Topics in QFT

Si Li @ Tsinghua 2012

Rough Plan:

① DGLA, Deformation theory

② classical field theory, BV geometry

③ Perturbative quantization, renormalization

Lecture 1: Deformation functor

We work in the category of algebraic geometry over $\mathbb{C}$.

X: algebraic scheme, glued by affine schemes

Locally, $X = \operatorname{Spec} A$ where

A = finitely generated $\mathbb{C}$-algebra

If $A = \mathbb{C}[x_1, \cdots, x_n] / (f_1, \cdots . f_m)$

then X = Zero locus of $f_1, \cdots, f_m$

If X, Y are two schemes, let

$\mathrm{Mor}(X, Y)$ = Space of morphisms from X to Y

Let <u>Sch</u> be the category of schemes

<u>Set</u> be the category of sets

<u>Idea</u>: Study X by looking at all the possible maps $Y \longmapsto X$, i.e.,

$$\begin{aligned} \mathrm{Mor}(\cdot, X) : \underline{\mathrm{Sch}} &\longmapsto \underline{\mathrm{Set}} \\ Y &\longmapsto \mathrm{Mor}(Y, X) \end{aligned}$$

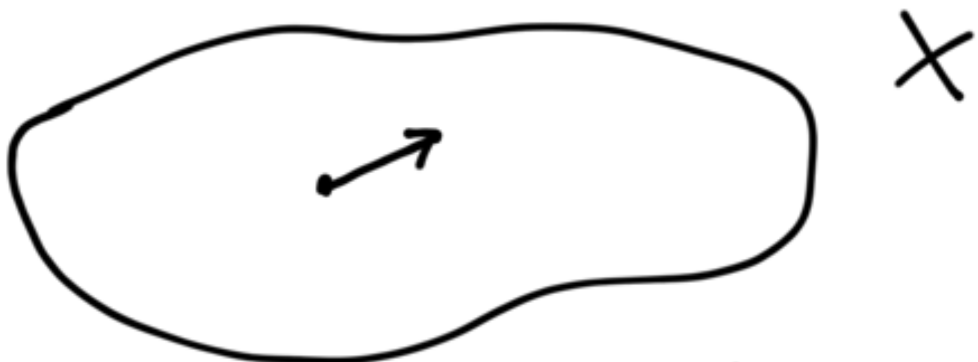
recovers the full information of X

Notation: $X(A) = \mathrm{Mor}(\mathrm{Spec}\,A, X)$

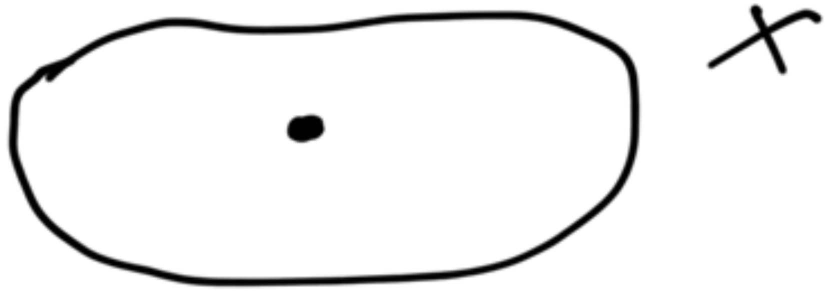
the space of A-pt.

Example ① $X(\mathbb{C})$ is the space of all closed pts of X.

② $X(\mathbb{C}[\varepsilon]/\varepsilon^2)$ is the space of tangent vectors



We'll be interested in **Artinian ring** A, which has a unique max ideal m_A. An element of $X(A)$ is a fat point.



Let $p \in X$ be a closed pt, the local geometry of X around p can be studied by looking at maps

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] & X \\ \operatorname{Spec} \mathbb{C} \arrow[u, hook] \arrow[r] & p \arrow[u, hook] \end{tikzcd}$$

Def'n. The $\boxed{\text{deformation functor}}$ of p in X is

$$\mathrm{Def}_{p,X} : \underline{\mathrm{Art}} \longmapsto \underline{\mathrm{set}}$$
$$A \longrightarrow X(A)$$

which maps the closed pt of A to p.

Prop. p is a smooth pt of X iff $\forall$ surjective Artinian rings $A \twoheadrightarrow B$, the map $\mathrm{Def}_{p,X}(A) \longrightarrow \mathrm{Def}_{p,X}(B)$ is surjective.

$$\begin{tikzcd}
\mathrm{Spec}\, B \arrow[r] \arrow[d, hook] & X \\
\mathrm{Spec}\, A \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

lifting property

Example. $X = \mathbb{C}^n = \mathrm{Spec}(\mathbb{C}[x_1, \cdots, x_n])$

p is the origin. Then

$$\mathrm{Def}_{p,X}(A) = A \times \cdots \times A$$

The lifting property obviously holds!

Example. $X = \mathrm{Spec}\left(\mathbb{C}[\varepsilon]/\varepsilon^2\right)$, $p = (\varepsilon = 0)$

Let $A = \mathbb{C}[\varepsilon]/\varepsilon^2$, $B = \mathbb{C}[\varepsilon]/\varepsilon^3$

Consider $\mathrm{Spec}\,A \mapsto X$ identity

then the lifting

$$
\begin{tikzcd}
\mathrm{Spec}\,A \arrow[r, "\simeq"] \arrow[d, hook] & X \\
\mathrm{Spec}\,B \arrow[ur, dashed, "?"'] &
\end{tikzcd}
\iff
\begin{tikzcd}
\mathbb{C}[\varepsilon]/\varepsilon^2 & \mathbb{C}[\varepsilon]/\varepsilon^2 \arrow[l] \arrow[dl, dashed, "?"] \\
\mathbb{C}[\varepsilon]/\varepsilon^3 \arrow[u] &
\end{tikzcd}
$$

doesn't exist. p is a singular pt of X.

· Moduli functors

In algebraic geometry, moduli space of geometric objects is studied via the functors.

Example: the moduli space of curves of genus g: $\mathcal{M}_g$. A morphism

$$Y \longmapsto \mathcal{M}_g$$

$\Longleftrightarrow$ a smooth family of genus g curves over Y (up to isomorphism)

$$\mathrm{Mor}(Y, \mathcal{M}_g) = \left\{ \begin{array}{c} \Sigma_g \to \mathcal{C} \\ \downarrow \\ Y \end{array} \right\} \Big/ \sim$$

Example: Let X be a projective variety. The moduli space of subschemes of X is represented by the Hilbert scheme Hilb_X.

$$\mathrm{Mor}(Y, \mathrm{Hilb}_X) \Longleftrightarrow \left\{ \begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[dr, "\text{flat}"'] & X \times Y \arrow[d] \\ & Y \end{tikzcd} \right\}$$

flat family of subschemes

Example. The moduli functor for coherent sheaves on X is given by

$$Y \longmapsto \left\{ \begin{matrix} \mathcal{E} \\ \downarrow \\ X \times Y \end{matrix} \quad \begin{matrix} \text{coherent sheaves on} \\ X \times Y. \ \text{flat} / Y \end{matrix} \right\} / \sim$$

· Local moduli functor

Defn: A formal (local) moduli functor is a functor

$$F : \underline{\text{Art}} \longmapsto \underline{\text{Sets}}$$

$F(\mathbb{C})$ = one element

(+ some technical assumption)

· DG world

Following the spirit of homological algebra, we'd like to consider scheme with a "differential"

$\rightsquigarrow$ "derived scheme"

<u>Example</u> [Derived zero locus] Let $f_1, \cdots, f_k \in \mathbb{C}[x_1, \cdots, x_n]$, the derived zero locus is given by graded algebra

$$A = \mathbb{C}[x_1, \cdots, x_n, \varepsilon_1, \cdots, \varepsilon_k]$$

where $\deg(x_i) = 0 \quad \deg(\varepsilon_i) = -1$

with the differential

$$d(\varepsilon_i) = f_i \qquad dx_i = 0$$

It's easy to see that

$$H^0(A) = \mathbb{C}[x_1, \cdots, x_n]/(f_1, \cdots, f_k)$$

is the ordinary zero locus.

If we write

$$F = (f_1, \cdots, f_k) : \mathbb{C}^n \to \mathbb{C}^k$$

then

<u>zero locus</u>: $F^{-1}(0) = \mathcal{O}(\mathbb{C}^n) \otimes_{\mathcal{O}(\mathbb{C}^k)} \mathbb{C}$

$$\begin{tikzcd} F^{-1}(0) \arrow[r] \arrow[d] & \mathbb{C}^n \arrow[d] \\ 0 \arrow[r] & \mathbb{C}^k \end{tikzcd}$$

<u>Derived zero locus</u>: $A = \mathcal{O}(\mathbb{C}^n) \overset{L}{\otimes}_{\mathcal{O}(\mathbb{C}^k)} \mathbb{C}$

- Local moduli functor extends naturally to Artinian dg algebra.

<u>Goal</u>: Geometry of perturbative QFT in the framework of dg moduli functor

Lecture 2
Maurer–Cartan functor

- **DGLA**

__Def'n__ A $\boxed{\text{dgLa}}$ is a graded vector space

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k[-k]$$

with a differential d of $\deg = 1$, and a ~~bracket~~ of $\deg = 0$: $[\,,\,] : \mathfrak{g} \otimes \mathfrak{g} \mapsto \mathfrak{g}$ s.t.

① (graded) anti-commutative

$$[a,b] = -(-1)^{|a||b|}[b,a]$$

where $|a|$ is the degree of a.

② (graded) Leibnitz rule

$$d[a,b] = [da,b] + (-1)^{|a|}[a,db]$$

③ (graded) Jacobi-Identity

$$[[a,b],c] = [a,[b,c]] - (-1)^{|a||b|} [b,[a,c]]$$

Example:

① An ordinary Lie algebra is a dgLa concentrated at deg $=0$ w/ $d=0$

② Let M be a smooth mfd. T_M the tangent bundle. The space of smooth tangent vectors

$$\Gamma(M, T_M) \text{ is a Lie algebra.}$$

Let $T^{\cdot}_{poly}(M) = \Gamma(M, \wedge^* T_M)[1]$

So $T^n_{poly}(M) = \Gamma(M, \wedge^{n+1} T_M)$

$T_{poly}(M)$ is a graded vector space.

We can define a bracket as follows
(Schouten-Nijenhuis bracket)

$$[\xi_0\wedge\xi_1\wedge\cdots\wedge\xi_k,\ \eta_0\wedge\eta_1\wedge\cdots\wedge\eta_l]$$

$$= \sum_{i,j}(-1)^{i+j+k}[\xi_i,\eta_j]\,\xi_0\wedge\cdots\wedge\hat{\xi}_i\wedge\cdots\xi_k\wedge\eta_0\wedge\cdots\hat{\eta}_j\cdots\eta_l$$

$$\Downarrow$$

$T_{poly}(M)$ is a dgLa (w/ $d=0$)

This will describe the deformation of Poisson structures.

③ Let X be a complex mfd. Consider

$$\Omega^{0,*}(X, T_X^{1,0})$$

differential: $\bar{\partial}$

$[\,,\,]$: bracket induced from $T_X^{1,0}$

$\Longrightarrow$ $\Omega^{0,*}(X, T_X^{1,0})$ is a dgLa. This will describe the deformation of complex structures on X.

④ Let X be complex mfd. E holomorphic vector bundle. Then $\Omega^{0,*}(X, \mathrm{End}(E))$

differential: $\bar{\partial}$

$[\,,\,]$: induced from matrix $\mathrm{End}(E)$

This controls the deformation of E as hol. vector bundle.

· <u>Chevalley-Eilenberg Complex</u>

Let $\mathfrak{g}$ be a dgLa, $\mathfrak{g}^\vee$ the dual.

We define

$$C^*(\mathfrak{g}) \triangleq \mathrm{Sym}^*(\mathfrak{g}^\vee[-1])$$

w/. CE differential where

$$d_{CE} = d_{\mathfrak{g}} + d_{[,]}$$

$d_{\mathfrak{g}} : \mathfrak{g}^\vee[-1] \to \mathfrak{g}^\vee[-1]$ is the dual of

$d : \mathfrak{g} \to \mathfrak{g}$. And

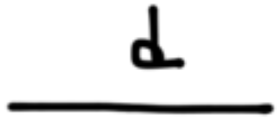
$$d_{[,]} : \mathfrak{g}^\vee[-1] \longrightarrow \mathrm{Sym}^2(\mathfrak{g}^\vee[-1]) \simeq \Lambda^2 \mathfrak{g}^\vee[-2]$$

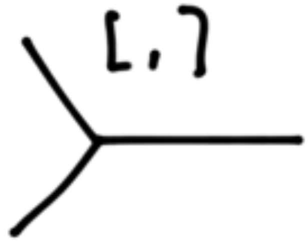
is the dual of the bracket:

$$[,] : \Lambda^2 \mathfrak{g} \mapsto \mathfrak{g}$$

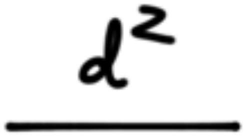
$\boxed{\text{Claim}}$: $\boxed{d_{CE}^2 = 0}$

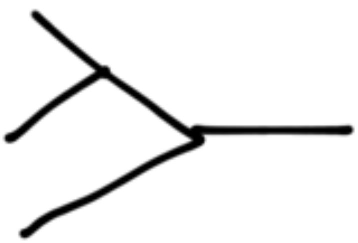
in fact, if we represent

d_{CE} : 

$+$ 

then

d_{CE}^2 : 

$+$ 

then $d_{CE}^2 = 0 \iff$ defining properties of dgLa.

$(C^*(\mathfrak{g}), d_{CE})$ is called C-E. Complex.

Example. Let M be a mfd.

$\Omega^*(M)$ is the de Rham complex.

$\forall\, \omega \in \Omega^k(M)$, it can be viewed as

$$\omega : \Lambda^k(T_M) \longmapsto C^\infty(M)$$

The de Rham differential can be represented as

$$d\omega(\xi_0 \wedge \cdots \wedge \xi_k)$$

$$= \sum_i (-1)^i \xi_i \left[\omega(\xi_0, \cdots, \hat{\xi}_i, \cdots \xi_n)\right]$$

$$+ \sum_{i<j} (-1)^{i+j} \omega\left([\xi_i, \xi_j] \wedge \xi_0 \wedge \cdots \hat{\xi}_i \cdots \hat{\xi}_j \cdots \xi_n\right)$$

Now suppose $M = G$ a compact Lie group.

$\mathfrak{g}$ = right inv. vector fields on G (Lie algebra of G)

$\mathfrak{g}^\vee$ = right inv. 1-forms on G

then $C^*(\mathfrak{g}) \hookrightarrow \Omega^*(G)$

$$d_{CE} \longrightarrow d_{DR}$$

In fact, $H^*(\mathfrak{g}) \triangleq H^*(C^*(\mathfrak{g}), d_{CE})$

$$= H^*_{DR}(G) \quad \#$$

· <u>$\mathfrak{g}$ · Module</u>

Let $\mathfrak{g}$ be a dg La. M be a $\mathfrak{g}$-mod.

i.e. M has a differential d_M s.t.

$$\begin{cases} d_M(\alpha m) = (d\alpha) m + (-1)^\alpha \alpha\, d_M m \\ [\alpha, \beta] m = \alpha \cdot \beta \cdot m - \beta \cdot \alpha \cdot m \end{cases}$$

$\forall \alpha, \beta \in \mathfrak{g}, \quad m \in M.$

We can similarly consider the complex

$$C^*(\mathfrak{g}, M) = \mathrm{Sym}^*(\mathfrak{g}^\vee[-1]) \otimes M$$

with differential

$$d_{CE} = d_{\mathfrak{g}} + d_{[\mathfrak{g},\mathfrak{g}]} + d_M + d_{[\mathfrak{g},M]}$$

where $d_{\mathfrak{g}} + d_{[\mathfrak{g}.\mathfrak{g}]} = d_{CE}$ acting on $\mathrm{Sym}^*(\mathfrak{g}^\vee[-1])$,

$$d_M : M \longrightarrow M \quad \text{and}$$

$$d_{[\mathfrak{g},M]} : M \longmapsto \mathfrak{g}^\vee[-1] \otimes M$$

is the dual of the module str.

$$\mathfrak{g} \otimes M \longmapsto M$$

Claim $d_{CE}^2 = 0$ on $C^*(\mathfrak{g}, M)$.

Moreover, $C^*(\mathfrak{g}, M)$ is a module over the dga $C^*(\mathfrak{g})$

Def'n. The Lie algebra cohomology of $\mathfrak{g}$ valued in M is given by

$$H^*(\mathfrak{g}, M) \stackrel{\Delta}{=} H^*(C^*(\mathfrak{g}, M), d_{CE})$$

If $M = \mathbb{C}$ trivial rep. Then

$$H^*(\mathfrak{g}, \mathbb{C}) = H^*(\mathfrak{g})$$

is the Lie algebra cohomology of $\mathfrak{g}$ modeling the de Rham cohomology.

Example. Let X be a mfd.

$$\mathfrak{g} = \Gamma(X, T_X)$$

the Lie alg. of vector fields. Then $M = C^\infty(X)$ is a $\mathfrak{g}$-module, and

$$C^*(\mathfrak{g}, M) = \mathrm{Hom}(\Lambda^\bullet T_X, C^\infty(X)) = \Omega^*(X)$$

and $d_{CE} = d_{DR}$. #

· Maurer-Cartan functor

Def'n. Let $\mathfrak{g}$ be a dgLa. The $\boxed{\text{Maurer-Cartan functor}}$ is the functor

$$MC_{\mathfrak{g}} : \underline{\mathrm{dg\,Art}} \longmapsto \underline{\mathrm{Set}}$$

If R is a dg Artinian ring, then

$$MC_{\mathfrak{g}}(R) = \left\{ \begin{array}{c} \alpha \in m_R \otimes \mathfrak{g} \\ \deg \alpha = 1 \end{array} \middle| \begin{array}{c} d\alpha + \frac{1}{2}[\alpha, \alpha] \\ = 0 \end{array} \right\}$$

<u>Prop</u>. $MC_{\mathfrak{g}}(R) = \mathrm{Hom}_{\text{Aug-dga}}(C^*(\mathfrak{g}), R)$

where $\mathrm{Hom}_{\text{Aug-dga}}$ is the space of homomorphisms between dga which preserves the closed pt.

We should think of $C^*(\mathfrak{g})$ as the structure sheaf of "dg scheme" with differential d_{CE}. We formally write the space as

$\mathbb{B}\mathfrak{g}$ the classifying space.

Then

$$MC_{\mathfrak{g}} = \mathrm{Hom}_{dga}(-, \mathbb{B}\mathfrak{g})$$

Is the moduli functor corresponding to $\mathbb{B}\mathfrak{g}$.

$\mathfrak{g}$-module $\Longleftrightarrow$ Coherent sheaf $C^*(\mathfrak{g}, M)$ on $\mathbb{B}\mathfrak{g}$

Lecture 3
Deformation Theory

We study the local geometry of B_g via deformation theory.

- <u>Deformation functor</u>

Let Δ^n be the n-simplex. It's modelled by the dga:

$$\Omega^*(\Delta^n) = \mathbb{C}[t_0, \cdots, t_n, dt_0, \cdots, dt_n] \Big/ \sim$$

where $\sim$: $\begin{cases} t_0 + \cdots + t_n = 1 \\ dt_0 + \cdots + dt_n = 0 \end{cases}$

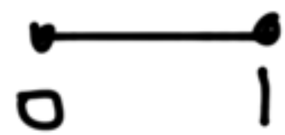
For natural map of simplices

$$i : \Delta^m \longmapsto \Delta^n$$

We have the natural map

$$i^* : \Omega^*(\Delta^n) \longmapsto \Omega^*(\Delta^m)$$

We'll write $I = \Delta^1$



with two maps

$$0, 1 : pt = \Delta^0 \hookrightarrow \Delta^1$$

Def'n: Let $\alpha_0, \alpha_1 : \mathrm{Spec}\, R \longmapsto B_g$

(equivalently $\alpha_0, \alpha_1 \in \mathcal{M}_g(R)$)

We say that α_0, α_1 are "gauge equiv"

if $\exists \ \beta : \mathrm{Spec}\, R \times I \longmapsto B_g$ s.t.

$$\beta\big|_{\mathrm{Spec}(R)\times\{0\}} = \alpha_0, \quad \beta\big|_{\mathrm{Spec}(R)\times\{1\}} = \alpha_1$$

Def'n. We define the deformation functor of $\mathfrak{g}$ as

$$\mathrm{Def}_{\mathfrak{g}} = MC_{\mathfrak{g}} / \text{gauge equiv.}$$

· Tangent space

Let X be a scheme, then it's tangent space is identified via $X(\mathbb{C}[\varepsilon]/\varepsilon^2)$

Def'n. The tangent space of $B\mathfrak{g}$ is defined by $\mathrm{Def}_{\mathfrak{g}}(\mathbb{C}[\varepsilon]/\varepsilon^2)$.

$\forall \alpha \in \mathrm{Def}_{\mathfrak{g}}(\mathbb{C}[\varepsilon]/\varepsilon^2)$. it's rep. by

$$\alpha \in \varepsilon \otimes \mathfrak{g}^1, \quad d\alpha = 0$$

Let $\beta \in MC_{\mathfrak{g}}(\mathrm{Spec}\,\mathbb{C}[[\varepsilon]]/\varepsilon^2 \times I)$

$\beta \in (\varepsilon \otimes \mathfrak{g} \otimes \mathbb{C}[t, dt])^1 \quad d\beta = 0$

degree one element.

$$\beta = \beta_0 + \beta_1 dt$$

$$\begin{cases} \beta_0 \in \varepsilon \otimes \mathfrak{g}^1 \otimes \mathbb{C}[t] \\ \beta_1 \in \varepsilon \otimes \mathfrak{g}^0 \otimes \mathbb{C}[t] \end{cases}$$

$d\beta = 0$ is equivalent to

$$\begin{cases} d\beta_0 = 0 \\ d\beta_1 = -\partial_t \beta_0 \end{cases}$$

It follows that

$$\beta\big|_{t=1} - \beta\big|_{t=0} = \int_0^t \partial_t \beta_0 \, dt$$

$$= -d \int_0^1 \beta_1 dt$$

It follows that

α_0, α_1 are gauge equivalent

$\Longrightarrow$ $\alpha_1 - \alpha_0$ is d-exact.

It's easy to see that the converse is also true.

Prop. The tangent space of $B_{\mathfrak{g}}$ is

$$H^1(\mathfrak{g}, d)$$

where d is the differential on $\mathfrak{g}$.

· **Obstruction theory**

Let $0 \to I \to \widetilde{R} \to R \to 0$ be an extension of dg Art ring R by square zero ideal I ($I^2 = 0$). We want to understand the lifting property

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, "\alpha"] \arrow[d, hook] & B_{\mathfrak{g}} \\ \operatorname{Spec} \widetilde{R} \arrow[ur, dotted, "?\ \widetilde{\alpha}"'] & \end{tikzcd}$$

(I is R-mod)

Let $\alpha \in MC_{\mathfrak{g}}(R)$. $d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$

$\alpha \in (m_R \otimes \mathfrak{g})'$ deg $= 1$

Let $\hat{\alpha} \in (m_{\hat{R}} \otimes g)'$ be an arbitrary lifting of α.

$$\begin{array}{ccc} (m_{\hat{R}} \otimes g)' & \longmapsto & (m_R \otimes g)' \\ \hat{\alpha} & \longrightarrow & \alpha \end{array}$$

$\hat{\alpha}$ may not satisfy MC eqn.

Let $\gamma = d\hat{\alpha} + \frac{1}{2}[\hat{\alpha}, \hat{\alpha}]$, then

$$\gamma\big|_R = 0 \quad \left(\gamma\big|_R = d\alpha + \frac{1}{2}[\alpha, \alpha] \right)$$

$$\Longrightarrow \gamma \in I \otimes g$$

Claim: $d\gamma + [\alpha . \gamma] = 0$

in fact

$$d\gamma + [\alpha, \gamma]$$
$$= \underbrace{d\left(\tfrac{1}{2}[\tilde{\alpha}, \tilde{\alpha}]\right) + [\alpha, d\tilde{\alpha}]}_{=0} + \underbrace{\tfrac{1}{2}[\tilde{\alpha}, [\tilde{\alpha}, \tilde{\alpha}]]}_{\substack{=0 \\ \text{by Jacobi}}}$$

$\Longrightarrow$ γ gives rise to an element

$$ob(\alpha) \in H^2(\underline{I} \otimes \mathfrak{g}, d + [\alpha, -])$$

If we consider a different lifting $\tilde{\alpha}' = \tilde{\alpha} + \beta$ where

$$\beta \in (\underline{I} \otimes \mathfrak{g})^1$$

then

$$\gamma' = d\hat{\alpha}' + \frac{1}{2}[\tilde{\alpha}', \tilde{\alpha}']$$

$$= \gamma + d\beta + [\alpha, \beta]$$

$$\Downarrow$$

$$ob(\alpha) \in H^2(I \otimes g, d + [\alpha, -])$$

doesn't depend on the choice.

Moreover, we find

$\exists$ lifting $\tilde{\alpha}$ satisfying MC $\Longleftrightarrow$ $ob(\alpha) = 0$ in $H^2(I \otimes g . d + [\alpha, -])$

In particular, if we consider the minimal extension $I = \mathbb{C}$, then $ob(\alpha) \in H^2(\mathfrak{g}, d)$

If $H^2(\mathfrak{g}, d) = 0$, then the lifting always exists, and $\mathcal{B}_{\mathfrak{g}}$ is smooth. $H^2(\mathfrak{g}, d)$ is called "<u>Obstruction space</u>".

In Summary: For $\mathcal{B}_{\mathfrak{g}}$

- Tangent space: $H^1(\mathfrak{g}, d)$
- obstruction space: $H^2(\mathfrak{g}, d)$
- Automorphism: $H^0(\mathfrak{g}, d)$

Example. Let X be a complex mfd.

We consider dgLa

$$\left(\mathfrak{g}=\Omega^{0,*}(X,T_X^{1,0}),\ \bar\partial,\ [\,,\,]\right)$$

MC element: R Artinian ring

$$\mu\in m_R\otimes\Omega^{0,1}(X,T_X^{1,0})$$

$$\bar\partial\mu+\frac{1}{2}[\mu,\mu]=0$$

μ gives a deformation of complex str. parametrized by R. The "new" holomorphic function is

$$\{f \mid \bar\partial f+\mu\lrcorner df=0\}$$

Tangent space : $H^1(X, T_X)$

Obstruction space : $H^2(X, T_X)$

If $X = \Sigma_g$ a Riemann surface of genus g, $\dim X = 1$

$\Rightarrow H^2(X, T_X) = 0$

$\Rightarrow$ $\underline{\mathcal{M}_g \text{ is smooth}}$.

$\underline{\underline{\text{Example}}}$. $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ be a hol. vector bundle,

$$\mathfrak{g} = \Omega^{0,*}(X, \mathrm{End}(E))$$

describes the deformation of E.

Tangent space: $H^1(X, \mathrm{End}(E)) = \mathrm{Ext}^1(E, E)$

obstruction space : $\mathrm{Ext}^2(E, E)$

Lecture 4
Classical field theory

- **Symplectic form**:

Let $\mathfrak{g}$ be a dgLa. An invariant pairing of $\deg = k$ on $\mathfrak{g}$ is a $\deg = k$ cochain map of $\mathfrak{g}$-modules

$$\langle \, , \, \rangle : \mathfrak{g} \otimes \mathfrak{g} \mapsto \mathbb{C}$$

Equivalently.

$$\begin{cases} \langle d\alpha, \beta \rangle + (-1)^{\alpha} \langle \alpha, d\beta \rangle = 0 \\ \langle [\alpha, \beta], \gamma \rangle = \langle \alpha, [\beta, \gamma] \rangle \end{cases}$$

Example: $\mathfrak{g} = \mathfrak{sl}_n$, consider trace map

$$\mathrm{Tr}: A \otimes B \longmapsto \mathrm{Tr}(AB)$$

defines an invariant pairing of deg $= 0$

Let ω be a symmetric non-degenerate invariant pairing of deg $= k$

$$\Rightarrow \omega \in \mathrm{Sym}^2(\mathfrak{g}^\vee) \simeq \wedge^2(\mathfrak{g}^\vee[-1])[2]$$

Therefore, ω can be viewed as a symplectic form on $B_{\mathfrak{g}}$ of deg $= k+2$

Def'n: A classical field theory is given by a dgL (L_∞) algebra with symplectic form ω of deg $= -1$

Let $(\mathfrak{g}, \omega)$ be such a str.

$C^*(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}[1]^\vee)$ can be viewed as the space of functions on $\mathfrak{g}[1]$.

d_{CE} is a deg=1 vector field w.

$$d_{CE}^2 = 0$$

ω induces a deg=1 Poisson bracket $\{ , \}$.

d_{CE} corresponds to a Hamiltonian function $S \in C^*(\mathfrak{g})$ s.t.

$$d_{CE} = \{S, -\}$$

$$d_{CE}^2 = 0 \Longleftrightarrow \{S, S\} = 0$$

In physics, $\{S,S\}=0$ is called "classical master equation"

$\{S,-\}$ generates the gauge symmetry in the BV-formalism

<u>Example</u>: Let X be a mfd.

$$f: X \mapsto \mathbb{C}$$

We consider the derived critical locus of f:

$$\mathrm{Crit}^{0}(f) = \mathrm{Sym}^{*}(T_X[1]) = \mathcal{O}(T_X^{*}[-1])$$

with a differential $\lrcorner\, df$

It corresponds to a L_∞ algebra

$$\mathfrak{g}[1] = T_X \oplus (T_X[1])^{\vee}$$

$= T_X \oplus T_X^\vee[-1]$

with a natural deg $= -1$ symplectic pairing:

$$\omega : T_X \oplus T_X^\vee[-1] \longrightarrow \mathbb{C}$$

It induces a Poisson str. $\{\ ,\ \}$

and
$$\lrcorner\, df = \{f, -\}$$

If we assume that
$$f = \text{quadratic} + \text{cubic}$$

then $\lrcorner\, df = d_{CE}$ gives the dgLa str. on $\mathfrak{g}$. For simplicity, we consider $X = V$ linear vector space and consider such case

$$f = \underset{\text{quadratic}}{f_2} + \underset{\text{cubic}}{f_3}$$

$$\mathfrak{g} = V[-1] \oplus V^*[-2]$$ then

the dgLa str. is given by

$$\begin{cases} df_2 : V[-1] \xrightarrow{d} V^*[-2] \\ df_3 : V[-1] \otimes V[-1] \xmapsto{[\,,\,]} V^*[-2] \end{cases}$$

$$\langle\,,\,\rangle : V[-1] \otimes V^*[-2] \longmapsto \mathbb{C}$$

invariant pairing of deg $= -3$

s.t. $\forall\, \alpha, \beta \in V[-1]$

$$\langle d\alpha + \tfrac{1}{2}[\alpha, \alpha], \beta \rangle = \frac{\partial f}{\partial \beta}(\alpha)$$

In general, if we have higher deg terms

$$f = f_2 + f_3 + f_4 + \cdots$$

then $df_{k+1} : (V[-1])^{\otimes k} \xrightarrow{l_k} V[-2]$

which can be viewed as L_∞ str.

· Field theory Examples

· Scalar field theory :

Let M be a compact Riem. mfd.

The space of fields is

$$\phi \in V = C^\infty(M)$$

We consider the classical action

functional

$$S[\phi]=\int_M d\mathrm{Vol}\left(\frac{1}{2}\phi\Delta\phi+\frac{1}{3!}\phi^3\right)$$

which can be viewed as "Polynomial"

on the infinite dim'l space V

<u>Critical locus</u>:

$$\Delta\phi+\frac{1}{2}\phi^2=0$$

If we consider $\mathrm{Crit}^D(f)$, we get

the dgLa $\mathfrak{g}=V[-1]\oplus V[-2]$ with

$$d: V[-1] \longmapsto V[-2]$$
$$\phi \longrightarrow \Delta\phi$$

and $[\cdot\,]: V[-1]^{\otimes 2} \longmapsto V[-1]$

$$\phi_1\otimes\phi_2 \longrightarrow \phi_1\phi_2$$

Euler-Lagrangian Eq^n $\Longleftrightarrow$ MC Eq^n

$\Delta\phi + \frac{1}{2}\phi^2 = 0$ $\qquad$ $d\phi + \frac{1}{2}[\phi,\phi] = 0$

· Chern-Simons theory

Let M be 3-dim'l Riem. mfd.

G Lie group w/. Lie algebra $\mathfrak{g}$. We consider the following dgLa

$$\mathfrak{g}_{CS} = \left(\Omega^*(M,\mathfrak{g}),\ d,\ [\,,\,]\right)$$

where d: de Rham differential

$[\,,]$: Lie bracket on $\mathfrak{g}$

There's a natural deg $= -3$ invariant

pairing

$$\langle\, , \rangle : \alpha \otimes \beta \longmapsto \langle \alpha, \beta \rangle = \int_M \mathrm{Tr}(\alpha\beta)$$

where $\mathrm{Tr}: \mathfrak{g} \otimes \mathfrak{g} \longmapsto \mathbb{C}$ is the normalized Killing form.

$$\Downarrow$$

natural Poisson bracket $\{\, , \}$.

d_{CE} leads to a local functional on $\mathfrak{g}_{cs}[1]$:

$$CS[A] = \frac{1}{2}\langle A, dA \rangle + \frac{1}{3!}\langle A, [A, A] \rangle$$

$$\forall\ A \in \Omega^*(X, \mathfrak{g})[1]$$

which is called the Chern–Simons functional.

· **Classical field theory**

Def'n: A local dgLa on a mfd M is given by a graded locally free sheaf $\mathcal{E}$ equipped with

$$d : \mathcal{E} \mapsto \mathcal{E}$$

a differential operator of deg $=1$. and

$$[\,,\,] : \mathcal{E} \otimes \mathcal{E} \mapsto \mathcal{E}$$

a bi-differential operator of deg $=0$ satisfying the usual def'n of dgLa.

We say $\mathcal{E}$ is "elliptic" if $(\mathcal{E}, d)$ is an elliptic complex.

<u>Def'n</u>: An invariant pairing on a local dgLa $\mathcal{E}$ is given by a map of bundles

$$\langle\ ,\ \rangle : \mathcal{E}\otimes\mathcal{E} \longmapsto \mathrm{Dens}_M \ \left(\simeq \Omega^{top}(M)\right)$$

density Line bundle

s.t.

(1) ~~the~~ the pairing is non-degenerate.

(2) the pairing $\alpha\otimes\beta \mapsto \int\langle\alpha,\beta\rangle$ is invariant for α,β compactly supp.

<u>Example</u>. $\mathfrak{g} = \Omega^{*}(M,\mathfrak{g})$ CS theory

<u>Example</u>. Let X be CY 3-fold, with hol. top form Ω_X. $\begin{array}{c}E\\ \downarrow\\ X\end{array}$ be a holomorphic vector bundle.

We consider the local dgLa

$$\mathfrak{g} = \Omega^{0,*}(X, \mathrm{End}\, E)$$

with $d = \bar{\partial}$ and $[\,,\,]$

There's a natural invariant pairing

$$\alpha \otimes \beta \longmapsto \int \mathrm{Tr}(\alpha \wedge \beta) \wedge \Omega_X$$

The dgLa corresponds to the holomorphic Chern-Simons functional

$$HCS[A] = \frac{1}{2} \int \mathrm{Tr}(A \wedge \bar{\partial} A) \wedge \Omega_X$$

$$+ \frac{1}{3} \int \mathrm{Tr}(A^3) \wedge \Omega_X$$

To model the construction of finite dim'l case on manifolds.

<u>Def'n</u>: A <u>perturbative classical field theory</u> is an elliptic dgLa $\mathcal{E}$ with an invariant pairing of deg $= -3$.

<u>Example</u> [Cotangent theory] Let $\mathcal{E}$ be an elliptic dgLa on M, let

$$\mathcal{E}^! = \mathcal{E}^\vee \otimes \mathrm{Dens}_M$$

where $\mathcal{E}^\vee$ is the dual bundle. Then

$\mathcal{E} \oplus \mathcal{E}^![-3]$ is a perturbative classical field theory.

Lecture 5

Gauge theory in the BV-formalism

· Finite dim'l model

Let V be a finite dim space w/. a Lie group G acting on V.

$V \rightsquigarrow$ space of "fields"

$G \rightsquigarrow$ gauge group

Let $S \in \mathcal{O}(V)$ be a function on V.

$S \rightsquigarrow$ Action functional

which is G-invariant. We want to model

$$\int_{V/G} e^{S/\hbar}$$

by integration on linear space

$$\mathfrak{g}[1] \oplus V$$

This is called the <u>BV-formalism</u>.

Note that

$$\mathcal{O}(V/G) = \mathcal{O}(V)^G$$

The BV-formalism is a derived construction,

① the derived quotient is modelled by

$$\mathcal{O}(V)^G \longmapsto C^*(\mathfrak{g}, \mathcal{O}(V))$$

$$= \mathrm{Sym}\left((V \oplus \mathfrak{g}[1])^\vee \right)$$

We view $\mathfrak{g} \oplus V[-1]$ as a dgLa

S extends to a functional on $\mathfrak{g}[1] \oplus V$

$$S \in \mathcal{O}(\mathfrak{g}[1] \oplus V)$$

S is gauge inv $\Rightarrow d_{CE} S = 0$

Critical locus of S

$\Rightarrow$ solution of EL

<u>BV formalism</u>: Derived critical locus of S on $\mathfrak{g}[1] \oplus V$

Therefore we're led to consider

$$E = T^*_{\mathfrak{g}[1] \oplus V}[-1]$$

$$= \mathfrak{g}[1] \oplus V \oplus V^\vee[-1] \oplus \mathfrak{g}^\vee[-2]$$

which has a canonical symplectic form of deg $= -1$.

$\Rightarrow$
- Poisson bracket $\{ , \}$ of deg $= 1$
- Differential d_{CE} on $C^*(\mathfrak{g}[1] \oplus V)$ extends to the CE differential on $C^*(E)$, which we denote by X_{CE}
- The function f which is gauge invariant $X_{CE}(f) = 0$

$X_{CE} \Rightarrow$ Hamiltonian function H_{CE} s.t. $X_{CE} = \{H_{CE}, -\}$

and $\{H_{CE}, S\} = 0$

Moreover

$$d_{CE}^2 = 0 \Rightarrow \{H_{CE}, H_{CE}\} = 0$$

and $\{S, S\} = 0$

Let $S^{BV} = S + H_{CE}$, then S^{BV} is the BV extension of S, which satisfies the "classical master equation"

$$\{S^{BV}, S^{BV}\} = 0$$

The gauge symmetry in the BV formalism is represented by the BRST operator $\{S^{BV}, -\}$

In Physics

$\mathfrak{g}[1]$ $\quad V \quad V^\vee[-1] \quad \mathfrak{g}^\vee[-2]$

ghost field anti-field anti-ghost

<u>Example</u> : [CS theory]

Let M be Riem. mfd of $\dim = 3$.

$\mathfrak{g}$ a Lie algebra

<u>field</u> : $\Omega^1(M, \mathfrak{g}) \ni A$ (connection)

<u>gauge</u> : $\Omega^0(M, \mathfrak{g}) \ni \phi$

CS Action:

$$CS[A] = \int \mathrm{Tr}\, \frac{1}{2} A \wedge dA + \frac{1}{6} A \wedge [A, A]$$

Gauge transformation:

$$\delta_\phi A = d\phi + [A, \phi]$$

and $\delta_\phi CS = 0$ gauge invariance.

<u>anti-field</u>: $\Omega^2(M, g) \ni A^\vee$

which is naturally the dual of $\Omega^1(M, g)$

<u>anti-ghost</u>: $\Omega^3(M, g) \ni \phi^\vee$

which is the dual of $\Omega^0(M, g)$

After BV-extension,

$$E = \Omega^*(M, \mathfrak{g})[1]$$

and the CS in the BV formalism

$$\begin{aligned} CS + H_{CE} &= \frac{1}{2}\int A\wedge dA + \frac{1}{6}\int A\wedge[A,A] \\ &+ \int A^\vee\wedge d\phi + \int \phi^\vee\wedge[A,\phi] \end{aligned}$$

If we write $\mathcal{A} = \phi + A + A^\vee + \phi^\vee$, then

$$CS^{BV}[\mathcal{A}] = \frac{1}{2}\int \mathcal{A}\wedge d\mathcal{A} + \frac{1}{6}\int \mathcal{A}\wedge[\mathcal{A}, \mathcal{A}]$$

the same form as the original CS, but w/ more fields.

· AKSZ construction

Let (X, d_X), (Y, d_Y) be two dg spaces such that

$$\begin{cases} X \text{ is equip. w/. a volume form of } \deg = k \\ Y \text{ is equip. w/. a symplectic form of } \deg = k+1 \text{ compatible w/. } d_Y \end{cases}$$

We consider

$$\mathrm{Map}(X, Y)$$

Given $f \in \mathrm{Map}(X, Y)$, the tangent space is

$$T_f \mathrm{Map}(X, Y) = \Gamma(X, f^* T_Y)$$

We define a pairing on $T_f \mathrm{Map}(X, Y)$

$$\alpha\otimes\beta \mapsto \int_X (\alpha,\beta)_Y = \langle\alpha,\beta\rangle$$

which defines a symplectic form of deg $=-1$ on $Map(X,Y)$

The differential d_X, d_Y gives

$$d = d_X + d_Y$$

Then $(Map(X,Y), d, \langle\,,\,\rangle)$ is a dg space w/ odd symplectic form of deg $=-1$. $d = \{S, -\}$

and S gives the actional functional in the BV-formalism.

CS theory is such example:

$$X = (M, \Omega_M^{*}) \longmapsto Y = (Bg, C^*(g))$$

Then we check that the corresponding S is precisely the CS action in the BV-formalism.

Lecture 7
Path Integral

· Path Integral

Let $\mathcal{E}$ be the space of fields for a classical field theory.

$S[\phi]$: action functional

$$\phi \in \mathcal{E}$$

One approach to quantum field theory is to look at the integral

$$\int_{\phi \in \mathcal{E}} [D\phi]\, e^{-S[\phi]/\hbar}$$

However, if $\mathcal{E}$ is infinite dim'l, the integral is NOT well-defined.

One special case that we can make sense of it is

"perturbative theory"

- Feynman Diagrams.

We start w/. finite dim'l model.

Let $V = \mathbb{R}^N$ be finite dim'l v. space.

Consider

$$Z(a) = \int_V d^N x \exp\left\{\frac{1}{\hbar}\left(-\frac{1}{2}Q(x) + I(x+a)\right)\right\}$$

where $Q(x) = \sum_{i,j} Q_{ij} x^i x^j$, $\underset{\text{positive}}{(Q_{ij}) > 0}$

and $I(x) = \frac{1}{3!} \sum_{i,j,k} I_{ijk} x^i x^j x^k$ cubic

The integration is understood as formal power series

$$Z(a) \triangleq \sum_{k\geq 0} \frac{1}{\hbar^k} \int_L d^N x \frac{(I(x+a))^k}{k!} e^{-\frac{1}{2\hbar} Q(x)}$$

If we let

$$Z_J = \int_L d^N x \exp\left\{-\frac{1}{2\hbar} Q(x) + \sum_i J_i x^i\right\}$$

$$= Z_0 \, e^{\frac{\hbar}{2} Q^{-1}(J,J)}$$

where $Q^{-1}(J,J) = \sum_{i,j} (Q^{-1})^{ij} J_i J_j$

Then we can compute

$$Z(a) = \int_V d^N x \exp\left\{ -\frac{1}{2\hbar} Q(x) + \sum_i x_i \frac{\partial}{\partial a^i} \right\} e^{\frac{1}{\hbar} I(a)}$$

$$= Z_0 \, e^{\frac{\hbar}{2} Q^{-1}\left(\frac{\partial}{\partial a}, \frac{\partial}{\partial a}\right)} e^{\frac{1}{\hbar} I(a)}$$

From this we deduce that

Prop. $Z(a) = Z_0 \exp\left(F(a)/\hbar\right)$ then

$$Z(a)/Z_0 = \sum_\Gamma \frac{w_\Gamma(Q^{-1}, I)}{|Aut(\Gamma)|}(a)$$

and
$$F(a) = \sum_{\Gamma:\,\mathrm{conn}} \hbar \frac{w_\Gamma(Q^{-1}, I)}{|Aut(\Gamma)|}(a)$$

Here $W_\Gamma(Q^{-1}, I)$ is the Feynman integral associated to a graph Γ

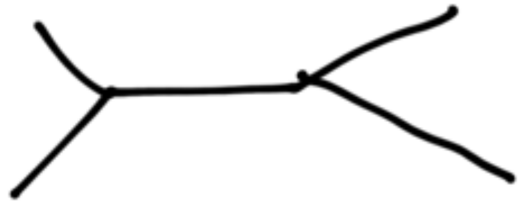
Propagator: $\hbar Q^{-1}$ is put on the edge

vertex: $\frac{1}{\hbar} I$ is put on each vertex

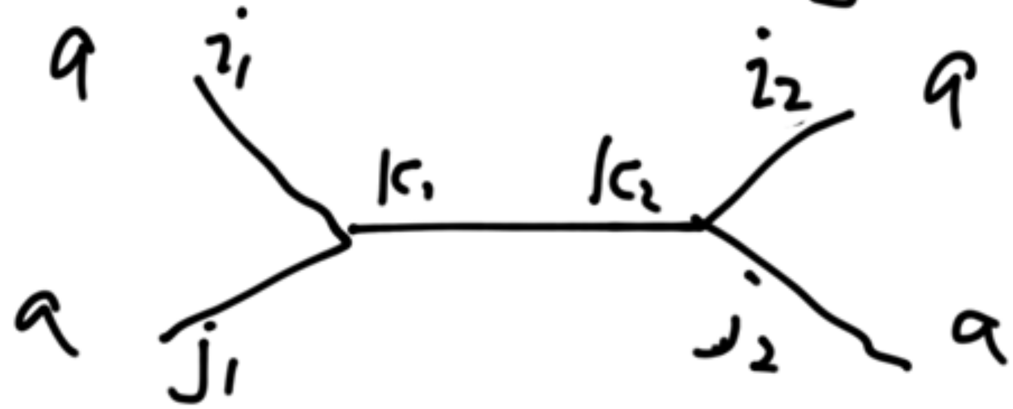
external edge: a is put on the external legs

In the above formula,

$\sum\limits_{\Gamma:\text{conn}}$ is the summation over all possible connected cubic graphs.

Example. Γ : 

we associate the weight by

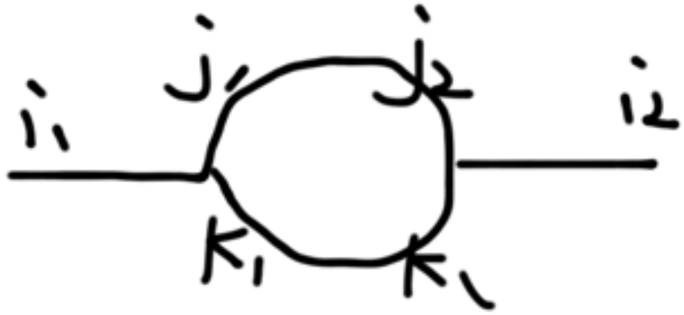


$$W_\Gamma(Q^{-1}, I)(a)$$

$$= \sum_{\substack{i_1, j_1, k_1 \\ i_2, j_2, k_2}} a_{i_1} a_{j_1} a_{i_2} a_{j_2} \hbar (Q^{-1})^{k_1 k_2} \left(\frac{1}{\hbar} I_{i_1 j_1 k_1}\right)\left(\frac{1}{\hbar} I_{i_2 j_2 k_2}\right)$$

this is an example of

"Tree diagram"

Example Γ: 

$$W_\Gamma(Q^{-1}, I)(a)$$

$$= \sum a_{i_1} a_{i_2} \hbar (Q^{-1})^{j_1 j_2} \hbar (Q^{-1})^{k_1 k_2} \left(\frac{1}{\hbar} I_{i_1 j_1 k_1}\right)\left(\frac{1}{\hbar} I_{i_2 j_2 k_2}\right)$$

this is an example of

"One-Loop diagram"

In general,

$$F(a) = \sum_{g \geq 0} \hbar^g F_g(a) \quad \text{where}$$

$$F_g(a) = \sum_{\substack{\Gamma : \text{conn} \\ g\text{-Loop}}} \frac{W_\Gamma(Q^{-1}, I)(a)}{|Aut(\Gamma)|}$$

· <u>Field theory and divergence</u>

Now we consider the field theory case, which is inf'l dim'l.

Let's start w/. scalar field theory on $M = \mathbb{R}^4$. The space of fields is

$$\mathcal{E} = C^\infty(M)$$

$$S[\phi] = \int_M -\frac{1}{2}\phi\Delta\phi + \frac{\lambda}{3!}\phi^3$$

$$\phi \in \mathcal{E}$$

We want to consider

$$e^{F(\lambda)/\hbar} \overset{?}{=} \int_{\phi\in\mathcal{E}} [D\phi]\, e^{\frac{1}{\hbar}\int -\frac{1}{2}\phi\Delta\phi + \frac{\lambda}{3!}\phi^3}$$

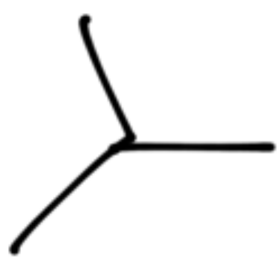
Following the finite dim'l construction, we need the following

<u>propagator</u>, Δ^{-1}

this operator is represented by the Green function

$$G(x,y) \in C^\infty(M\times M\setminus\Delta)$$

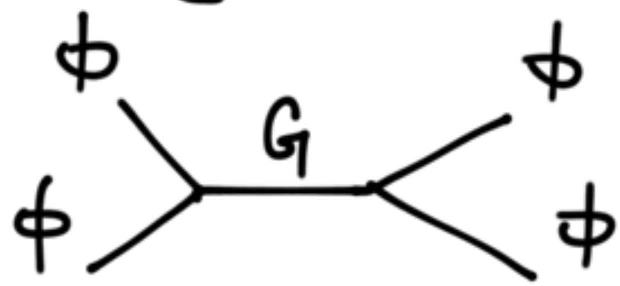
which is singular along the diagonal $\Delta \hookrightarrow M\times M$.

<u>Vertex</u>.  $\frac{\lambda}{3!}\int\phi^3$

We can write down the Feynman diagrams to model the path integral.

· Ultra-Violet divergence

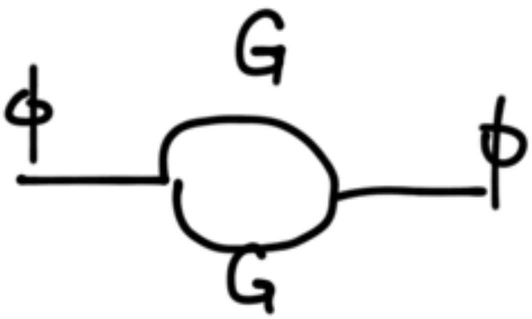
· **Tree diagram**.



$$\int_{M\times M} \phi^2(x)\phi^2(y)\, G(x,y)$$

$$= \int_M \phi^2 \frac{1}{\Delta}(\phi^2) \quad \text{which is OK}$$

· **One-Loop diagram**



$$= \int \phi(x)\phi(y)\, G^2(x,y)$$

which is divergent!

$$G(x,y) \sim \frac{1}{|x-y|^2}$$

In general, if you look at diagrams w/. loops, the graph integral is always divergent!

This is the ultra-Violet divergence in quantum field theory, related to the inf'l dim'l.

· Regularization

Let's summarize the trouble for scalar field theory example

$$S = \int -\frac{1}{2}\phi\Delta\phi + \frac{\lambda}{3!}\phi^3$$

propagator: $\Delta^{-1} \to G(x,y)$ which has singularity

$\Downarrow$

Break down the Feynman diagram expansion beyond tree diagrams

We can write

$$G = \int_0^\infty e^{-t\Delta} dt \text{ where } e^{-t\Delta}$$

is the Heat kernel for Δ.

For $M = \mathbb{R}^4$, flat metric,

$$e^{-t\Delta} = \frac{1}{(4\pi t)^2} e^{-|x-y|^2/t}$$

The diagonal singularity comes from $t \mapsto 0$.

There's another divergence due to the non-compactness of M, for $t \mapsto \infty$ (IR divergence)

We introduce the cut-off

$$P_\epsilon^L = \int_\epsilon^L e^{-t\Delta} dt$$

and consider instead the graph integrals

$W_\Gamma(P_\epsilon^L, I)$ which is well-defined

Since P_ϵ^L is smooth for $0<\epsilon<L<\infty$

The UV-divergence comes from the limit $\lim_{\epsilon\to 0} W_\Gamma(P_\epsilon^L, I)$

However, the asymptotic behavior of $e^{-t\Delta}$ as $t\to 0$ implies the asymptotic behavior of $W_\Gamma(P_\epsilon^L, I)$

as $\epsilon \mapsto 0$.

<u>Prop</u>. $\exists$ local actional functional $I^{CT}(\epsilon)$, which is singular as $\epsilon \to 0$, s.t. $\lim_{\epsilon\to 0} W_{\Gamma}(P_\epsilon^L, I + I^{CT}(\epsilon))$ exists

- $I^{CT}(\epsilon)$ is called the "<u>Counter-term</u>". which is used to "cancel" all the UV divergence to give finite answer. Let

- $$F[L] = \lim_{\epsilon\to 0} \sum_{\Gamma:\text{conn}} \frac{W_\Gamma(P_\epsilon^L, I + I^{CT}(\epsilon))}{|\mathrm{Aut}(\Gamma)|}$$

which depends on a parameter L.

$F[L]$ is called the "effective action at scale L"

- $\lim\limits_{L\to\infty} F[L]$ is the quantum limit, which models the full path integral.

<u>Example</u>: Γ: 

$$W_\Gamma(P_\epsilon^L, I) = \int_{M\times M} \phi(x)\phi(y)\left(P_\epsilon^L(x,y)\right)^2$$

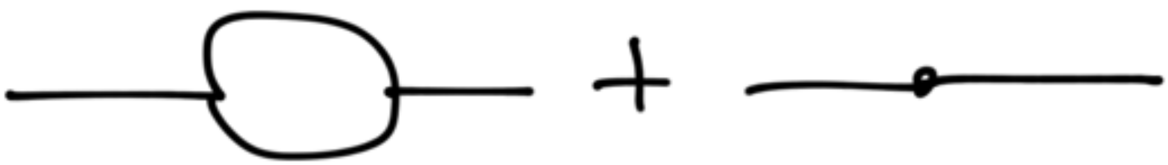
<u>Claim</u> $\Longrightarrow$ $= C \log\epsilon \int \phi^2$

$+$ (terms smooth as $\epsilon \to 0$)

Here c is a constant.
So we need to add the local functional

$$-c(\log\epsilon)\hbar\int_M \phi^2 \text{ into } I^{[1]}(\epsilon)$$

s.t.



is finite for $\epsilon \to 0$.

This procedure is called "renormalization"

Lecture 8
Renormalization of gauge theory

Now we consider the case for gauge theory. We consider a classical gauge theory in the BV formalism:

$$\begin{cases} \underline{\text{fields}}: \ \mathcal{E} = \Gamma(M, E) \ \text{elliptic complex} \\ \underline{\text{pairing}}: \ \langle \, , \, \rangle : E \otimes E \longmapsto \mathrm{Dens}_M \\ \qquad\qquad\qquad \text{of } \deg = -1 \end{cases}$$

Let $E^\vee$ be the complex of dual bundles, and

$$E^! = E^\vee \otimes \mathrm{Dens}_M$$

The pairing gives the isom

$$E \xrightarrow{\simeq} E^![-1]$$

We'll let $\overline{\mathcal{E}}$ denote the space of distributional sections of $\mathcal{E}$.

$\Downarrow$

natural pairing

$$\mathcal{E} \otimes \overline{\mathcal{E}}^! \longmapsto \mathbb{C}$$

So we can identify

$$\overline{\mathcal{E}}^! \simeq (\mathcal{E})^\vee \text{ the dual of } \mathcal{E}.$$

Let $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ $\begin{matrix} F \\ \downarrow \\ Y \end{matrix}$ be two bundles,

$\mathcal{E}, \mathcal{F}$ be the corresponding sections

$$\mathcal{E} = \Gamma(X, E) \qquad \mathcal{F} = \Gamma(X, F)$$

then we'll denote by

$$\mathcal{E} \hat{\otimes} \mathcal{F} \cong \Gamma(X \times Y, E \boxtimes F)$$

and similarly for distributions.

<u>Def'n</u>: The space of formal functionals on $\mathcal{E}$ is defined to be

$$\mathcal{O}(\mathcal{E}) = \prod_{n \geq 0} \mathcal{O}^{(n)}(\mathcal{E})$$

where $\mathcal{O}^{(n)}(\mathcal{E}) = \left(\overline{\mathcal{E}}^{!}\right)^{\hat{\otimes} n}$

$$= \overline{\left(E^{\vee} \boxtimes \cdots \boxtimes E^{\vee}\right)^{!}}$$

distributions on $X \times \cdots \times X$.

$\mathcal{O}^{(n)}(\mathcal{E})$ can be viewed as "deg $= n$ polynomial" on $\mathcal{E}$.

$\forall\, S \in \mathcal{O}^{(n)}(\mathcal{E})$, $\alpha_1, \cdots, \alpha_n \in \mathcal{E}$, we have the natural pairing

$$\langle S, \alpha_1 \otimes \cdots \otimes \alpha_n \rangle \stackrel{\Delta}{=} \frac{\partial}{\partial \alpha_1} \cdots \frac{\partial}{\partial \alpha_n} \Big|_0 S$$

<u>Def'n</u>: $\mathcal{O}_{loc}(\mathcal{E}) \subset \mathcal{O}(\mathcal{E})$ will denote the subspace of local functionals. Here S is <u>local</u> if

$$S = \int_M \mathcal{L} \quad \text{for some}$$

lagrangian density.

Example

① $S(\phi) = \int \phi^2 \partial_x \phi$ is local

② $S(\phi) = \int_{M \times M} \phi(x)\phi(y) e^{-|x-y|^2}$

is not local.

- Let $\langle \, , \rangle : E \otimes E \longrightarrow \mathrm{Dens}_M$

and $\omega = \int \langle \, , \rangle$

then $\omega^{-1} \in \overline{\mathcal{E}}^! \otimes \overline{\mathcal{E}}^!$ is the δ-function distribution along the diagonal $\Delta \hookrightarrow X \times X$

- Let $(\mathcal{E}, Q)$ be the elliptic complex

$Q^{GF} : \mathcal{E} \longrightarrow \mathcal{E}$ differential of $\deg = -1$

s.t. $\Delta = [Q, Q^{GF}]$ is generalized Laplacian.

Let $e^{-t\Delta} \in \mathcal{E} \otimes \mathcal{E}$ represents the Heat kernel, then

$$\boxed{\omega^{-1} = \lim_{t \to 0} e^{-t\Delta}}$$

which defines a <u>Poisson bracket</u> on $\mathcal{O}_{loc}(\mathcal{E})$, similar to symplectic geometry. Let S be the classical action for the gauge theory in the BV formalism, then S satisfies the <u>classical master eqn</u>

$$\{S, S\} = 0$$

This corresponds to $d_{CE}^2 = 0$

We'll write

$$S = S_2 + I$$

quadratic

higher deg (interaction)

then $\{S_2, -\}$ gives the differential Q.

$\{S, S\} = 0$ becomes

$$\begin{cases} Q^2 = 0 \\ QI + \frac{1}{2}\{I, I\} = 0 \end{cases}$$

- Renormalization

The propagator of S is

" $\frac{1}{Q}$ " which doesn't exist,

due to exactly gauge sym.

We can do the "gauge fixing"

$$\frac{1}{Q} \Longrightarrow \frac{Q^{GF}}{\Delta}$$ which now exists

Consider the regularization

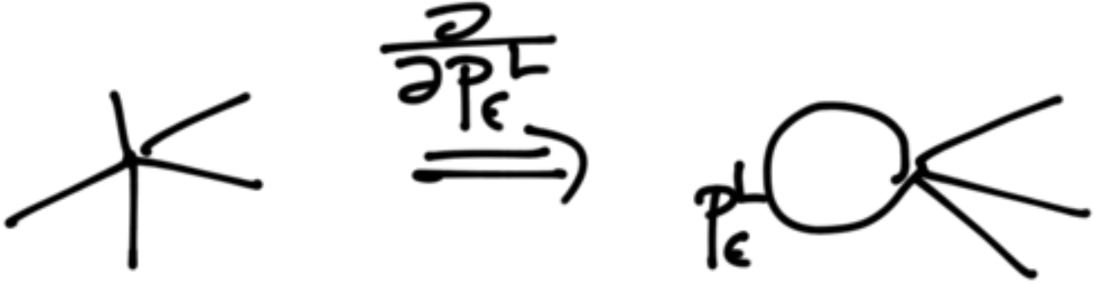
$$P_\epsilon^L = \int_\epsilon^L (Q^{GF} \otimes 1) e^{-t\Delta} dt \in \mathrm{Sym}^2(\mathcal{E})$$

which is called the <u>regularized propagator</u>.

We define

$$\frac{\partial}{\partial P_\epsilon^L} : \mathcal{O}^{(n)}(\mathcal{E}) \longmapsto \mathcal{O}^{(n-2)}(\mathcal{E})$$

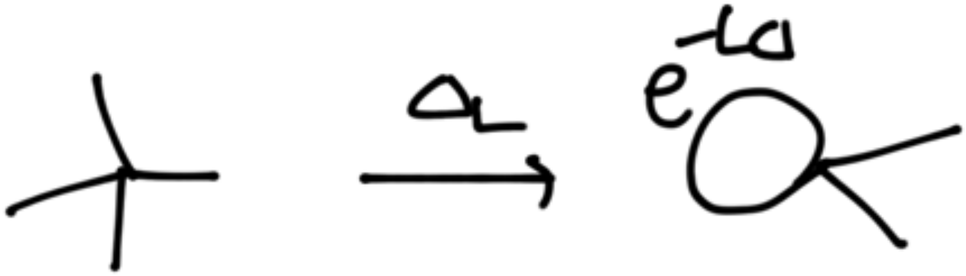
by the natural contraction



This is well-defined on distributions since P_ϵ^L is smooth for $0<\varepsilon<L<\infty$

Similarly, we define

$$\Delta_L = \text{Contraction w/. } e^{-LD} \quad (L>0)$$



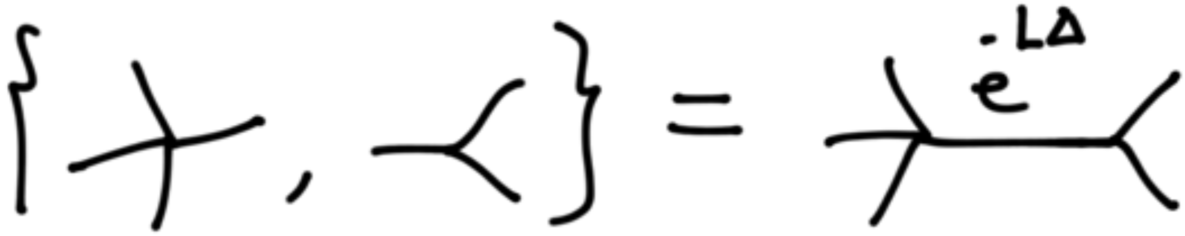
Then P_ϵ^L gives a homotopy

$$\left[Q, \frac{\partial}{\partial P_\epsilon^L}\right] = \Delta_\epsilon - \Delta_L$$

__Def'n__: The __regularized BV bracket__ at scale L is defined by

$$\{S_1, S_2\}_L = \Delta_L(S_1 S_2) - (\Delta_L S_1) S_2 - (-1)^{|S_1|} S_1 \Delta_L S_2$$

$$\forall\, S_1, S_2 \in \mathcal{O}(\mathcal{E})$$



__Rk__: If $S_1, S_2 \in \mathcal{O}_{loc}(\mathcal{E})$, then

$$\lim_{L\to 0} \{S_1, S_2\}_L = \{S_1, S_2\} \text{ exists.}$$

This is the classical bracket.

<u>Defn</u>: A perturbative quantization is given by a family of functional

$$F[L] = \sum_{g \geq 0} \hbar^g F_g[L] \quad \text{s.t.}$$

① Renormalization group flow equation

$$e^{F[L_2]/\hbar} = e^{\hbar \frac{\partial}{\partial P_\epsilon^L}} e^{F[L_1]/\hbar}$$

or equivalently

$$F[L_2] = \sum_{\Gamma: \text{conn}} \frac{w_\Gamma(P_{L_1}^{L_2}, F[L_1])}{|\text{Aut}\,\Gamma|}$$

② Quantum master equation

$$(Q + \hbar \Delta_L) e^{F[L]/\hbar} = 0$$

③ Classical limit: $\lim_{L \to 0} F_0[L] = I$

④ Locality: $F[L]$ is asymptotic local as $L \to 0$.

Rk ① RG gives the homotopy between QME at different scales.

② $F[\infty]$ models the full path integral in the perturbative sense.

Let $\mathcal{H} = \mathrm{Ker}([Q, Q^{GF}]) \subset \mathcal{E}$

be the subspace of Harmonic elements.

Then $F[\infty]$ solves the QME for Δ_∞ at $\mathcal{H}$. Since $\mathcal{H}$ is finite dim'l. In the case of cotangent theory

$$\mathcal{H} = T^*M[-1]$$

Costello) it corresponds to a Volume form on the moduli space $M = B_g$

· <u>Quantization via deformation</u>

How do we find such a family $F[L]$?

① RG flow is easy. We can always find counter terms s.t.

$$F[L] = \lim_{\varepsilon \to 0} \sum_{\Gamma : \text{conn}} \frac{W_\Gamma(P_\varepsilon^L, I + I^{CT}(\varepsilon))}{|\mathrm{Aut}\,\Gamma|}$$

then it satisfies RG

② QME may not be able to satisfy.

There's some intrinsic obstruction, which is called in physics "anomaly"

This is similar to the usual deformation that we discussed in the very beginning. The relevant complex is

$$\left(\mathcal{O}_{loc}(\mathcal{E}),\ Q + \{I, -\} \right)$$

$$\text{CME} \Rightarrow \left(Q + \{I, -\} \right)^2 = 0$$

The corresponding obstruction theory is developed by K. Costello

Prop. The obstruction space for quantization is

$$H^1\left(\mathcal{O}_{loc}(\mathcal{E}),\ Q + \{I, -\}\right)$$

The tangent space of moduli of quantization

$$H^0\left(\mathcal{O}_{loc}(\mathcal{E}), Q + \{I, -\}\right)$$