-k-1}.$$

$\{\tau_k^\alpha\}$ can be viewed as the linear coordinates on $H^*(X)[[z]]$. The transformation

$$\mathbf{s} \to \{\tau_k^\alpha(\mathbf{s})\}$$

is invertible, which leads to different choice of the coordinates $\{\tau_k^\alpha\}$ on $\mathcal{M}_X$. In terms of $\{\tau_k^\alpha\}$, we find

$$p_{k,\alpha} = \frac{\partial \mathbf{F}_0^{\mathcal{L}}(\tau)}{\partial \tau_k^\alpha}.$$

If we set

$$\tau_k^\alpha = 0, \quad k > 0,$$

then the expansion

$$e^{\mu(\mathbf{s}(\tau_0^\alpha, \tau_1^\alpha = 0, \cdots))/z} = 1 + \sum_\alpha \tau_0^\alpha \phi_\alpha z^{-1} + O(z^{-2})$$

plays the role of Givental's J-function, or Barannikov's generalized period map. In particular, $\{\tau_0^\alpha\}$ gives the flat coordinates of the underlying Frobenius manifold structure. As we will see, this is the analogue of K.Saito's primitive period map [40] in Landau-Ginzburg models.

The generating function $\mathbf{F}_0^{\mathcal{L}}$ depends on the choice of $\mathcal{L}$. In contrast to the A-model, there is a prior no canonical choice of the splitting $\mathcal{L}$. In fact, the freedom of the splitting is exactly responsible for the famous holomorphic anomaly equation [7] when $\mathcal{L}$ is the complex conjugate splitting. See also [11,13]. Around the large complex limit, there exists monodromy splitting from which $\mathbf{F}_0^{\mathcal{L}}$ gives the mirror of Gromov-Witten invariants [2,19,34].

2.5. **Feynman diagrams.** $\mathbf{F}_0^{\mathcal{L}}$ and the BCOV interaction I^{BCOV} are related by tree-level Feynman diagrams. In the absence of gravitational descendants, this is observed in [7], leading to the string field theory of B-twisted topological string by Kodaira-Spencer gauge theory. It is further illustrated in [5] as a homological perturbation related to the deformation theory on Calabi-Yau manifolds. We give a brief discussion here and refer to [33] for details.

For simplicity, we take $\mathcal{L}$ to be the complex conjugate splitting and write the generating function as $\mathbf{F}_0^X$. This splitting is related to the Harmonic theory as follows. Let g be a chosen Kähler metric on X. Let $\bar{\partial}^*$ be the adjoint of $\bar{\partial}$ on $\mathrm{PV}(X)$, and

$$\Delta = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

be the Laplacian. Let

$$h_u \in \mathrm{PV}(X) \otimes \mathrm{PV}(X), \quad u > 0$$

be the kernel of the heat operator $e^{-u\Delta}$. Formally,

$$e^{-u\Delta}(\alpha)(x) = \mathrm{Tr}(h_u(x,y) \wedge \alpha(y)), \quad \forall \alpha \in \mathrm{PV}(X),$$

where x, y represents two copies of coordinate on X, and the trace map is the integration over y with respect to the Calabi-Yau volume form as before. We define the *BCOV propagator* by the kernel

$$P = \int_0^\infty (\bar{\partial}^*\partial \otimes 1) h_u du,$$

which represents the operator

$$\bar{\partial}^*\partial\frac{1}{\Delta}.$$

Here $\frac{1}{\Delta}$ is Green's operator for the Laplacian Δ. Formally, P is the gauge fixed expression of $\frac{\partial}{\bar{\partial}}$, which is the inverse of the free part of the Kodaira-Spencer gauge action [7]. Let

$$\mathbf{H} = \{\mu \in \mathrm{PV}(X) | \Delta\mu = 0\}$$

be the harmonic elements. Since $\bar{\partial}\mathbf{H} = \partial\mathbf{H} = 0$, Hodge theory implies a natural isomorphism

$$\mathcal{H}_+ = \mathbf{H}[[z]], \quad \mathcal{H} = \mathbf{H}((z)),$$

which can be identified with the complex conjugate splitting [13].

Definition 2.1. The genus zero partition function $\mathbf{F}_0^{BCOV}$ of BCOV theory is the formal function on $\mathbf{H}[[z]]$ defined by

$$\mathbf{F}_0^{BCOV} = \sum_{\Gamma:\mathrm{Tree}} \frac{W_\Gamma(P, I^{BCOV})}{|Aut(\Gamma)|}$$

$W_\Gamma(P, I^{BCOV})$ is the Feynman diagram integrals with I^{BCOV} as the vertices and P as the propagator. $Aut(\Gamma)$ is the size of the automorphism group as graphs. The summation is over all connected tree diagrams with external edges where we put harmonic polyvector fields $\mathbf{H}[[z]]$.

We can also view $\mathbf{F}_0^X$ as a function on $\mathbf{H}[[z]]$. Then $\mathbf{F}_0^X$ has the following perturbative tree diagram expansion

$$\mathbf{F}_0^X = \mathbf{F}_0^{BCOV}.$$

This formula says that $\mathbf{F}_0^X$ arising from the deformation theory of Calabi-Yau manifolds can be identified with the tree-level amplitude of a gauge theory (BCOV theory). The importance of this observation is that it allows us to generalize to construct the higher genus generating function via higher loop Feynman diagrams, though suitable renormalization of the ultraviolet divergence is required. This is the point of view fully developed in [13].

3. LANDAU-GINZBURG MODEL

Now we move on to the Landau-Ginzburg B-model. We will focus on an isolated singularity defined by a weighted homogeneous polynomial

$$f : X = \mathbb{C}^n \to \mathbb{C}, \quad f(\lambda^{q_1} x_1, \cdots, \lambda^{q_n} x_n) = \lambda f(x_1, \cdots, x_n).$$

q_i are called the weights of x_i, and the central charge of f is defined by

$$\hat{c}_f = \sum_i (1 - 2q_i).$$

Associated to f, K.Saito has introduced the concept of a primitive form [40], which induces a Frobenius manifold structure (originally called a flat structure) on the local universal deformation space of f. The generalization to arbitrary isolated singularities is later fully established by M. Saito [43]. See also [4,14,15, 45] for a certain class of Laurent polynomials. This gives rise to the genus zero correlation functions in the Landau-Ginzburg B-model. For string theoretical point of view, we refer to [35].

In the rest of this section, we will give a brief review of primitive forms. Our presentation will base on the work [28], which exhibits a unified geometry of two models. We will also describe the perburbative formula of primitive forms [28] which is fully developed in [29] for applications to mirror symmetry between Landau-Ginzburg models.

3.1. **Residue revisited.** We would like to extend the discussion on Calabi-Yau manifolds to Landau-Ginzburg models on the pair $(X = \mathbb{C}^n, f)$. Let us fix the holomorphic volume form

$$\Omega = dx_1 \wedge \cdots \wedge dx_n$$

for convenience. Different choice of Ω will lead to essentially equivalent results in the following discussions. In the Landau-Ginzburg model, we have the twisted operator

$$\bar{\partial}_f := \bar{\partial} + \{f, -\} : \mathrm{PV}(X) \to \mathrm{PV}(X).$$

Its cohomology is given by

$$H^*(\mathrm{PV}(X), \bar{\partial}_f) = H^0(\mathrm{PV}(X), \bar{\partial}_f) \cong \mathrm{Jac}_0(f),$$

where $\mathrm{Jac}_{\mathbf{0}}(f) = \mathbb{C}\{x_i\}/\{\partial_i f\}$ is the Milnor ring of the isolated singularity. Let $\Omega^k_{X,\mathbf{0}}$ be the germ of holomorphic k-forms at $\mathbf{0}$. Let us define

$$\Omega_f := \Omega^n_{X,\mathbf{0}}/df \wedge \Omega^{n-1}_{X,\mathbf{0}}.$$

With our choice of Ω, we can identify

$$\mathrm{Jac}_{\mathbf{0}}(f) \to \Omega_f, \quad [\phi] \to [\phi\Omega].$$

There exists a classical residue pairing defined on Ω_f:

$$\eta_f : \Omega_f \otimes \Omega_f \to \mathbb{C}.$$

Our first task is to understand the residue pairing η_f in terms of trace map on $\mathrm{PV}(X)$. This is done in [28] as follows. Since X is noncompact, the naive trace map via integration doesn't work. Let

$$\mathrm{PV}_c(X) \subset \mathrm{PV}(X)$$

be the subspace of compactly supported polyvector fields. The trace map is defined on $\mathrm{PV}_c(X)$

$$\mathrm{Tr}(\mu) = \int_X (\mu \lrcorner \Omega) \wedge \Omega, \quad \mu \in \mathrm{PV}_c(X).$$

As in the Calabi-Yau case, Tr descends to cohomologies

$$\mathrm{Tr} : H^*(\mathrm{PV}_c(X), \bar{\partial}_f) \to \mathbb{C}.$$

The key observation is that the embedding of complexes

$$(\mathrm{PV}_c(X), \bar{\partial}_f) \hookrightarrow (\mathrm{PV}(X), \bar{\partial}_f)$$

is in fact a quasi-isomorphism, inducing a canonical isomorphism

$$H^*(\mathrm{PV}_c(X), \bar{\partial}_f) \simeq H^*(\mathrm{PV}(X), \bar{\partial}_f).$$

It follows that we have a induced trace map

$$\mathrm{Tr} : \mathrm{Jac}_0(f) \to \mathbb{C}.$$

It is proved in [28] that given $[\alpha], [\beta] \in \mathrm{Jac}_0(f)$,

$$\eta_f([\alpha\Omega], [\beta\Omega]) = \mathrm{Tr}([\alpha\beta]).$$

To get a feeling on how this works, let us look at the simplest example when $X = \mathbb{C}$ is one-dimensional. Let ρ be a cut-off function, which is 1 around the isolated singularity 0 and vanishes outside a big ball. Let us define maps

$$T_\rho : \mathrm{PV}(X) \to \mathrm{PV}_c(X), \quad \mu \to \rho\mu + (\bar\partial\rho)\frac{\partial_x}{f'} \wedge \alpha.$$

and

$$R_\rho : \mathrm{PV}(X) \to \mathrm{PV}(X), \quad \mu \to (1-\rho)\frac{\partial_x}{f'} \wedge \alpha.$$

Note that both T_ρ and R_ρ are well-defined since $\bar\partial\rho$ and $(1-\rho)$ vanishes around the critical point of f. It is straightforward to check that

$$\bar\partial_f R_\rho + R_\rho \bar\partial_f = 1 - T_\rho.$$

This in fact proves the quasi-isomorphic embedding $\mathrm{PV}_c(X) \hookrightarrow \mathrm{PV}(X)$ with an explicit homotopic inverse. Let g be a holomorphic function representing an element $[g] \in \mathrm{Jac}_0(f)$. Via the above explicit homotopy, we have

$$\mathrm{Tr}([g]) = \int_{\mathbb{C}} (T_\rho(g) \lrcorner dx) \wedge dx = \int_{\mathbb{C}} \bar\partial\rho \frac{g dx}{f'} = \oint \frac{g dx}{f'} = \mathrm{Res}_0(\frac{g dx}{f'}).$$

In higher dimensions, the proof is similar though via a more complicated homotopy. We refer to [28] for details.

We remark that $\mathrm{PV}_c(X)$ can be replaced by L^2 space, where Hodge theory and L^2-cohomology could be developed. The construction first appeared in [24], which is also fully developed recently in [16].

3.2. **Higher residues.** The advantage of revisiting residue via polyvector fields is that it allows us to generalize easily to higher residues discovered in [41]. The construction is parallel to Calabi-Yau models. Let us introduce a descendant variable z and consider the extended complex

$$(\mathrm{PV}(X)[[z]], Q_f = \bar\partial_f + z\partial).$$

The cohomology is given by

$$H^*(\mathrm{PV}(X)[[z]], Q_f) \simeq \mathbb{C}\{x_i\}[[z]]/(\partial_{x_i} f + z\frac{\partial}{\partial x_i})\mathbb{C}\{x_i\}[[z]].$$

Under Ω, this can be identified with the (formally completed) Brieskorn lattice

$$\mathcal{H}_f^{(0)} := \Omega^n_{X,\mathbf{0}}[[z]]/(df + zd)\Omega^{n-1}_{X,\mathbf{0}}[[z]].$$

Note that

$$\Omega_f = \mathcal{H}_f^{(0)}/z\mathcal{H}_f^{(0)}.$$

There is a natural $\mathbb{Q}$-grading on $\mathcal{H}_f^{(0)}$ defined by assigning the degrees

$$\deg(x_i) = q_i, \quad \deg(dx_i) = q_i, \quad \deg(z) = 1.$$

For a homogeneous element of the form $\varphi = z^k g(x_i) dx_1 \wedge \cdots \wedge dx_n$, we define

$$\deg(\varphi) = \deg(g) + k + \sum_i q_i.$$

In [41], K. Saito constructed the *higher residue pairing*

$$K_f : \mathcal{H}_f^{(0)} \otimes \mathcal{H}_f^{(0)} \to z^n \mathbb{C}[[z]]$$

which satisfies the following properties

(1) K_f is equivariant with respect to the $\mathbb{Q}$-grading, i.e.,

$$\deg(K_f(\alpha, \beta)) = \deg(\alpha) + \deg(\beta)$$

for homogeneous elements $\alpha, \beta \in \mathcal{H}_f^{(0)}$.

(2) $K_f(\alpha, \beta) = (-1)^n \overline{K_f(\beta, \alpha)}$, where the $-$ operator takes $z \to -z$.

(3) $K_f(v(z)\alpha, \beta) = K_f(\alpha, v(-z)\beta) = v(z) K_f(\alpha, \beta)$ for $v(z) \in \mathbb{C}[[z]]$.

(4) The leading z-order of K_f defines a pairing

$$\mathcal{H}_f^{(0)}/z\mathcal{H}_f^{(0)} \otimes \mathcal{H}_f^{(0)}/z\mathcal{H}_f^{(0)} \to \mathbb{C}, \quad \alpha \otimes \beta \mapsto \lim_{z \to 0} z^{-n} K_f(\alpha, \beta)$$

which coincides with the usual residue pairing

$$\eta_f : \Omega_f \otimes \Omega_f \to \mathbb{C}.$$

The last property implies that K_f defines a semi-infinite extension of the residue pairing, which explains the name "higher residue". An alternate way to understand the higher residue pairing is through the trace map in the spirit of our construction for residue. The natural embedding

$$(\mathrm{PV}_c(X)[[z]], Q_f) \hookrightarrow (\mathrm{PV}(X)[[z]], Q_f)$$

is again a quasi-isomorphism. Let us define a pairing

$$\tilde{K}_f : \mathrm{PV}_c(X)[[z]] \times \mathrm{PV}_c(X)[[z]] \to z^n \mathbb{C}[[z]], \quad \tilde{K}_f(g(z)\alpha, h(z)\beta) = z^n g(z) h(-z) \operatorname{Tr}(\alpha\beta).$$

It is easy to see that $\tilde{K}_f$ descends to $H^*(\mathrm{PV}_c(X)[[z]], Q_f)$ which is canonically isomorphic to $H^*(\mathrm{PV}(X)[[z]], Q_f)$. Under the identification of $H^*(\mathrm{PV}(X)[[z]], Q_f)$ with $\mathcal{H}_f^{(0)}$, we obtain a pairing

$$\tilde{K}_f : \mathcal{H}_f^{(0)} \otimes \mathcal{H}_f^{(0)} \to z^n \mathbb{C}[[z]].$$

It is proved in [28] that $\tilde{K}_f$ is precisely the higher residue pairing

$$\tilde{K}_f = K_f.$$

3.3. **Universal unfolding.** In the Calabi-Yau case, the (extended) moduli space of complex structures is controlled by the differential graded Lie algebra $(\mathrm{PV}(X), \bar\partial, \{,\})$. In the Landau-Ginzburg case, this is twisted to be $(\mathrm{PV}(X), \bar\partial_f, \{,\})$. The universal solutions of the associated Maurer-Cartan equation is greatly simplified, and can be represented as a deformation of $f(\mathbf{x})$ via the universal unfolding:

$$F : \mathbb{C}^n \times \mathbb{C}^\mu \to \mathbb{C}, \quad F = f(\mathbf{x}) + \sum_{\alpha=1}^{\mu} s^\alpha \phi_\alpha(\mathbf{x}),$$

where $\mu = \dim_{\mathbb{C}} \mathrm{Jac}(f)$, and $\{\phi_\alpha(\mathbf{x})\}$ is a basis of $\mathrm{Jac}_{\mathbf{0}}(f)$.

In our case f being weighted homogenous, we can further assume that ϕ_α are all weighted homogeneous with increasing degrees

$$0 = \deg(\phi_1) \leq \deg(\phi_2) \leq \cdots \leq \deg(\phi_\mu) = \hat{c}_f, \quad \text{where } \deg(x_i) = q_i.$$

The Brieskorn lattice and the higher residue pairing can be extended to the family case on the germ $\mathcal{M} = (\mathbb{C}^\mu, 0)$ associated to the unfolding F. We have

$$\mathcal{H}_F^{(0)} := \Omega^n_{X\times\mathcal{M}/\mathcal{M},\mathbf{0}}[[z]]/(dF + zd)\Omega^{n-1}_{X\times\mathcal{M}/\mathcal{M},\mathbf{0}}[[z]]$$

where $\Omega^*_{X\times\mathcal{M}/\mathcal{M}}$ is the sheaf of relative holomorphic differential forms. It can be viewed as a free sheaf of rank μ on $\mathcal{M}\times\hat\Delta$, where $\hat\Delta$ is the formal disk with parameter z. $\mathcal{H}_F^{(0)}$ is equipped with a flat Gauss-Manin connection on $\mathcal{M}\times\hat\Delta$, denoted by ∇^{GM}. The higher residue pairing extends to

$$K_F : \mathcal{H}_F^{(0)} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{H}_F^{(0)} \to z^n \mathcal{O}_\mathcal{M}[[z]]$$

satisfying the following properties [41]

(1) $K_F(s_1, s_2) = (-1)^n \overline{K_F(s_2, s_1)}$, where $-$ is the operator $z \to -z$.
(2) $K_F(g(z)s_1, s_2) = K_F(s_1, g(-z)s_2) = g(z)K_F(s_1, s_2)$ for any $g \in \mathcal{O}_\mathcal{M}[[z]]$.
(3) $\partial_V K_F(s_1, s_2) = K_F(\nabla^{GM}_V s_1, s_2) + K_F(s_1, \nabla^{GM}_V s_2)$ for any $V \in T_\mathcal{M}$.
(4) $z\partial_z K_F(s_1, s_2) = K_F(\nabla^{GM}_{z\partial_z} s_1, s_2) + K_F(s_1, \nabla^{GM}_{z\partial_z} s_2)$.
(5) The induced pairing

$$\mathcal{H}_F^{(0)}/z\mathcal{H}_F^{(0)} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{H}_F^{(0)}/z\mathcal{H}_F^{(0)} \to \mathcal{O}_\mathcal{M}$$

coincides with the classical residue pairing.

In a similar fashion, K_F can be constructed via fiberwise cut-off function [28].

3.4. **Good basis and Generating function.** Let us extend the higher residue pairing to

$$K_f : \mathcal{H}_f \otimes \mathcal{H}_f \to \mathbb{C}((z)).$$

This defines a symplectic pairing ω_f on $\mathcal{H}_f$ by

$$\omega_f(\alpha, \beta) := \mathrm{Res}_{z=0}\, z^{-n} K_f(\alpha, \beta) dz,$$

with $\mathcal{H}_f^{(0)}$ being an isotropic subspace. At this page, we are in completely analogue situation as in Calabi-Yau models. The next step is to use deformation theory to describe a lagrangian cone inside $\mathcal{H}_f$ from which we obtain the generating function as the genus zero invariants in Landau-Ginzburg B-model.

The first thing we need is a splitting $\mathcal{L}$

$$\mathcal{H}_f = \mathcal{H}_f^{(0)} \oplus \mathcal{L}$$

such that:

(1) $\mathcal{L}$ preserves the $\mathbb{Q}$-grading;
(2) $\mathcal{L}$ is an isotropic subspace;
(3) $z^{-1} : \mathcal{L} \to \mathcal{L}$.

Here for f being weighted homogeneous, (1) is equivalent to the conventional condition that $\nabla^{GM}_{z\partial_z}$ preserves $\mathcal{L}$. The choice of $\mathcal{L}$ is essentially equivalent to the notion of a good section [40]. Let

$$B = \mathcal{H}_f^{(0)} \cap z\mathcal{L}.$$

We can represent a basis of B by homogeneous polynomials $\{\phi_\alpha\}_{\alpha=1}^{\mu}$ such that

$$B = \mathrm{Span}_{\mathbb{C}}\{[\phi_\alpha \Omega]\}.$$

$\{\phi_\alpha\}$ can be viewed as a "good" representative of $\mathrm{Jac}_0(f)$, defining a lifting

$$\Omega_f \to \mathcal{H}_f^{(0)}.$$

The isotropic property of $\mathcal{H}_f^{(0)}$ and $\mathcal{L}$ implies that the higher residue paring

$$K_f(B, B) \in z^n \mathbb{C}.$$

In fact, $B \subset \mathcal{H}_f^{(0)}$ implies that

$$K_f(B, B) \subset z^n \mathbb{C}[[z]]$$

while $B \subset z\mathcal{L}$ implies that

$$K_f(B, B) \subset z^n \mathbb{C}[z^{-1}].$$

This says that for such choice of representatives $\{\phi_\alpha\}$, the higher residue pairings vanish beyond the leading ordinary residue pairing. Such vanishing property is the key property of so-called "*good basis*" [40].

Given a good basis B, we can identify

$$\mathcal{H}_f^{(0)} = B[[z]], \quad \mathcal{L} = z^{-1}B[z^{-1}], \quad \mathcal{H}_f = B((z)).$$

Now we follow the Calabi-Yau situation to describe the lagrangian cone. A universal solution of the Maurer-Cantan equation

$$Q_f\mu + \frac{1}{2}\{\mu, \mu\} = 0, \quad \mu \in \mathrm{PV}(X)[[z]],$$

can be described by

$$\mu(\mathbf{s}) = \sum_{k\geq 0}\sum_\alpha s_k^\alpha \phi_\alpha z^k,$$

parametrized by $\mathbf{s} = \{s_k^\alpha\}$. $\{s_0^\alpha\}$ parametrizes the universal unfolding, while s_k^α for $k > 0$ corresponds to descendant deformations.

Let $\phi^\alpha \in B$ be a dual basis of B such that

$$K_f([\phi_\alpha\Omega], [\phi^\beta\Omega]) = z^n\delta_\alpha^\beta.$$

The lagrangian cone is then specified by

$$z - ze^{\mu/z}$$

from which we can always expand

$$z[e^{\mu(\mathbf{s})/z}\Omega] = z[\Omega] + \sum_{k\geq 0}\sum_\alpha \tau_k^\alpha(\mathbf{s})[\phi_\alpha\Omega](-1)^k z^k + \sum_{k\geq 0}\sum_\alpha p_{k,\alpha}(\mathbf{s})[\phi^\alpha\Omega]z^{-k-1} \in \mathcal{H}_f[[\mathbf{s}]]$$

where [] refers to the class in $\mathcal{H}_f$. $\{\tau_k^\alpha\}$ can be viewed as the linear coordinates on $B[[z]]$. The transformation

$$\mathbf{s} \to \{\tau_k^\alpha(\mathbf{s})\}$$

defines a coordinate transformation. In terms of $\{\tau_k^\alpha\}$, we find

$$p_{k,\alpha} = \frac{\partial \mathbf{F}_0^{\mathcal{L}}(\tau)}{\partial \tau_k^\alpha},$$

where $\mathbf{F}_0^{\mathcal{L}}$ is the generating function of genus zero invariants in Landau-Ginzburg B-model with respect to the splitting $\mathcal{L}$, or the good basis B.

3.5. **Primitive form and perturbation theory.** A section $\zeta \in \mathcal{H}_F^{(0)}$ is called a *primitive form* if it satisfies the following conditions:

(1) (Primitivity) The section ζ induces an $\mathcal{O}_{\mathcal{M}}$-module isomorphism
$$z\nabla^{GM}\zeta : T_{\mathcal{M}} \to \mathcal{H}_F^{(0)}/z\mathcal{H}_F^{(0)}; \quad V \mapsto z\nabla_V^{GM}\zeta.$$
(2) (Orthogonality) For any local sections V_1, V_2 of $T_{\mathcal{M}}$,
$$K_F\big(\nabla_{V_1}^{GM}\zeta, \nabla_{V_2}^{GM}\zeta\big) \in z^{n-2}\mathcal{O}_{\mathcal{M}}.$$
(3) (Holonomicity) For any local sections V_1, V_2, V_3 of $T_{\mathcal{M}}$,
$$K_F\big(\nabla_{V_1}^{GM}\nabla_{V_2}^{GM}\zeta, \nabla_{V_3}^{GM}\zeta\big) \in z^{n-3}\mathcal{O}_{\mathcal{M}} \oplus z^{n-2}\mathcal{O}_{\mathcal{M}};$$
$$K_F\big(\nabla_{z\partial_z}^{GM}\nabla_{V_1}^{GM}\zeta, \nabla_{V_2}^{GM}\zeta\big) \in z^{n-3}\mathcal{O}_{\mathcal{M}} \oplus z^{n-2}\mathcal{O}_{\mathcal{M}}.$$
(4) (Homogeneity) There is a constant $r \in \mathbb{C}$ such that
$$\left(\nabla_{z\partial_z}^{\Omega} + \nabla_E^{\Omega}\right)\zeta = r\zeta.$$
where E is the Euler vector field. In our case of weighted homogeneous singularity, we have $r = \sum_i q^i$.

It is proved in [40] that the space of primitive forms of f up to a constant rescaling is isomorphic to the space of good basis. The generalization to arbitrary isolated singularities is established by M. Saito [43, 44].

Closed formula for primitive forms are known only for ADE singularities ($\hat{c}_f < 1$) and simple elliptic singularities ($\hat{c}_f = 1$) [40]. In the following we will describe a perturbative theory of primitive forms [28] from which we can compute the primitive form recursively order by order.

Givental's J-function is essentially the analogue of primitive form. Starting from the formula for the lagrangian cone
$$z[e^{\mu(\mathbf{s})/z}\Omega] = z[\Omega] + \sum_{k\geq 0}\sum_{\alpha} \tau_k^\alpha(\mathbf{s})[\phi_\alpha\Omega](-1)^k z^k + \sum_{k\geq 0}\sum_{\alpha} p_{k,\alpha}(\mathbf{s})[\phi^\alpha\Omega]z^{-k-1} \in \mathcal{H}_f[[\mathbf{s}]].$$
If we set
$$\tau_k^\alpha = 0, \quad k > 0,$$
then we find the expansion
$$[e^{\mu(\mathbf{s}(\tau_0^\alpha, \tau_1^\alpha=0,\cdots))/z}\Omega] = [\Omega] + \sum_{\alpha} \tau_0^\alpha[\phi_\alpha\Omega]z^{-1} + O(z^{-2}).$$
It is shown in [28] that the left hand side
$$e^{\mu(\mathbf{s}(\tau_0^\alpha, \tau_1^\alpha=0,\cdots))/z}\Omega$$

gives exactly the expression of primitive form with respect to the good basis $\{\phi_\alpha\}$ under the trivialization map via Gauss-Manin connection

$$\mathcal{H}_F \stackrel{e^{(F-f)/z}}{\to} \mathcal{H}_f \otimes_{\mathbb{C}} \mathbb{C}[[\tau_0^\alpha]].$$

This is translated into the following algorithm. Let $s = \{s^\alpha\}$ parametrizes the universal unfolding with respect to the good basis ϕ_α

$$F = f + \sum_\alpha s^\alpha \phi_\alpha.$$

Let us represent the primitive ζ as a power series

$$\zeta = \sum_{k\geq 0} \zeta_{(k)} = \sum_{k\geq 0}\sum_\alpha \zeta^\alpha_{(k)}\phi_\alpha \Omega, \quad \zeta^\alpha_{(k)} \in \mathbb{C}[[z]] \otimes_{\mathbb{C}} \mathbb{C}[s]_k$$

where $\mathbb{C}[s]_k$ is the space of homogenous polynomials in $\{s^\alpha\}$ of degree k. Then ζ is characterized by the equation [28]

$$(\dagger) \qquad [e^{(F-f)/z}\zeta] \in [\Omega] + z^{-1}B[z^{-1}][[s]].$$

This equation can be solved recursively in terms of the order k as follows.

Since $e^{(F-f)/z} \equiv 1 \mod (s)$, the leading order of (†) is

$$\zeta_{(0)} \in [\Omega] + z^{-1}B[z^{-1}]$$

which is uniquely solved by $\zeta_{(0)} = \Omega$. Suppose we have solved (†) up to order N, i.,e, $\zeta_{(\leq N)} := \sum_{k=0}^N \zeta_{(k)}$ such that

$$[e^{(F-f)/t}\zeta_{(\leq N)}] \in [\Omega] + z^{-1}B[z^{-1}][[s]] \mod (s^{N+1}).$$

Let $R_{N+1} \in B((z)) \otimes_{\mathbb{C}} \mathbb{C}[s]_{(N+1)}$ be the $(N+1)$-th order component of $e^{(F-f)/t}\zeta_{(\leq N)}$. Let

$$R_{N+1} = R^+_{N+1} + R^-_{N+1}$$

where $R^+_{N+1} \in B[[z]] \otimes_{\mathbb{C}} \mathbb{C}[s]_{(N+1)}$, $R^-_{N+1} \in z^{-1}B[z^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[s]_{(N+1)}$. Let $\tilde{R}^+_{N+1} \in B_F[[z]] \otimes_{\mathbb{C}} \mathbb{C}[s]_{(N+1)}$ correspond to R^+_{N+1} under the manifest identification between B and B_F. Then

$$\zeta_{(\leq N+1)} := \zeta_{(\leq N)} - \tilde{R}^+_{N+1}$$

gives the unique solution of $(\dagger)$ up to order $N+1$. This algorithm allows us to solve ζ perturbatively to arbitrary order. Moreover, we can find the expansion

$$[e^{(F-f)/z}\zeta] = [\Omega] + \sum_\alpha \tau^\alpha(s)[\phi_\alpha\Omega]z^{-1} + \sum_\alpha p_\alpha(s)[\phi^\alpha\Omega]z^{-2} + O(z^{-3}).$$

Then the coordinate transformation

$$s^\alpha \to \tau^\alpha(s)$$

gives the flat coordinate τ^α of the underlying Frobenius manifold structure. In terms of τ^α, we have

$$p_\alpha(s(\tau)) = \frac{\partial \mathbf{F}_0(\tau)}{\partial \tau^\alpha}$$

where $\mathbf{F}_0(\tau)$ is the potential function satisfying the WDVV equation.

3.6. **Examples of primitive forms.** Let us illustrate how the perturbative formula of primitive form works in some examples.

3.6.1. *ADE singularity.* Let f be a weighted homogenous polynomial of ADE type. This is equivalent to $\hat{c}_f < 1$. There exists a unique choice of good basis, and any representative of homogenous basis $\{\phi_\alpha\}$ of $\mathrm{Jac}_0(f)$ are equivalent. Their degrees are bounded by

$$\deg(\phi_\alpha) \leq \hat{c}_f < 1.$$

The fact that an arbitrary representative $\{\phi_\alpha\}$ being a good basis is a very special property of ADE singularities and follows from a degree argument as follows [40]. We need to show that

$$K_f(\phi_\alpha \Omega, \phi_\beta \Omega) \in \mathbb{C} z^n, \forall \alpha, \beta.$$

In fact, since K_f preserves the grading, we have

$$\deg(K_f(\phi_\alpha \Omega, \phi_\beta \Omega)) = \deg(\phi_\alpha) + \deg(\phi_\alpha) + 2\deg(\Omega) = \deg(\phi_\alpha) + \deg(\phi_\alpha) + n - \hat{c}_f.$$

It follows that

$$n \leq \deg(K_f(\phi_\alpha \Omega, \phi_\beta \Omega)) \leq n + \hat{c}_f.$$

Since $0 < \hat{c}_f < 1$ and $\deg(K_f(\phi_\alpha \Omega, \phi_\beta \Omega))$ is an integer, it follows that $\deg(K_f(\phi_\alpha \Omega, \phi_\beta \Omega)) = n$, i.e.,

$$K_f(\phi_\alpha \Omega, \phi_\beta \Omega) \in \mathbb{C} z^n.$$

Let F be the universal unfolding

$$F = f + \sum_\alpha s^\alpha \phi_\alpha.$$

We extend the weighted homogeneous degree to s^α, z by

$$\deg(s^\alpha) = 1 - \deg(\phi_\alpha), \quad \deg(z) = 1$$

such that F is homogenous of degree 1. Equation (†) respects the degree. By the fact that $\deg(s^\alpha) > 0$ and simple degree counting, we find that

$$[e^{(F-f)/z}\Omega] \in [\Omega] + z^{-1} B[z^{-1}][[s^\alpha]],$$

i.e. $\zeta = \Omega$ is the primitive form. It is again a special property of ADE singularity that the primitive form doesn't depend on the deformation parameter.

3.6.2. *Simple elliptic singularity.* Simple elliptic singularities are characterized by $\hat{c}_f = 1$. We consider an example of type $E_6^{(1,1)}$ with

$$f = \frac{1}{3}(x_1^3 + x_2^3 + x_3^3).$$

A monomial basis is given by

$$\{\phi_1, \ldots, \phi_8\} = \{1, x_1, x_2, x_3, x_1x_2, x_2x_3, x_3x_1, x_1x_2x_3\}.$$

Lemma 3.1. *The above basis $\{\phi_i\}$ is a good basis.*

Proof. Similar to the ADE singularities, we have the degree estimate

$$3 = n \leq \deg(K_f(\phi_i\Omega, \phi_j\Omega) \leq n + \hat{c}_f = 4.$$

The only possibility that the above degree reaches 4 is when $\phi_i = \phi_j = \phi_8$. Therefore we only need to show that

$$K_f(\phi_8\Omega, \phi_8\Omega) = 0.$$

This follows from the symmetry of the higher residue pairing

$$K_f(\phi_8\Omega, \phi_8\Omega) = (-1)^3\overline{K_f(\phi_8\Omega, \phi_8\Omega)} = -K_f(\phi_8\Omega, \phi_8\Omega).$$

□

We consider the universal unfolding

$$F = f + \sigma\phi_8 + \sum_{\alpha=1}^{7} u^\alpha \phi_\alpha$$

where $(u^1, \cdots, u^7, \sigma)$ are the deformation parameters. There exists a one-parameter choice of good basis for simple elliptic singularities. In this example, they are

$$B(c) = \mathrm{Span}_{\mathbb{C}}\{\phi_1, \cdots, \phi_7, \phi_8 + cz\}, \quad c \in \mathbb{C}.$$

$B(c)$ being a good basis follows immediately from the above lemma.

We extend the weighted homogenous degree as before. Then

$$\deg(u^\alpha) > 0, \quad \deg(\sigma) = 0.$$

We can use similar degree argument as the ADE case, with some extra care on the degree zero parameter σ. The first observation is that as elements in $\mathcal{H}_f$,

$$[e^{\sigma\phi_8/z}\Omega] = g(\sigma)[\Omega] + z^{-1}h(\sigma)[\phi_8\Omega],$$

where $g(\sigma)$ and $h(\sigma)$ are respectively given by

$$g(\sigma) = 1 + \sum_{r=1}^{\infty} \frac{(-1)^r \sigma^{3r} \prod_{j=1}^{r}(3j-2)^3}{(3r)!}, \quad h(\sigma) = \sigma + \sum_{r=1}^{\infty} \frac{(-1)^r \sigma^{3r+1} \prod_{j=1}^{r}(3j-1)^3}{(3r+1)!}.$$

This can be computed explicitly by expanding $e^{\sigma\phi_8/z}$ and using the equivalence relation in $\mathcal{H}_f$. $g(\sigma), h(\sigma)$ are the fundamental solutions to

$$\left((1+\sigma^3)\partial_\sigma^2 + 3\sigma^2\partial_\sigma + \sigma\right)\nu(\sigma) = 0,$$

which is the the Picard-Fuchs equation for the period integrals on the elliptic curve inside $\mathbb{P}^2$ defined by the cubic equation $f + \sigma\phi_8 = 0$.

By a similar degree argument, we have

$$\begin{aligned}
[e^{(F-f)/z}\Omega] &\in [e^{\sigma\phi_8/z}\Omega] + z^{-1}B(c)[z^{-1}] \\
&= (g(\sigma) - ch(\sigma))[\Omega] + z^{-1}h(\sigma)[(\phi_8 + cz)\Omega] + z^{-1}B(c)[z^{-1}] \\
&= (g(\sigma) - ch(\sigma))[\Omega] + z^{-1}B(c)[z^{-1}].
\end{aligned}$$

It follows that

$$\left[e^{(F-f)/z}\frac{\Omega}{g(\sigma) - ch(\sigma)}\right] \in \Omega + z^{-1}B(c)[z^{-1}].$$

In particular, the primitive form associated to the good basis $B(c)$ is given by [40]

$$\zeta(c) = \frac{\Omega}{g(\sigma) - ch(\sigma)}.$$

In other words, the normalization is given by a period on the elliptic curve. Different choices of primitive forms correspond to the choices of the period.

3.6.3. *Exceptional unimodular singularity.* For singularities with $\hat{c}_f > 1$, there exists parameters of the deformation which have negative degree. The degree argument that we have used in ADE and simple elliptic singularity doesn't work well here since there would exist nontrivial mixing between positive degree and negative degree parameters. This is the essential difficult to find the expression of primitive forms in such situation. Nevertheless, we can use the perturbative algorithm to find order expansion of the primitive form.

The first nontrivial examples beyond simple elliptic singularities are given by Arnold's exceptional unimodular singularities. There are in total 14 types, and let us consider an example of type E_{12}

$$f = x^3 + y^7.$$

The central charge is given by $\hat{c}_f = \frac{22}{21}$. There is a unique good basis given by

$$\{\phi_1, \cdots, \phi_{12}\} := \{1, y, y^2, x, y^3, xy, y^4, xy^2, y^5, xy^3, xy^4, xy^5\}.$$

This follows from a similar degree argument as ADE singularities and we leave the details to the readers. See also Section 4.2.

We represent the universal unfolding by

$$F = x^3 + y^7 + \sum_{i=1}^{12} u_i \phi_i.$$

By direct calculations (with a computer), the primitive form ζ up to order 10 is given by [28]

$$\begin{aligned}
\zeta/\Omega = 1 &+ \frac{4}{3 \cdot 7^2} u_{11} u_{12}^2 - \frac{64}{3 \cdot 7^4} u_{11}^2 u_{12}^4 - \frac{76}{3^2 \cdot 7^4} u_{10} u_{12}^5 + \frac{937}{3^3 \cdot 7^5} u_9 u_{12}^6 + \frac{218072}{3^4 \cdot 5 \cdot 7^6} u_{11}^3 u_{12}^6 \\
&+ \frac{1272169}{3^4 \cdot 5 \cdot 7^7} u_{10} u_{11} u_{12}^7 + \frac{28751}{3^4 \cdot 7^7} u_8 u_{12}^8 - \frac{1212158}{3^4 \cdot 7^8} u_9 u_{11} u_{12}^8 - \frac{38380}{3^3 \cdot 7^8} u_7 u_{12}^9 \\
&+ \Big(\frac{1}{7^2} u_{12}^3 - \frac{101}{5 \cdot 7^4} u_{11} u_{12}^5 + \frac{1588303}{3^4 \cdot 5 \cdot 7^7} u_{11}^2 u_{12}^7 + \frac{378083}{3^4 \cdot 5 \cdot 7^7} u_{10} u_{12}^8 - \frac{108144}{3 \cdot 7^8} u_9 u_{12}^9\Big) x \\
&+ \Big(\frac{1447}{3^3 \cdot 7^6} u_{12}^7 - \frac{71290}{3^3 \cdot 7^8} u_{11} u_{12}^9\Big) y - \frac{45434}{3^4 \cdot 7^8} u_{12}^{10} xy \\
&- \Big(\frac{53}{3^2 \cdot 7^4} u_{12}^6 - \frac{46244}{3^3 \cdot 7^7} u_{11} u_{12}^8\Big) x^2 + \frac{22054}{3^4 \cdot 7^7} u_{12}^9 x^3 + O(u^{10}).
\end{aligned}$$

4. Mirror symmetry between Landau-Ginzburg models

4.1. **Mirror singularities.** The main application of the perturbative primitive form is to mirror symmetry. The Calabi-Yau B-model around the large complex limit is mirror to Gromov-Witten theory. For Landau-Ginzburg B-models that we have discussed, they are expected to be mirror to FJRW theory [17]. The computation tools that we have developed will enable us to approach this.

The Landau-Ginzburg mirror pairs originate from an old physical construction of Berglund-Hübsch [6] that was completed by Krawitz [25]. Let

$$W : \mathbb{C}^N \to \mathbb{C}$$

be a weighted homogeneous polynomial with an isolated critical point at the origin. We define its *maximal group of diagonal symmetries* to be

$$G_W = \left\{ (\lambda_1, \ldots, \lambda_N) \in (\mathbb{C}^*)^N \,\middle|\, W(\lambda_1 x_1, \ldots, \lambda_N x_N) = W(x_1, \ldots, x_N) \right\}.$$

In the BHK mirror construction, the polynomial W is required to be *invertible* [9, 25], i.e., the number of variables must equal the number of monomials of

W and it contains no monomial of the form $x_i x_j$ for $i \neq j$. By rescaling the variables, we can always write W as

$$W = \sum_{i=1}^{N} \prod_{j=1}^{N} x_j^{a_{ij}}.$$

We denote its *exponent matrix* by $E_W = \left(a_{ij}\right)_{N\times N}$. The mirror polynomial of W is

$$W^T = \sum_{i=1}^{N} \prod_{j=1}^{N} x_j^{a_{ji}},$$

i.e., the exponent matrix E_{W^T} of the mirror polynomial is the transpose of E_W.

All invertible polynomials have been classified by Kreuzer and Skarke.

Theorem 4.1 ([27], Theorem 1)**.** *A polynomial is invertible if and only if it is a disjoint sum of the three following atomic types, where $a \geq 2$ and $a_i \geq 2$:*

- Fermat: x^a.
- Chain: $x_1^{a_1} + x_1 x_2^{a_2} + \cdots + x_{N-1} x_N^{a_N}$.
- Loop: $x_1^{a_1} x_N + x_1 x_2^{a_2} + \cdots + x_{N-1} x_N^{a_N}$.

The inverse matrix E_W^{-1} plays an important role in the mirror map constructed by Krawitz in [25]. Let us write

$$E_W^{-1} = \begin{pmatrix} q_{11} & \cdots & q_{1N} \\ \vdots & \vdots & \vdots \\ q_{N1} & \cdots & q_{NN} \end{pmatrix},$$

and define

$$\rho_j^W := \left(\exp(2\pi\sqrt{-1}q_{1j}), \dots, \exp(2\pi\sqrt{-1}q_{Nj})\right),$$
$$\rho_j^{W^T} := \left(\exp(2\pi\sqrt{-1}q_{j1}), \dots, \exp(2\pi\sqrt{-1}q_{jN})\right).$$

According to [25], the group G_W is generated by $\{\rho_j^W\}_{j=1}^N$ and G_{W^T} is generated by $\{\rho_j^{W^T}\}_{j=1}^N$. Recall q_i is the weight of x_i in W. Let q_i^T be the weight of x_i in W^T. We remark that

$$q_i = \sum_{j=1}^{N} q_{ij} \quad \text{and} \quad q_i^T = \sum_{j=1}^{N} q_{ji}.$$

4.2. **Good basis of invertible polynomials.** To obtain the Frobenius manifold structure on the deformation space of the singularity, we need to identify a good basis. The abstract existence of such a good basis is proved in [40] for the quasi-homogenous cases and in [43] for general isolated singularity. Moreover, such a good basis is generally not unique. However, mirror symmetry favors for a particular one as the mirror of FJRW theory, and we need explicit information about the good basis to obtain the enumerative data. This involves the exact computation of higher residue pairing which is usually hard to perform.

In case of ADE and simple elliptic singularities, the justification of good basis follows from a simple degree counting argument (see Section 3.6). The effectivity of this method fails when the central charge of the polynomial gets large. The way out is that we have to incorporate the full symmetry group G_W to improve the degree counting argument. This is developed in [23] to find explicit good basis for all invertible polynomials. We review this result here.

In the rest of this section, we will use f instead of W to adopt the conventional notation in singularity theory.

Firstly, if $f(\mathbf{x},\mathbf{y}) = f_1(\mathbf{x}) + f_2(\mathbf{y})$ is the disjoint sum of two polynomials f_1, f_2, and $\{\eta_i(\mathbf{x})\}_{i\in I}$ and $\{\varphi_\alpha(\mathbf{y})\}_{\alpha\in A}$ are good basis for f_1 and f_2 respectively, then it is easy to show that $\{\eta_i(\mathbf{x})\varphi_\alpha(\mathbf{y})\}_{(i,\alpha)\in I\times A}$ is a good basis of W. This reduces the search of good basis for invertible polynomials to atomic types by Theorem 4.1. They are found in [23] as follows.

Theorem 4.2 ([23]). *Let f be an invertible polynomial of atomic types. Then the following choice $\{\phi_\alpha\}$ is a good basis of f.*

- *Let $f = x^a$ be a Fermat, then $\{\phi_\alpha\} = \{x^r \mid 0 \le r \le a-2\}$.*
- *Let $f = x_1^{a_1} + x_1x_2^{a_2} + \cdots + x_{N-1}x_N^{a_N}$ be a chain, then*

$$\{\phi_\alpha\} = \left\{\prod_{i=1}^N x_i^{r_i}\right\}_{\mathbf{r}}$$

  where $\mathbf{r} = (r_1, \cdots, r_N)$ with $r_i \le a_i - 1$ for all i and $\mathbf{r}$ is not of the form $(*, \cdots, *, k, a_{N-2l}-1, \cdots, 0, a_{N-2}-1, 0, a_N - 1)$ with $k \ge 1$.
- *Let $f = x_1^{a_1}x_N + x_1x_2^{a_2} + \cdots + x_{N-1}x_N^{a_N}$ be a loop, then*

$$\{\phi_\alpha\} = \left\{\prod_{i=1}^N x_i^{r_i} \;\middle|\; 0 \le r_i < a_i\right\}.$$

We will call the above monomials the *standard basis*.

The Fermat type $f = x^a$ is of type A_{a-1}-singularity, and we already know that its standard basis is a good basis. We will focus on the remaining two cases.

Recall G_f is the diagonal symmetry group of f. $H_f^{(0)}$ inherits the structure of G_f-representation. Explicitly, let $\{\rho_j^f = (e^{2\pi\sqrt{-1}q_{1j}}, \cdots, e^{2\pi\sqrt{-1}q_{Nj}})\}_{j=1}^N$ be the generator of G_f, where $E_f^{-1} = (q_{ij})_{N\times N}$ is the inverse of the exponent matrix E_f. Then the G_f-action on $H_f^{(0)}$ is generated by

$$\rho_j^f : x_i \to e^{2\pi\sqrt{-1}q_{ij}} x_i, \quad dx_i \to e^{2\pi\sqrt{-1}q_{ij}} dx_i, \quad z \to z.$$

Since G_f preserves f, it also preserves the higher residue pairing K_f. Therefore we have a G_f-invariant pairing

$$K_f : H_f^{(0)} \otimes H_f^{(0)} \to z^n \mathbb{C}[[z]].$$

Let $x_1^{r_1} \cdots x_N^{r_N}$ and $x_1^{r'_1} \cdots x_N^{r'_N}$ be monomials in the standard basis for either the chain or loop type. We need to show that

$$K_f(x_1^{r_1} \cdots x_N^{r_N} d^N x, x_1^{r'_1} \cdots x_N^{r'_N} d^N x) \in z^N \mathbb{C}.$$

Let

$$(m_1, \cdots, m_N) = (r_1 + r'_1, \cdots, r_N + r'_N).$$

The ρ_j^f-invariance of K_f implies the *integral conditions*

$$\sum_{i=1}^N (m_i + 2) q_{ij} = k_j \in \mathbb{Z}, \quad \text{for all } j, \quad \text{if} \quad K_f(x_1^{r_1} \cdots x_N^{r_N} d^N x, x_1^{r'_1} \cdots x_N^{r'_N} d^N x) \neq 0.$$

(the extra 2 comes from two copies of $d^N x$). This is equivalent to

$$(\star) \qquad (k_1, k_2, \cdots, k_N)\, E_f = (m_1 + 2, m_2 + 2, \cdots, m_N + 2).$$

Remark 4.1. The integral conditions is mirror to the Integer Degree Axiom for the degree of orbifold Line bundles in FJRW theory [23].

4.2.1. *chain type.* Let $f = x_1^{a_1} + x_1 x_2^{a_2} + \cdots + x_{N-1} x_N^{a_N}$. The exponent matrix has the form

$$E_f = \begin{pmatrix} a_1 & & & & \\ 1 & a_2 & & & \\ & \ddots & \ddots & & \\ & & & a_{N-1} & \\ & & & 1 & a_N \end{pmatrix}.$$

Equation ($\star$) in this case becomes

$$m_1 = k_1a_1 + k_2 - 2, \quad m_2 = k_2a_2 + k_3 - 2, \quad \ldots, \quad m_{N-1} = k_{N-1}a_{N-1} + k_N - 2, \quad m_N = k_Na_N - 2$$

where $0 \le m_i \le 2a_i - 2$ and the k_i are integers. Tracing all possible values of k_i starting from k_N, the only possibilities are

(1) $(k_1, \cdots, k_N) = (1, 1, \cdots, 1)$.
(2) $(k_1, \cdots, k_N) = (1, \cdots, 1, 0, 2, \cdots, 0, 2)$.
(3) $(k_1, \cdots, k_N) = (1, \cdots, 1, 2, 0, 2, \cdots, 0, 2)$.

In case (3), we have

$$(m_1, \cdots, m_N) = (a_1 - 1, \cdots, a_{N-2l-2} - 1, a_{N-2l-1}, 2a_{N-2l} - 2, \cdots, 0, 2a_{N-2} - 2, 0, 2a_N - 2).$$

This can not appear if both $x_1^{r_1} \cdots x_N^{r_N}$ and $x_1^{r_1'} \cdots x_N^{r_N'}$ are in the standard basis. For cases (1) and (2), we check directly that

$$\deg(x_1^{m_1} \cdots x_N^{m_N}) = \deg(x_1^{r_1} \cdots x_N^{r_N}) + \deg(x_1^{r_1'} \cdots x_N^{r_N'}) = \hat{c}_f.$$

Since K_f preserves the $\mathbf{Q}$-grading, we have

$$\deg K_f(x_1^{r_1} \cdots x_N^{r_N} d^N x, x_1^{r_1'} \cdots x_N^{r_N'} d^N x) = \deg(x_1^{r_1} \cdots x_N^{r_N}) + \deg(x_1^{r_1'} \cdots x_N^{r_N'}) + 2\sum_i q_i = N.$$

It follows that $K_f([x_1^{r_1} \cdots x_N^{r_N}]_f, [x_1^{r_1'} \cdots x_N^{r_N'}]_f)$ lies in $z^N\mathbb{C}$.

4.2.2. *loop type.* Let $f = x_1^{a_1}x_N + x_1x_2^{a_2} + \cdots + x_{N-1}x_N^{a_N}$. The exponent matrix is

$$E_f = \begin{pmatrix} a_1 & & & & 1 \\ 1 & a_2 & & & \\ & \ddots & \ddots & & \\ & & & a_{N-1} & \\ & & & 1 & a_N \end{pmatrix}.$$

With the convention $k_1 \equiv k_{N+1}$, Equation ($\star$) above implies

$$m_i + 2 = k_ia_i + k_{i+1}, \quad i = 1, \cdots, N.$$

Let $h_i = k_i - 1$ for each i. The above equation becomes

$$m_i + 2 = (h_i + 1)a_i + h_{i+1} + 1.$$

Since $0 \le m_i \le 2a_i - 2$, we get

$$(a) \qquad 1 - a_i \le h_ia_i + h_{i+1} \le a_i - 1, \quad i = 1, \cdots, N.$$

If there is some $h_{i+1} = 0$, then the above equation implies $h_i = 0$, and recursively,

$$(h_1, \cdots, h_N) = (0, 0, \cdots, 0).$$

Otherwise, we can assume none of the h_i is zero. There are two situations. Either there is one h_i with $|h_i| = 1$ or all $|h_i| \geq 2$. For the first case, we assume some $h_{i+1} = \pm 1$. Since $h_i \neq 0$ by assumption, the inequality (a) implies $h_i = \mp 1$. We can repeat this process and get the following solution when N is an even number:

$$(h_1, \cdots, h_N) = (\pm 1, \mp 1, \cdots, \pm 1, \mp 1).$$

Finally we prove it is impossible to have all $|h_i| \geq 2$. Equation (a) implies

$$-1 + \frac{1 - h_{i+1}}{a_i} \leq h_i \leq 1 - \frac{1 + h_{i+1}}{a_i}. \tag{b}$$

If all $|h_{i+1}| \geq 2$, this implies

$$|h_i| < |h_{i+1}|. \tag{c}$$

In fact, if $h_{i+1} \geq 2$, then the RHS of inequality (b) implies $h_i < 1$. By assumption, we know $h_i \leq -2$. However, since

$$-h_{i+1} < -1 + \frac{1 - h_{i+1}}{a_i},$$

inequality (c) follows from the LHS of (b). A similar argument works for $h_{i+1} \leq -2$. We repeat this process and we find

$$|h_i| = |h_{i+N}| < \cdots < |h_{i+1}| < |h_i|,$$

which is absurd. Thus the only possibilities for the k_i's are

(1) $(k_1, \cdots, k_N) = (1, 1, \cdots, 1)$, and
(2) $(k_1, \cdots, k_N) = (1 \pm 1, 1 \mp 1, \cdots, 1 \pm 1, 1 \mp 1)$, if N is even.

In each case, again we have

$$\deg(x_1^{m_1} \cdots x_N^{m_N}) = \deg(x_1^{r_1} \cdots x_N^{r_N}) + \deg(x_1^{r_1'} \cdots x_N^{r_N'}) = \hat{c}_f.$$

By the same degree reason as in the chain case, we know $K_f(x_1^{r_1} \cdots x_N^{r_N} d^N x, x_1^{r_1'} \cdots x_N^{r_N'} d^N x)$ lies in $z^N \mathbb{C}$.

4.3. **A Landau-Ginzburg Mirror theorem.** The standard basis for an invertible polynomial determines a primitive form, hence a Frobenius manifold structure on the germ of its universal unfolding. In particular, we obtain the genus zero potential function $\mathbf{F}_0$ of the associated Landau-Ginzburg B-model. The exact formula of $\mathbf{F}_0$ is not known in general. The difficulty goes back to the lack of knowledge on its primitive form, which is constructed from the good basis via an abstract Riemann-Hilbert-Birkhoff problem [40]. Nevertheless, the perturbative algorithm in Section 3.5 allows us to compute $\mathbf{F}_0$ order by order recursively.

The perturbative theory becomes extremely useful when integrability is taken into account. In fact, WDVV equation reduces the computation of $\mathbf{F}_0$ to only a finite order computation. The most general form of such reconstruction type theorem is established in [23].

Theorem 4.3 ([23]). *Let W be an invertible polynomial with no chain variables of weight $1/2$. Then for both FJRW theory of (W, G_W) and primitive form of W^T with respect to the standard good basis, the genus 0 invariants are completely determined by 2-point, 3-point and 4-point functions accompanied with WDVV equation, String equation, Dimension Axiom and Integer Degree Axiom.*

Here the selection rules of Dimension Axiom and Integer Degree Axiom are natural geometric properties (see [23] for details). There is a minor situation for chain types with weight $1/2$ not covered, i.e., $W = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{N-1}^{a_{N-1}}x_N + x_N^{a_N}$ with $a_N = 2$. This is a technical difficulty of missing information about certain FJRW 3-point functions due to the non-algebraic nature of FJRW theory. This theorem says that $\mathbf{F}_0$ is completely determined by symmetries and its Taylor series up to order 4. In particular, the enumerative data of genus 0 Landau-Ginzburg models can be fully computed.

A straight-forward application of our reconstruction theorem is to prove a general form of the Landau-Ginzburg mirror symmetry conjecture between invertible polynomials. At genus zero, the Landau-Ginzburg mirror symmetry conjecture says that the generating function of orbifold FJRW theory of the pair (W, G_W) is equivalent to the generating function of the Landau-Ginzburg B-model of W^T associated to a primitive form. This is established for ADE singularities [17] and simple elliptic singularities [26, 37]. The method of perturbative primitive form is first applied to exceptional unimodular singularities [29]. Those are the first examples of Landau-Ginzburg mirror symmetry for singularites of central charge greater than 1. In general, we have

Theorem 4.4 (Landau-Ginzburg Mirror Symmetry Theorem [23])**.** *Let W be an invertible polynomial with no chain variables of weight $1/2$. Then the FJRW theory of (W, G_W) is equivalent to Saito-Givental theory of W^T at all genera.*

The proof is just reduced to a finite check of 4-point functions, thanks to Theorem 4.3. We refer to [23] for details.

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