SOME CLASSICAL / QUANTUM ASPECTS OF CALABI-YAU MODULI

SI LI

ABSTRACT. We review some classical and quantum geometry of Calabi-Yau moduli related to B-model aspects of closed string mirror symmetry. This note comes out of the author's lectures in the workshop "B-model aspects of Gromov-Witten theory" held at University of Michigan in 2013.

CONTENTS

1. Introduction 1
2. Classical geometry 2
2.1. Deformation theory on Calabi-Yau and local moduli 2
2.2. Generalized period map and Frobenius manifold structure 6
2.3. Landau-Ginzburg model 12
2.4. Perturbative theory of primitive forms 17
3. Quantum geometry 20
3.1. A toy model of Weyl quantization 20
3.2. Symlectic geometry and BCOV theory 23
3.3. Givental's formalism via renormalization 24
3.4. Quantum BCOV theory 28
References 31

1. INTRODUCTION

Mirror symmetry is a physics-motivated duality between symplectic geometry (or the *A-model*) and complex geometry (or the *B-model*). In contrast to the A-model, Calabi-Yau condition is necessary for a well-defined B-model. In this article we discuss several aspects of local geometry on the moduli space in the B-model related to closed string mirror symmetry, focusing on compact Calabi-Yau models and Landau-Ginzburg models.

This article consists of two main parts: classical geometry (or the genus zero theory) and quantum geometry (or the higher genus theory). The geometry of genus zero theory can be summarized as defining the Frobenius manifold structure [14] on the local moduli space of Calabi-Yau geometry. It originated (called the *flat structure*) around early 1980's from K. Saito's theory of primitive forms [32, 33] in his study of period integrals over vanishing cycles associated to an isolated singularity. This has now become the geometric content of Landau-Ginzburg B-model encoding the genus zero correlation functions. K. Saito's construction was extended by Barannikov and Kontsevich [5] to compact Calabi-Yau models via tools of deformation theory, and packaged into the framework of variation of semi-infinite Hodge structures [2, 3]. The first part will be mainly reviewing this classical story. The quantum B-model on Calabi-Yau manifolds has a candidate in physics via the quantization of a gauge theory [6] (*Kodaira-Spencer gauge theory*) whose classical limit describes the deformation of complex structures. Geometrically, such quantization can be obtained as the infinite dimensional Weyl quantization with the help of renormalization techniques in quantum field theory [11]. This is a realization of the topological B-twisted closed string field theory in the sense of Zwiebach [38]. The second part will be focused on explaining this subject.

Acknowledgement: The authors would like to thank the organizers and participants of the workshop on B-model aspects of Gromov-Witten theory, and the hospitality of the mathematics department at University of Michigan.

2. Classical geometry

2.1. **Deformation theory on Calabi-Yau and local moduli.** We start with the deformation theory on Calabi-Yau manifolds via polyvector fields following [5].

2.1.1. *Polyvector fields.* Let X be a compact Calabi-Yau manifold of dimension d. Ω_X will be a fixed holomorphic volume form which is unique up to a constant. We consider

$$\mathrm{PV}(X) = \bigoplus_{0\leq i,j\leq d} \mathrm{PV}^{i,j}(X), \quad \mathrm{PV}^{i,j}(X) = \mathcal{A}^{0,j}(X, \wedge^i T_X)$$

the space of polyvector fields on X. Here T_X is the holomorphic tangent bundle, and $\mathcal{A}^{0,j}(X, \wedge^i T_X)$ is the space of smooth $(0,j)$-forms valued in $\wedge^i T_X$. $\mathrm{PV}(X)$ is a differential bi-graded commutative algebra: the differential is

$$\bar{\partial} : \mathrm{PV}^{i,j}(X) \to \mathrm{PV}^{i,j+1}(X),$$

and the algebra structure arises from wedge product. Our degree convention is that elements of $\mathrm{PV}^{i,j}(X)$ are of degree $j-i$. The graded-commutativity says

$$\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha$$

where $|\alpha|, |\beta|$ denote the degree of α, β respectively. Ω_X induces an identification between the space of polyvector fields and differential forms

$$\begin{aligned} \mathrm{PV}^{i,j}(X) &\overset{\lrcorner\Omega_X}{\cong} \mathcal{A}^{d-i,j}(X) \\ \alpha &\to \alpha \lrcorner \Omega_X \end{aligned}$$

where $\lrcorner$ is the contraction, and $\mathcal{A}^{i,j}(X)$ denotes smooth differential forms of type (i,j). The holomorphic de Rham differential ∂ on forms defines an operator on $\mathrm{PV}(X)$ via the above isomorphism, which we still denote by

$$\partial : \mathrm{PV}^{i,j}(X) \to \mathrm{PV}^{i-1,j}(X)$$

i.e.

$$(\partial\alpha)\lrcorner\Omega_X \equiv \partial(\alpha\lrcorner\Omega_X), \quad \alpha \in \mathrm{PV}(X).$$

The definition of ∂ doesn't depend on the choice of Ω_X on compact Calabi-Yau manifolds. It induces a bracket on polyvector fields (Bogomolov-Tian-Todorov lemma)

$$\{\alpha, \beta\} := \partial\left(\alpha\beta\right) - \left(\partial\alpha\right)\beta - (-1)^{|\alpha|}\alpha(\partial\beta)$$

which coincides with the Schouten-Nijenhuis bracket (up to a sign). The fundamental algebraic structures of polyvector fields on Calabi-Yau geometry can be summarized by saying that the tuple $\{(\mathrm{PV}(X), \bar{\partial}), \wedge, \partial, \{-,-\}\}$ defines a differential Gerstenhaber-Batalin-Vilkovisky (GBV) algebra.

We can integrate polyvector fields by the *trace map* $\mathrm{Tr} : \mathrm{PV}(X) \to \mathbb{C}$

$$\mathrm{Tr}(\alpha) := \int_X (\alpha\lrcorner\Omega_X) \wedge \Omega_X. \tag{2.1}$$

This is only non-vanishing on $\mathrm{PV}^{d,d}(X)$. Let $\langle -,- \rangle$ be the induced pairing $\mathrm{PV}(X) \otimes \mathrm{PV}(X) \to \mathbb{C}$

$$\alpha \otimes \beta \to \langle \alpha, \beta \rangle \equiv \mathrm{Tr}\left(\alpha\beta\right).$$

It is easy to see that $\bar{\partial}$ is (graded) skew self-adjoint for this pairing and ∂ is (graded) self-adjoint.

2.1.2. *Deformation of complex structures.* We are interested in the moduli space of complex structures on compact Calabi-Yau manifolds. The main local result is the smoothness of the moduli space (Bogomolov-Tian-Todorov Theorem), which is also a direct consequence of the differential GBV structure.

Let us fix a choice of Kähler metric on X. Locally, the deformation space of complex structure of X can be described by the space

$$\mathcal{M}^{cx} := \left\{ \mu \in \mathrm{PV}^{1,1}(X), \|\mu\| < \epsilon \,\middle|\, \bar{\partial}\mu + \frac{1}{2}\{\mu, \mu\} = 0, \bar{\partial}^*\mu = 0 \right\},$$

where ϵ is a sufficiently small number. Let $\mu_1 \in H^1(X, T_X)$ be a harmonic element with respect to the Kähler metric. μ_1 represents a tangent vector of the moduli space at the point X, i.e. a first order deformation. It can be extended to a genuine deformation

$$\mu_t = \sum_{k=1}^{\infty} t^k \mu_k \in \mathrm{PV}^{1,1}(X), \quad |t| << 1$$

by solving recursively (in order of powers of t)

$$\bar{\partial}\mu_t = -\frac{1}{2}\{\mu_t, \mu_t\}, \quad \bar{\partial}^*\mu_t = 0,$$

or equivalently by solving

$$\bar{\partial}\mu_i = -\frac{1}{2}\sum_{k=1}^{i-1}\{\mu_i, \mu_{k-i}\}, \quad i > 1, \quad \bar{\partial}^*\mu_i = 0.$$

For a general complex manifold, the harmonic part of the RHS may not be vanishing, representing the obstructions for solving the above equation. However, this does not happen for Calabi-Yau manifolds. Indeed, we can solve μ_t with the additional property that $\partial\mu = 0$. Suppose we have solved μ_k for $k < i$, with $\bar{\partial}^*\mu_k = \partial\mu_k = 0$. Bogomolov-Tian-Todorov lemma implies that

$$\{\mu_k, \mu_{i-k}\} = \partial(\mu_k \wedge \mu_{i_k}),$$

which has no harmonic component. It follows that μ_i can be solved by

$$\mu_i = -\frac{1}{2}\bar{\partial}^* G\partial(\sum_{k=1}^{i-1} \mu_i \wedge \mu_{k-i})$$

which satisfies $\bar{\partial}^*\mu_i = \partial\mu_i = 0$. Here $G = \frac{1}{\Delta}$ is the Green's operator for the Laplacian $\Delta = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ acting on $\mathrm{PV}(X)$. It can be further shown that the power series μ_t is convergent given t sufficiently small. This implies that the local deformation of complex structures on Calabi-Yau manifolds is unobstructed.

2.1.3. *Extended deformation space and the Formality Theorem.* We can consider the extended deformation space $\mathcal{M}$ [5] by solving

$$\bar{\partial}\mu + \frac{1}{2}\{\mu, \mu\} = 0$$

modulo gauge equivalence. Here μ is allowed to be polyvectors of all types. By the same argument as above, the deformation problem is unobstructed.

Remark 2.1. In this paper, we treat $\mathcal{M}$ as a formal graded manifold [5].

In [5], Barannikov and Kontsevich have introduced a remarkable way to organize the above argument via the *Formality Theorem*. The deformation problem is controlled by the differential graded Lie algebra (DGLA)

$$(\mathrm{PV}(X), \bar{\partial}, \{,\}).$$

There are two closely related DGLA's. The first one is

$$(\ker\partial, \bar{\partial}, \{,\}),$$

where $\ker\partial \subset \mathrm{PV}(X)$ is the subspace of polyvector fields annihilated by ∂. Bogomolov-Tian-Todorov lemma implies that $\{,\}$ is a well-defined Lie bracket on $\ker\partial$. In fact,

$$\{,\} : \ker\partial \times \ker\partial \to \mathrm{im}\,\partial \subset \ker\partial.$$

The second DGLA is

$$(\mathbb{H}, 0, 0)$$

where $\mathbb{H} \subset \mathrm{PV}(X)$ is the subspace of harmonic elements. We associate the trivial differential and Lie bracket. There is a well-defined diagram of morphisms of DGLA's

$$\begin{tikzcd} & (\ker\partial, \bar{\partial}, \{,\}) \arrow[dl, "j"'] \arrow[dr, "\pi"] & \\ (\mathrm{PV}(X), \bar{\partial}, \{,\}) & & (\mathbb{H}, 0, 0) \end{tikzcd}$$

where j is the natural embedding, and π is the harmonic projection. By Hodge theory, both j and π induce isomorphisms on the cohomology of the differential complex, hence quasi-isomorphisms of DGLA's. Since quasi-isomophisms can be inverted via L_∞ morphisms, we obtain the following Formality Theorem

Theorem 2.1 ([5]). *The DGLA $(\mathrm{PV}(X), \bar\partial, \{,\})$ is L_∞ quasi-isomorphic to the DGLA $(\mathbb{H}, 0, 0)$ of its cohomology.*

Quasi-isomorphic DGLA's have equivalent moduli functors. It follows that the extended deformation space is smooth, being locally parametrized by $\mathbb{H}$.

2.2. **Generalized period map and Frobenius manifold structure.** There is a line bundle $\mathcal{L}$ over $\mathcal{M}_X^{cx}$ whose fiber parametrizes the holomorphic volume forms. It gives rise to the period map (locally)

$$\mathcal{M}^{cx} \to \mathbb{P}(H^n(X, \mathbb{C}))$$

by sending $[X_t] \in \mathcal{M}^{cx}$ to the line in $H^n(X, \mathbb{C})$ representing the fiber of $\mathcal{L}$.

Period map here can be viewed as varying the holomorphic volume form along with the deformation of the complex structure. The choice of the deformation of the pair (X, Ω_X) can be described by a pair $(\mu, \rho) \in \mathrm{PV}^{1,1}(X) \oplus \mathrm{PV}^{0,0}(X)$ as follows. μ defines a deformation of complex structure solving

$$\bar\partial\mu + \frac{1}{2}\{\mu, \mu\} = 0.$$

It is easy to see that $e^{\mu} \lrcorner \Omega_X$ is of type $(n, 0)$ in the new complex structure μ. It differs from the new holomorphic volume form by a factor e^{ρ}, which solves the equation

$$d(e^{\rho} e^{\mu} \lrcorner \Omega_X) = 0.$$

This can be also read by

$$\bar\partial\mu + \frac{1}{2}\{\mu, \mu\} = 0, \quad \bar\partial\rho + \partial\mu + \{\mu, \rho\} = 0,$$

or simply

$$Q(\mu + z\rho) + \frac{1}{2}\{\mu + z\rho, \mu + z\rho\} = 0,$$

where $Q = \bar\partial + z\partial$ and z is a formal parameter.

Barannikov [2,3] extended the period map to the "generalized period" on the extended moduli space $\mathcal{M}$. It can be viewed as the compact Calabi-Yau analogue of Saito's primitive period map [32] for isolated singularities. We briefly review his construction here. Consider the new DGLA

$$(\mathrm{PV}(X)[[z]], Q, \{,\}).$$

Remark 2.2. The formal variable z is the same as $\hbar$ in [2,3].

Notation 2.1. Given a vector space A, $A[[z]]$ ($A((z))$ respectively) will denote the formal power series (Laurent series respectively) in z valued in A. $A[[\mathbf{u}]]$ will denote the formal power series in $\mathbf{u} = \{u^\alpha\}$ valued in A. If both sets of variables are involved, the topology is understood as follows: $A((z))[[\mathbf{u}]] \equiv B[[\mathbf{u}]]$ for $B = A((z))$, while $A[[\mathbf{u}]]((z)) \equiv C((z))$ for $C = A[[\mathbf{u}]]$, etc.

There exists universal solutions [3] (modulo gauge equivalence)

$$\mu(u,z) = \sum_\alpha \mu_\alpha(z) u^\alpha + \frac{1}{2}\sum_{\alpha,\beta} \mu_{\alpha\beta}(z) u^\alpha u^\beta + \cdots \in \mathrm{PV}(X)[[z]][[\mathbf{u}]]$$

to the associated Maurer-Cantan equation

$$Q\mu(u,z) + \frac{1}{2}\{\mu(u,z), \mu(u,z)\} = 0,$$

where u^α are the deformation parameters as coordinates on $\mathcal{M}$, and $\mu_\alpha(z)$ forms a $\mathbb{C}[[z]]$-basis of $H^*(\mathrm{PV}(X)[[z]], Q)$. It is direct to check that the Maurer-Cantan equation is formally equivalent to

$$Qe^{\mu(u,z)/z} = 0.$$

Note that in our notation, $e^{\mu(u,z)/z} \in \mathrm{PV}(X)((z))[[\mathbf{u}]]$.

Notation 2.2. Given $\mu \in \mathrm{PV}(X)[[z]]$ with $Q\mu = 0$, we will use $[\mu]$ to represent its cohomology class in $H^*(\mathrm{PV}(X)[[z]], Q)$. Similar notations apply to other cohomologies.

Let us define an isomorphism

$$\Gamma_\Omega : \mathrm{PV}(X)((z)) \to \mathcal{A}(X)((z)), \quad z^k\alpha \to z^{k+i-1}\alpha \lrcorner \Omega_X, \quad \alpha \in \mathrm{PV}^{i,j}(X). \tag{2.2}$$

It transfers Q to the de Rham differential

$$\Gamma_\Omega \circ Q = d \circ \Gamma.$$

As a result, the universal solutions $\mu(u,z)$ defines a cohomology class

$$\Gamma_\Omega(\left[ze^{\mu(u,z)/z}\right]) \in H^*(X, \mathbb{C})((z))[[\mathbf{u}]].$$

Definition 2.1. For simplicity, let us denote from now on by

$$S(X) := \mathrm{PV}(X)((z)), \quad S_+(X) := \mathrm{PV}(X)[[z]], \quad S_-(X) := z^{-1}\,\mathrm{PV}(X)[z^{-1}].$$

Lemma 2.1. *Under the isomorphism Γ_Ω, we have*

$$\Gamma_\Omega(S_+(X)) = \prod_{p\in\mathbb{Z}} z^{d-p+1} F^p \mathcal{A}(X),$$

where $F^p\mathcal{A}(X) = \mathcal{A}^{\geq p,*}(X)$. At the cohomology level we have an isomorphism

$$\Gamma_\Omega : H^*(S_+(X), Q) \overset{\cong}{\to} \prod_{p\in\mathbb{Z}} z^{d-p+1} F^p H^*(X, \mathbb{C}).$$

Similarly

$$\Gamma_\Omega : H^*(S(X), Q) \overset{\cong}{\to} H^*(X, \mathbb{C})((z)).$$

Definition 2.2. We define a symplectic pairing on $S(X)$ by

$$\omega(f(z)\alpha, g(z)\beta) := \mathrm{Res}_{z=0}\,(f(z)g(-z)dz)\,\mathrm{Tr}(\alpha\beta).$$

The differential Q is (graded) skew-symmetric with respect to the symplectic pairing ω. Therefore ω descends to define a symplectic pairing on the cohomology $H^*(S(X), Q)$, where $H^*(S_+(X), Q)$ becomes an isotropic subspace.

Definition 2.3. An opposite filtration of $H^*(S(X), Q)$ is a linear isotropic subspace $\mathcal{L} \subset H^*(S(X), Q)$ such that

(1) $H^*(S(X), Q) = H^*(S_+(X), Q) \oplus \mathcal{L}$,
(2) $\mathcal{L}$ is preserved by the operator $z^{-1} : H^*(S(X), Q) \to H^*(S(X), Q)$.

The subspaces $z^k H^*(S_+(X), Q) \subset H^*(S_+(X), Q), k \geq 0$, defines a decreasing filtration, whose associated graded space is

$$\mathrm{Gr}\, H^*(S_+(X), Q) \cong H^*(X, \wedge^* T_X)[[z]].$$

It is easy to see that under Γ_Ω, this filtration can be identified with the Hodge filtration, and $\mathcal{L}$ is equivalent to an opposite splitting filtration. Given an opposite filtration $\mathcal{L}$, it defines us a splitting projection

$$\pi_+^{\mathcal{L}} : H^*(S(X), Q) \to H^*(S_+(X), Q),$$

and an isomorphism of vector spaces

$$H^*(S_+(X), Q)/zH^*(S_+(X), Q) \cong H^*(S_+(X), Q) \cap z\mathcal{L},$$

which further induces an isomorphism of $\mathbb{C}[[z]]$-modules

$$\mathrm{Gr}\, H^*(S_+(X), Q) \cong H^*(S_+(X), Q).$$

Definition 2.4. $\mathcal{L}$ leads to a choice of $\mathbb{C}[[z]]$-basis of $H^*(S_+(X), Q)$ by $H^*(S_+(X), Q) \cap z\mathcal{L}$. We will let $\{\mu_\alpha^{\mathcal{L}}\}_\alpha$ denote such a basis that

$$H^*(S_+(X), Q) \cap z\mathcal{L} = \mathrm{Span}_{\mathbb{C}}\{[\mu_\alpha^{\mathcal{L}}]\}.$$

Proposition 2.1. *Given an opposite filtration $\mathcal{L}$, there exists a universal solution of the form*

$$\mu^{\mathcal{L}}(\tau,z)=\sum_{\alpha}\mu^{\mathcal{L}}_{\alpha}\tau^{\alpha}_{z}+\frac{1}{2}\sum_{\alpha,\beta}\mu_{\alpha\beta}(z)\tau^{\alpha}_{z}\tau^{\beta}_{z}+\cdots\in \mathrm{PV}(X)[[z]][[\boldsymbol{\tau}]],$$

where $\boldsymbol{\tau}=\{\tau^{\alpha}\}$ are coordinates on $\mathcal{M}$, $\tau^{\alpha}_{z}=\tau^{\alpha}+O(\tau^{2})\in\mathbb{C}[[z]][[\boldsymbol{\tau}]]$ such that

$$\pi^{\mathcal{L}}_{+}(\left[ze^{\mu^{\mathcal{L}}(\tau,z)/z}-z\right])=\sum_{\alpha}\mu^{\mathcal{L}}_{\alpha}\tau^{\alpha}.$$

Proof. Up to a (z-dependent) linear change of coordinates on u, we can assume that the universal solution $\mu(u,z)$ is of the form

$$\mu(u,z)=\sum_{\alpha}\mu^{\mathcal{L}}_{\alpha}u^{\alpha}+O(u^{2}).$$

Consider the projection

$$\pi^{\mathcal{L}}_{+}(\left[ze^{\mu(u,z)/z}-z\right])\in H^{*}(S_{+}(X),Q)[[\mathbf{u}]].$$

Since $\mu^{\mathcal{L}}_{\alpha}$ forms a $\mathbb{C}[[z]]$-basis of $H^{*}(S_{+}(X),Q)$, we can write

$$\pi^{\mathcal{L}}_{+}(\left[ze^{\mu(u,z)/z}-z\right])=\sum_{\alpha}\mu^{\mathcal{L}}_{\alpha}\tau^{\alpha}(u,z)$$

where

$$\tau^{\alpha}(u,z)=u^{\alpha}+O(u^{2})\in\mathbb{C}[[z,\mathbf{u}]].$$

In particular, u^{α} can be solved in terms of τ^{α},z by

$$u^{\alpha}(\tau,z)=\tau^{\alpha}+O(\tau^{2})\in\mathbb{C}[[z,\boldsymbol{\tau}]].$$

Then $\mu^{\mathcal{L}}(\tau,z)=\mu(u(\tau,z),z)$. □

In particular, we find the relation

$$\left[ze^{\mu^{\mathcal{L}}(\tau,z)/z}\right]\in z+\sum_{\alpha}\left[\mu^{\mathcal{L}}_{\alpha}\right]\tau^{\alpha}+\mathcal{L}[[\boldsymbol{\tau}]], \tag{2.3}$$

where $\mathcal{L}=z^{-1}\mathrm{Span}_{\mathbb{C}}\{[\mu^{\mathcal{L}}_{\alpha}]\}[z^{-1}]$.

Definition 2.5. [3] Given an opposite filtration $\mathcal{L}$, we define the generalized period map

$$\Pi^{\mathcal{L}}:\mathcal{M}\to H^{*}(X,\mathbb{C})$$

as the map of formal (graded) manifolds from $(\mathcal{M},0)$ to $(H^{*}(X,\mathbb{C}),\Omega_{X})$ by

$$\tau^{\alpha}\to\Gamma_{\Omega}(\left[ze^{\mu^{\mathcal{L}}(\tau,z)/z}\right])|_{z=1}.$$

It is easy to see that $\Pi^{\mathcal{L}}$ is an isomorphism of formal graded manifolds.

2.2.1. *Frobenius manifold structure.* Now we explain Barannikov's formulation [2,3] of Frobenius manifold structure on $\mathcal{M}$ associated to an opposite filtration.

Definition 2.6. Let $\mathcal{H} \equiv H^*(S(X), Q)$, and let $\mathcal{H}^{(0)}_{\mathcal{M}} \subset \mathcal{H}[[\boldsymbol{\tau}]]$ be the free $\mathbb{C}[[z]][[\boldsymbol{\tau}]]$-module generated by $[z\partial_{\tau^\alpha} e^{\mu(\tau,z)/z}]$. The symplectic pairing ω extends $\mathbb{C}[[\boldsymbol{\tau}]]$-linearly to

$$\omega : \mathcal{H}[[\boldsymbol{\tau}]] \otimes_{\mathbb{C}[[\boldsymbol{\tau}]]} \mathcal{H}[[\boldsymbol{\tau}]] \to \mathbb{C}[[\boldsymbol{\tau}]]$$

which we denote by the same symbol.

Lemma 2.2. $\mathcal{H}[[\boldsymbol{\tau}]] = \mathcal{H}^{(0)}_{\mathcal{M}} \oplus \mathcal{L}[[\boldsymbol{\tau}]]$. *Moreover, this is an isotropic decomposition, i.e.* $\omega(\mathcal{H}^{(0)}_{\mathcal{M}}, \mathcal{H}^{(0)}_{\mathcal{M}}) = 0$.

Proof. The decomposition $\mathcal{H}[[\boldsymbol{\tau}]] = \mathcal{H}^{(0)}_{\mathcal{M}} \oplus \mathcal{L}[[\boldsymbol{\tau}]]$ follows from (2.3) since $\left[z\partial_{\tau^\alpha} e^{\mu(\tau,z)/z}\right] \in \mu^{\mathcal{L}}_\alpha + \mathcal{L}[[\boldsymbol{\tau}]]$. To see $\mathcal{H}^{(0)}_{\mathcal{M}}$ is an isotropic subspace,

$$\begin{aligned} &\omega(a(z)z\partial_{\tau^\alpha} e^{\mu(\tau,z)/z}, b(z)z\partial_{\tau^\beta} e^{\mu(\tau,z)/z}) \\ =& \operatorname{Res}_{z=0} \operatorname{Tr}(a(z)b(-z)\partial_{\tau^\alpha}\mu(\tau,z)\partial_{\tau^\beta}\mu(\tau,-z)e^{(\mu(\tau,z)-\mu(\tau,-z))/z})dz. \end{aligned}$$

If $a(z), b(z)$ contains only non-negative powers of z, the expression inside Tr has only non-negative powers of z whose residue vanishes. □

Lemma 2.3 (Transversality)**.** $\partial_{\tau^\alpha} : \mathcal{H}^{(0)}_{\mathcal{M}} \to z^{-1}\mathcal{H}^{(0)}_{\mathcal{M}}$.

Proof. By Lemma 2.2, we only need to show that $\omega(z\partial_{\tau^\alpha}\mathcal{H}^{(0)}_{\mathcal{M}}, \mathcal{H}^{(0)}_{\mathcal{M}}) = 0$, which follows from a similar calculation as in Lemma 2.2. □

Corollary 2.1. *There exists* $A^\gamma_{\alpha\beta}(\boldsymbol{\tau}) \in \mathbb{C}[[\boldsymbol{\tau}]]$ *such that*

$$(\partial_{\tau^\alpha}\partial_{\tau^\beta} - z^{-1}A^\gamma_{\alpha\beta}(\boldsymbol{\tau})\partial_{\tau^\gamma})\left[e^{\mu(\tau,z)/z}\right] = 0.$$

Proof. By Equation (2.3) and Lemma 2.2, $[z\partial_{\tau^\alpha} e^{\mu(\tau,z)/z}]$ forms a $\mathbb{C}[[\boldsymbol{\tau}]]$-basis of $\mathcal{H}^{(0)}_{\mathcal{M}} \cap z\mathcal{L}[[\boldsymbol{\tau}]]$. By Equation (2.3) and Lemma 2.3,

$$z\partial_{\tau^\beta}[z\partial_{\tau^\alpha} e^{\mu(\tau,z)/z}] \in \mathcal{H}^{(0)}_{\mathcal{M}} \cap z\mathcal{L}[[\boldsymbol{\tau}]],$$

hence a $\mathbb{C}[[\boldsymbol{\tau}]]$-linear combination of $\{[z\partial_{\tau^\alpha} e^{\mu(\tau,z)/z}]\}_\alpha$. □

The following corollary is a direct consequence.

Corollary 2.2. *The generalized period satisfies*

$$(\partial_{\tau^\alpha}\partial_{\tau^\beta} - A^\gamma_{\alpha\beta}(\boldsymbol{\tau})\partial_{\tau^\gamma})\Pi^{\mathcal{L}} = 0.$$

Let us define a metric by

$$g_{\alpha\beta} := \omega(\partial_{\tau^\alpha} e^{\mu(\tau,z)/z}, z\partial_{\tau^\beta} e^{\mu(\tau,z)/z}).$$

Lemma 2.4. *$g_{\alpha\beta}$ is a non-degenerate constant matrix.*

Proof. It follows from Equation (2.3) that $g_{\alpha\beta} = \mathrm{Tr}(\mu_\alpha^{\mathcal{L}} \wedge \mu_\beta^{\mathcal{L}})$. □

Corollary 2.3. *Let $A_{\alpha\beta\gamma}(\boldsymbol{\tau}) := \sum_\delta A_{\alpha\beta}^\delta(\boldsymbol{\tau}) g_{\delta\gamma}$. Then $A_{\alpha\beta\gamma}(\boldsymbol{\tau})$ is (graded) symmetric in α, β, γ.*

Proof. This follows from $\partial_\gamma g_{\alpha\beta} = 0$. □

Lemma 2.5. *$A_{\alpha\beta}^\gamma(\boldsymbol{\tau}) \in \mathbb{C}[[\boldsymbol{\tau}]]$ satisfies the WDVV equation.*

Proof. Define the Dubrovin connection

$$\nabla_{\tau^\alpha} = \partial_{\tau^\alpha} - z^{-1} A_\alpha,$$

where A_α is the $\mathbb{C}[[z]][[\tau]]$-linear transformation on $\mathcal{H}_{\mathcal{M}}^{(0)}$ defined on the basis by

$$A_\alpha : \left[z\partial_{\tau^\beta} e^{\mu(\tau,z)/z} \right] \to \sum_\gamma A_{\alpha\beta}^\gamma \left[z\partial_{\tau^\gamma} e^{\mu(\tau,z)/z} \right].$$

Then $\nabla \left[z\partial_{\tau^\alpha} e^{\mu(\tau,z)/z} \right] = 0$ on the basis. The WDVV equation is equivalent to $\nabla^2 = 0$, which follows from the curvature condition. □

The properties above can be summarized as follows. The triple $(\partial_{\tau^\alpha}, A_{\alpha\beta}^\gamma, g_{\alpha\beta})$ defines a (formal) Frobenius manifold structure on $\mathcal{M}$, with τ^α being the flat coordinates. In particular, there exists a function $\mathcal{F}_0^{\mathcal{L}}(\boldsymbol{\tau})$ satisfying

$$A_{\alpha\beta\gamma}(\boldsymbol{\tau}) = \partial_{\tau^\alpha} \partial_{\tau^\beta} \partial_{\tau^\gamma} \mathcal{F}_0^{\mathcal{L}}(\boldsymbol{\tau}).$$

There also exists the Euler vector field and identity vector field.

$\mathcal{F}_0^{\mathcal{L}}(\boldsymbol{\tau})$ is called the prepotential, encoding the genus zero correlation functions in the Calabi-Yau B-model. It depends on the choice of the opposite filtration $\mathcal{L}$. When X is around the large complex limit, the degeneration leads to an opposite Monodromy weight filtration, and $\mathcal{F}_0^{\mathcal{L}}(\boldsymbol{\tau})$ is identified with the genus zero Gromov-Witten invariants of the mirror Calabi-Yau for a large class of examples [3, 17, 29].

2.3. **Landau-Ginzburg model.** Now we move to the Landau-Ginzburg B-model. We will focus on an isolated singularity defined by a weighted homogeneous polynomial

$$f : X = \mathbb{C}^n \to \mathbb{C}, \quad f(\lambda^{q_1} x_1, \cdots, \lambda^{q_n} x_n) = \lambda f(x_1, \cdots, x_n).$$

q_i are called the weights of x_i, and the central charge of f is defined by

$$\hat{c}_f = \sum_i (1 - 2q_i).$$

Associated to f, K. Saito has introduced the concept of a primitive form [32], which induces a Frobenius manifold structure (originally called a flat structure) on the local universal deformation space of f. The construction of primitive forms for arbitrary isolated singularities is later fully established by M. Saito [35]. See also [4, 12, 13, 37] for generalizations to certain class of Laurent polynomials. This gives rise to the genus zero correlation functions in the Landau-Ginzburg B-model. The generalized period map for compact Calabi-Yau manifolds can be viewed as the analogue of primitive period map.

In this rest of this section, we will give a brief review of primitive forms. Our presentation will base on the work [21], which exhibits a unified geometry of Landau-Ginzburg and Calabi-Yau models. We will also describe the perburbative formula of primitive forms [21] which is fully developed in [22, 27] to prove the mirror symmetry conjecture between Landau-Ginzburg models.

2.3.1. *Universal unfolding.* The DGLA controlling the deformation theory has a natural twisting in the Landau-Ginzburg case

$$\left(\mathrm{PV}(X), \bar{\partial}_f, \{,\}\right), \quad \bar{\partial}_f = \bar{\partial} + df \lrcorner,$$

where $df\lrcorner$ is the contraction with the holomorphic 1-form df. We will be also working with a subcomplex

$$\mathrm{PV}_c(X) \subset \mathrm{PV}(X)$$

of polyvector fields with compact support. Since $X = \mathbb{C}^n$ is Stein, we have

Lemma 2.6. *The embedding* $(\mathrm{PV}_c(X), \bar{\partial}_f) \hookrightarrow (\mathrm{PV}(X), \bar{\partial}_f)$ *is quasi-isomorphic. The cohomology is given by*

$$H^*(\mathrm{PV}(X), \bar{\partial}_f) \cong \mathrm{Jac}_0(f),$$

where $\mathrm{Jac}_{\mathbf{0}}(f) = \mathbb{C}\{x^i\}/\{\partial_i f\}$ *is the Milnor ring of the isolated singularity.*

It follows that in the Landau-Ginzburg case, the universal solutions of the associated Maurer-Cartan equation is greatly simplified, and can be represented as a deformation of $f(\mathbf{x})$ via the universal unfolding:

$$F:\mathbb{C}^n\times\mathbb{C}^\mu\to\mathbb{C},\quad F(\mathbf{x},\mathbf{s}):=f(\mathbf{x})+\sum_{\alpha=1}^{\mu}s_\alpha\phi_\alpha(\mathbf{x}),\quad \mathbf{s}=(s_1,\cdots,s_\mu).$$

where $\mu=\dim_{\mathbb{C}}\operatorname{Jac}_{\mathbf{0}}(f)$, and $\{\phi_\alpha(\mathbf{x})\}$ is a basis of $\operatorname{Jac}_{\mathbf{0}}(f)$.

In the case f being weighted homogenous, we can further assume that ϕ_α are all weighted homogeneous with increasing degrees

$$0=\deg(\phi_1)\le\deg(\phi_2)\le\cdots\le\deg(\phi_\mu)=\hat{c}_f,\quad \text{where }\deg(x_i)=q_i.$$

We will extend our weight degree assignment to the deformation parameter

$$\deg(s_\alpha):=1-\deg(\phi_\alpha)$$

such that F becomes weighted homogeneous of total degree 1. Let us denote by

$$\mathcal{M}:=(\mathbb{C}^\mu,\mathbf{0})$$

the germ around $\mathbf{0}\in\mathbb{C}^\mu$, parametrizing the local deformation space. $\{s_\alpha\}$ is viewed as a coordinate system on $\mathcal{M}$.

Let $\Omega:=dx^1\wedge\cdots\wedge dx^n$ be our fixed holomorphic volume form. Let $\Omega^k_{X,\mathbf{0}}$ be the germ of holomorphic k-forms at $\mathbf{0}$.

Definition 2.7. $\Omega_f:=\Omega^n_{X,\mathbf{0}}/df\wedge\Omega^{n-1}_{X,\mathbf{0}}$.

With our choice of Ω, we can identify

$$\operatorname{Jac}_{\mathbf{0}}(f)\to\Omega_f,\quad [\phi]\to[\phi\Omega].$$

There exists a classical residue pairing defined on Ω_f:

$$\eta_f:\Omega_f\otimes\Omega_f\to\mathbb{C}.$$

This has an alternate geometric description as follows. Recall the trace map

$$\operatorname{Tr}:\mathrm{PV}_c(X)\to\mathbb{C},\quad \mu\to\int_X\mu\lrcorner\Omega\wedge\Omega$$

is well-defined on $\mathrm{PV}_c(X)$. It is easy to see that it descends to cohomologies

$$\operatorname{Tr}:H^*(\mathrm{PV}_c(X),\bar\partial_f)\to\mathbb{C}.$$

Proposition 2.2. [21] *Let $\iota:H^*(\mathrm{PV}_c(X),\bar\partial_f)\to H^*(\mathrm{PV}(X),\bar\partial_f)$ denote the isomorphism as in Lemma 2.6. Then the residue pairing is related to the trace map by*

$$\eta_f([\phi_1\Omega],[\phi_2\Omega])=\operatorname{Tr}(\iota^{-1}([\phi_1])\wedge\iota^{-1}([\phi_2])),\quad\forall[\phi_i]\in\operatorname{Jac}_{\mathbf{0}}(f).$$

2.3.2. *Brieskorn lattice and higher residues.* Analogous to the Calabi-Yau case, we consider the following extended DGLA

$$(\mathrm{PV}(X)[[z]], Q_f, \{,\}), \quad Q_f := \bar{\partial}_f + z\partial_\Omega,$$

where ∂_Ω is defined with respect to the volume form Ω.

Definition 2.8. [33] Define $\mathcal{H}_f^{(0)} := \Omega^n_{X,\mathbf{0}}[[z]]/(df + zd)\Omega^{n-1}_{X,\mathbf{0}}$ the (formally completed) *Brieskorn lattice* associated to f.

Lemma 2.7. [21] *The embedding* $(\mathrm{PV}_c(X)[[z]], Q_f) \hookrightarrow (\mathrm{PV}(X)[[z]], Q_f)$ *is a quasi-isomorphism. It induces isomorphisms*

$$H^*(\mathrm{PV}(X)[[z]], Q_f) \cong H^0(\mathrm{PV}(X)[[z]], Q_f) \overset{z\Gamma_\Omega}{\cong} \mathcal{H}_f^{(0)},$$

where Γ_Ω *is defined the same as in* (2.2).

There is a similar semi-infinite Hodge filtration on $\mathcal{H}_f^{(0)}$ given by $\mathcal{H}_f^{(-k)} := z^k\mathcal{H}_f^{(0)}$, with graded pieces

$$\mathcal{H}_f^{(-k)}/\mathcal{H}_f^{(-k-1)} \cong \Omega_f.$$

In particular, $\mathcal{H}_f^{(0)}$ is a free $\mathbb{C}[[z]]$-module of rank μ. We will also denote the extension to Laurent series by

$$\mathcal{H}_f := \mathcal{H}_f^{(0)} \otimes_{\mathbb{C}[[z]]} \mathbb{C}((z)).$$

There is a natural $\mathbb{Q}$-grading on $\mathcal{H}_f^{(0)}$ defined by assigning the weight degrees

$$\deg(x_i) = q_i, \quad \deg(dx_i) = q_i, \quad \deg(z) = 1.$$

For a homogeneous element of the form $\varphi = z^k g(x_i) dx_1 \wedge \cdots \wedge dx_n$, we define

$$\deg(\varphi) = \deg(g) + k + \sum_i q_i.$$

In [33], K. Saito constructed a *higher residue pairing*

$$K_f : \mathcal{H}_f^{(0)} \otimes \mathcal{H}_f^{(0)} \to z^n\mathbb{C}[[z]]$$

which satisfies the following properties

(1) K_f is equivariant with respect to the $\mathbb{Q}$-grading, i.e.,

$$\deg(K_f(\alpha, \beta)) = \deg(\alpha) + \deg(\beta)$$

for homogeneous elements $\alpha, \beta \in \mathcal{H}_f^{(0)}$.

(2) $K_f(\alpha, \beta) = (-1)^n\overline{K_f(\beta, \alpha)}$, where the $-$ operator takes $z \to -z$.

(3) $K_f(v(z)\alpha, \beta) = K_f(\alpha, v(-z)\beta) = v(z)K_f(\alpha, \beta)$ for $v(z) \in \mathbb{C}[[z]]$.
(4) The leading z-order of K_f defines a pairing

$$\mathcal{H}_f^{(0)}/z\mathcal{H}_f^{(0)} \otimes \mathcal{H}_f^{(0)}/z\mathcal{H}_f^{(0)} \to \mathbb{C}, \quad \alpha \otimes \beta \mapsto \lim_{z\to 0} z^{-n} K_f(\alpha, \beta)$$

which coincides with the usual residue pairing

$$\eta_f : \Omega_f \otimes \Omega_f \to \mathbb{C}.$$

The last property implies that K_f defines a semi-infinite extension of the residue pairing, which explains the name "higher residue". An alternate way to understand the higher residue pairing is through the trace map in the spirit of Proposition 2.2. Let us define a pairing

$$\tilde{K}_f : \mathrm{PV}_c(X)[[z]] \times \mathrm{PV}_c(X)[[z]] \to z^n\mathbb{C}[[z]], \quad \tilde{K}_f(f(z)\alpha, g(z)\beta) = z^n f(z)g(-z)\,\mathrm{Tr}(\alpha\beta).$$

It is easy to see that $\tilde{K}_f$ descends to $H^*(\mathrm{PV}_c(X)[[z]], Q_f)$ which is canonically isomorphic to $H^*(\mathrm{PV}(X)[[z]], Q_f)$.

Proposition 2.3. [21] *$\tilde{K}_f$ coincides with K_f under the isomorphism $H^*(\mathrm{PV}(X)[[z]], Q_f) \cong \mathcal{H}_f^{(0)}$ as in Lemma 2.7.*

The Brieskorn lattice and the higher residue pairing can be extended to the family case on the germ $\mathcal{M}$ associated to the unfolding F. We have

$$\mathcal{H}_F^{(0)} := \Omega^n_{X\times\mathcal{M}/\mathcal{M},\mathbf{0}}[[z]]/(dF + zd)\Omega^{n-1}_{X\times\mathcal{M}/\mathcal{M},\mathbf{0}}$$

where $\Omega^*_{X\times\mathcal{M}/\mathcal{M},\mathbf{0}}$ is the germ of the sheaf of relative holomorphic differential forms at $\mathbf{0}$. It can be viewed as a free sheaf of rank μ on $\mathcal{M} \times \hat{\Delta}$, where $\hat{\Delta}$ is the formal disk with parameter z. $\mathcal{H}_F^{(0)}$ is equipped with a flat Gauss-Manin connection on $\mathcal{M} \times \hat{\Delta}$, denoted by ∇^{GM}. The higher residue pairing extends to

$$K_F : \mathcal{H}_F^{(0)} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{H}_F^{(0)} \to z^n \mathcal{O}_\mathcal{M}[[z]]$$

satisfying the following properties

(1) $K_F(s_1, s_2) = (-1)^n \overline{K_F(s_2, s_1)}$, where $-$ is the operator $z \to -z$.
(2) $K_F(g(z)s_1, s_2) = K_F(s_1, g(-z)s_2) = g(z)K_F(s_1, s_2)$ for any $g \in \mathcal{O}_\mathcal{M}[[z]]$.
(3) $\partial_V K_F(s_1, s_2) = K_F(\nabla^{GM}_V s_1, s_2) + K_F(s_1, \nabla^{GM}_V s_2)$ for any $V \in T_\mathcal{M}$.
(4) $z\partial_z K_F(s_1, s_2) = K_F(\nabla^{GM}_{z\partial_z} s_1, s_2) + K_F(s_1, \nabla^{GM}_{z\partial_z} s_2)$.
(5) The induced pairing

$$\mathcal{H}_F^{(0)}/z\mathcal{H}_F^{(0)} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{H}_F^{(0)}/z\mathcal{H}_F^{(0)} \to \mathcal{O}_\mathcal{M}$$

coincides with the classical residue pairing.

2.3.3. *Primitive forms.*

Definition 2.9. A section $\zeta \in \mathcal{H}_F^{(0)}$ is called a *primitive form* if it satisfies the following conditions:

(1) (Primitivity) The section ζ induces an $\mathcal{O}_{\mathcal{M}}$-module isomorphism
$$z\nabla^{GM}\zeta : T_{\mathcal{M}} \to \mathcal{H}_F^{(0)}/z\mathcal{H}_F^{(0)}; \quad V \mapsto z\nabla_V^{GM}\zeta.$$
(2) (Orthogonality) For any local sections V_1, V_2 of $T_{\mathcal{M}}$,
$$K_F\left(\nabla_{V_1}^{GM}\zeta, \nabla_{V_2}^{GM}\zeta\right) \in z^{n-2}\mathcal{O}_{\mathcal{M}}.$$
(3) (Holonomicity) For any local sections V_1, V_2, V_3 of $T_{\mathcal{M}}$,
$$K_F\left(\nabla_{V_1}^{GM}\nabla_{V_2}^{GM}\zeta, \nabla_{V_3}^{GM}\zeta\right) \in z^{n-3}\mathcal{O}_{\mathcal{M}} \oplus z^{n-2}\mathcal{O}_{\mathcal{M}};$$
$$K_F\left(\nabla_{z\partial_z}^{GM}\nabla_{V_1}^{GM}\zeta, \nabla_{V_2}^{GM}\zeta\right) \in z^{n-3}\mathcal{O}_{\mathcal{M}} \oplus z^{n-2}\mathcal{O}_{\mathcal{M}}.$$
(4) (Homogeneity) There is a constant $r \in \mathbb{C}$ such that
$$\left(\nabla_{z\partial_z}^{\Omega} + \nabla_E^{\Omega}\right)\zeta = r\zeta.$$
where E is the Euler vector field. In the case of weighted homogeneous singularity, we have $r = \sum_i q^i$.

The space of primitive forms has a geometric description. Let us extend the higher residue pairing to
$$K_f : \mathcal{H}_f \otimes \mathcal{H}_f \to \mathbb{C}((z)).$$
This defines a symplectic pairing ω_f on $\mathcal{H}_f$ by
$$\omega_f(\alpha, \beta) := \mathrm{Res}_{z=0}\, z^{-n}K_f(\alpha, \beta)dz,$$
with $\mathcal{H}_f^{(0)}$ being an isotropic subspace. Following [32],

Definition 2.10. A good section σ is a splitting of the quotient $\mathcal{H}_f^{(0)} \to \Omega_f$:
$$\sigma : \Omega_f \to \mathcal{H}_f^{(0)},$$
such that: (1) σ preserves the $\mathbb{Q}$-grading; (2) $K_f(\mathrm{Im}(\sigma), \mathrm{Im}(\sigma)) \subset z^n\mathbb{C}$. A basis of the image $\mathrm{Im}(\sigma)$ of a good section σ will be called a good basis of $\mathcal{H}_f^{(0)}$.

Definition 2.11. A good opposite filtration $\mathcal{L}$ is defined by a splitting
$$\mathcal{H}_f = \mathcal{H}_f^{(0)} \oplus \mathcal{L}$$
such that: (1) $\mathcal{L}$ preserves the $\mathbb{Q}$-grading; (2) $\mathcal{L}$ is an isotropic subspace; (3) $z^{-1} : \mathcal{L} \to \mathcal{L}$.

Remark 2.3. Here for f being weighted homogeneous, (1) is equivalent to the conventional condition that $\nabla^{GM}_{z\partial_z}$ preserves $\mathcal{L}$ (see e.g. [21] for an exposition).

The above two definitions are equivalent. In fact, a good opposite filtration $\mathcal{L}$ defines the splitting $\sigma : \Omega_f \xrightarrow{\cong} \mathcal{H}_f^{(0)} \cap z\mathcal{L}$. Conversely, a good section σ gives rise to the good opposite filtration $\mathcal{L} = z^{-1}\,\mathrm{Im}(\sigma)[z^{-1}]$. As shown in [32,35], the primitive forms associated to weighted homogeneous singularities are in one-to-one correspondence with good sections (up to a nonzero scalar). We remark that for general isolated singularities, we need the notion of *very good sections* [35,36] in order to incorporate with the monodromy.

Theorem 2.2. [32] *The space of primitive forms of f up to rescaling by a constant is isomorphic to the space of good sections.*

Remark 2.4. The generalization of this identification to arbitrary isolated singularities is established by M. Saito [35,36].

2.4. **Perturbative theory of primitive forms.** In this subsection, we describe the algebraic algorithm [21,22] to compute the primitive form, flat coordinates and the prepotential with respect to a good basis.

We start with a good basis $\{[\phi_\alpha \Omega]\}_{\alpha=1}^{\mu}$ of $\mathcal{H}_f^{(0)}$, where $\{\phi_\alpha\}_{\alpha=1}^{\mu}$ are weighted homogeneous polynomials in $\mathbb{C}[\mathbf{x}]$ that represent a basis of $\mathrm{Jac}_{\mathbf{0}}(f)$ and $\phi_1 = 1$.

2.4.1. *The exponential map.* Let F be a local universal unfolding of $f(\mathbf{x})$

$$F(\mathbf{x}, \mathbf{s}) := f(\mathbf{x}) + \sum_{\alpha=1}^{\mu} s_\alpha \phi_\alpha(\mathbf{x}), \quad \mathbf{s} = (s_1, \cdots, s_\mu).$$

Let $B := \mathrm{Span}_{\mathbb{C}}\{[\phi_\alpha \Omega]\} \subset \mathcal{H}_f^{(0)}$ be spanned by the chosen good basis. Then

$$\mathcal{H}_f^{(0)} = B[[z]], \quad \mathcal{H}_f = B((z)).$$

Let $B_F := \mathrm{Span}_{\mathbb{C}}\{\phi_\alpha \Omega\}$ be the vector space spanned by the forms $\phi_\alpha \Omega$. We use a different notation to distinguish it with B, since B_F should be viewed as a subspace of the Brieskorn lattice for the unfolding F. See [21,22] for more details. Consider the following exponential operator [21,22]

$$e^{(F-f)/z} : B_F \to B((z))[[\mathbf{s}]]$$

defined as a $\mathbb{C}$-linear map on the basis of B_F as follows. Let

$$\mathbb{C}[\mathbf{s}]_k := \mathrm{Sym}^k(\mathrm{Span}_{\mathbb{C}}\{s_1, \cdots, s_\mu\})$$

denote the space of k-homogeneous polynomial in $\mathbf{s}$ (not to be confused with the weighted homogeneous polynomials). As elements in $\mathcal{H}_f \otimes \mathbb{C}[\mathbf{s}]_k$, we can decompose

$$[z^{-k}(F-f)^k\phi_\alpha\Omega] = \sum_{m\geq -k}\sum_\beta h^{(k)}_{\alpha\beta,m} z^m[\phi_\beta\Omega],$$

where $h^{(k)}_{\alpha\beta,m} \in \mathbb{C}[\mathbf{s}]_k$. Then we define

$$e^{(F-f)/z}(\phi_\alpha\Omega) := \sum_{k=0}^\infty \sum_\beta \sum_{m\geq -k} h^{(k)}_{\alpha\beta,m}\frac{z^m}{k!}[\phi_\beta\Omega] \in B((z))[[\mathbf{s}]].$$

The exponential map extends to a $\mathbb{C}((z))[[\mathbf{s}]]$-linear isomorphism

$$e^{(F-f)/z} : B_F((z))[[\mathbf{s}]] \to B((z))[[\mathbf{s}]],$$

which plays the role of parallel transport with respect to the Gauss-Manin connection. Let

$$K_f : B((z))[[\mathbf{s}]] \times B((z))[[\mathbf{s}]] \to \mathbb{C}((z))[[\mathbf{s}]]$$

also denote the $\mathbb{C}[[\mathbf{s}]]$-linear extension of the higher residue pairing to $\mathcal{H}_f[[\mathbf{s}]]$.

Lemma 2.8. [21,22] *For any $\varphi_1, \varphi_2 \in B_F$, we have*

$$K_f(e^{(F-f)/z}\varphi_1, e^{(F-f)/z}\varphi_2) \in z^n\mathbb{C}[[z, \mathbf{s}]]$$

In particular, $e^{(F-f)/z}$ maps $B_F[[z]]$ to an isotropic subspace of $\mathcal{H}_f[[\mathbf{s}]]$.

Theorem 2.3. [21,22] *Given a good basis $\{[\phi_\alpha d^n\mathbf{x}]\}_{\alpha=1}^\mu \subset \mathcal{H}_f^{(0)}$, there exists a unique pair $(\zeta, \mathcal{J})$ satisfying the following: (1) $\zeta \in B_F[[z]][[\mathbf{s}]]$, (2) $\mathcal{J} \in [\Omega] + z^{-1}B[z^{-1}][[\mathbf{s}]] \subset \mathcal{H}_f[[s]]$, and*

$$e^{(F-f)/z}\zeta = \mathcal{J}. \tag{$\star$}$$

Moreover, both ζ and $\mathcal{J}$ are homogeneous of weight $\sum_i q_i$.

This is the analogue of (2.3) for Calabi-Yau. $\zeta(\mathbf{s})$ can be solved recursively with respect to the order in $\mathbf{s}$. We refer to [22] for details, and to [21] for a compact formula of this algorithm. The decomposition is a formal solution of the Riemann-Hilbert-Birkhoff problem for primitive forms [32]. The volume form $\zeta = \sum_{k=0}^\infty \sum_\alpha \zeta^\alpha_{(k)}[\phi_\alpha\Omega]$ gives the power series expansion of a representative of the primitive form associated to the good basis $\{[\phi_\alpha d^n\mathbf{x}]\}_{\alpha=1}^\mu$.

2.4.2. *Flat coordinates and potential function.* Let $(\zeta, \mathcal{J})$ be the unique solution of $(\star)$. ζ represents the power series expansion of a primitive form. However for the purpose of mirror symmetry, it is more convenient to work with $\mathcal{J}$, which plays the role of Givental's J-function (see [16] for an introduction). This allows us to read off the flat coordinates and the potential function of the associated Frobenius manifold structure.

With the embedding $z^{-1}\mathbb{C}[z^{-1}][[\mathbf{s}]] \hookrightarrow z^{-1}\mathbb{C}[[z^{-1}]][[\mathbf{s}]]$, we decompose

$$\mathcal{J} = [d^n x] + \sum_{m=-1}^{-\infty} z^m \mathcal{J}_m, \quad \text{where } \mathcal{J}_m = \sum_\alpha \mathcal{J}_m^\alpha[\phi_\alpha \Omega], \mathcal{J}_m^\alpha \in \mathbb{C}[[\mathbf{s}]].$$

We denote the z^{-1}-term by

$$t_\alpha(\mathbf{s}) := \mathcal{J}_{-1}^\alpha(\mathbf{s}).$$

It is easy to see that $t_\alpha = s_\alpha + O(\mathbf{s}^2)$ and is homogeneous of the same weight as s_α. Therefore t_α defines a set of new homogeneous local coordinates on the (formal) deformation space of f.

Proposition 2.4. *The function $\mathcal{J} = \mathcal{J}(\mathbf{s}(\mathbf{t}))$ in coordinates t_α satisfies*

$$\partial_{t_\alpha}\partial_{t_\beta}\mathcal{J} = z^{-1}\sum_\gamma A_{\alpha\beta}^\gamma(\mathbf{t})\partial_{t_\gamma}\mathcal{J}$$

for some homogeneous $A_{\alpha\beta}^\gamma(\mathbf{t}) \in \mathbb{C}[[\mathbf{t}]]$ of weighted degree $\deg\phi_\alpha + \deg\phi_\beta - \deg\phi_\gamma$. *Moreover, for any $\alpha, \beta, \gamma, \delta$,*

$$\partial_{t_\alpha} A_{\beta\gamma}^\delta = \partial_{t_\beta} A_{\alpha\gamma}^\delta, \quad \sum_\sigma A_{\alpha\sigma}^\delta A_{\beta\gamma}^\sigma = \sum_\sigma A_{\beta\sigma}^\delta A_{\alpha\gamma}^\sigma$$

Lemma 2.9. *In terms of the coordinates t_α, we have*

$$K_f(z\partial_{t_\alpha}\mathcal{J}, z\partial_{t_\beta}\mathcal{J}) = z^n g_{\alpha\beta}.$$

Here $g_{\alpha\beta}$ is the constant equal to the residue pairing $\eta_f(\phi_\alpha\Omega, \phi_\beta\Omega)$.

Similarly, the triple $(\partial_{t_\alpha}, A_{\alpha\beta}^\gamma, g_{\alpha\beta})$ defines a (formal) Frobenius manifold structure on a neighborhood S of the origin with $\{t_\alpha\}$ being the flat coordinates, together with the potential function $\mathcal{F}_0(\mathbf{t})$ satisfying

$$A_{\alpha\beta\gamma}(\mathbf{t}) = \partial_{t_\alpha}\partial_{t_\beta}\partial_{t_\gamma}\mathcal{F}_0(\mathbf{t}).$$

It is not hard to see that $\mathcal{F}_0(\mathbf{t})$ is homogeneous of degree $3 - \hat{c}_f$. The potential function $\mathcal{F}_0(\mathbf{t})$ can also be computed perturbatively.

Remark 2.5. ζ is in fact an analytic primitive form [21]. Therefore, both t_α and $\mathcal{F}_0(\mathbf{t})$ are analytic functions of $\mathbf{s}$ at the germ $\mathbf{s} = 0$.

Remark 2.6. The closed formula of primitive forms for weighted homogenous singularities only exists for ADE ($\hat{c}_f < 1$) and simple elliptic singularities $\hat{c}_f = 1$ [32]. They can be easily obtained via the perburbative method [21]. For $\hat{c}_f > 1$, expressions of primitive forms are unknown and this has become long one of the major obstacles toward understanding mirror symmetry between Landau-Ginzburg models. It turns out that the perturbative formula, together with the WDVV equation, is enough to compute the full data of the Landau-Ginzburg B-model. The first non-trivial examples are Arnold's unimodular exceptional singularities, whose mirror symmetry with FJRW-theory [15] (Landau-Ginzburg A-model) is established [22] via the perburbative method. Such mirror symmetry between singularity theories is fully established in [27] for almost all weighted homogeneous polynomials when Landau-Ginzburg mirrors exist.

3. Quantum geometry

In this section, we will quantize the symplectic structure that appears in the previous section for the generalized period maps, or the primitive forms. We analyze Givental's symplectic loop space formalism in the context of B-model geometry, and explain the Fock space construction via the renormalization techniques of gauge theory. It leads to the quantum BCOV theory developed in [11]. This is parallel to another categorical approach [9, 10, 28] to the quantum B-model associated to a Calabi-Yau categories of D-branes. Our quantum field theory approach has the advantage of manifest physics intuitions and is related to methods of background symmetries and integrable hierarchies.

3.1. **A toy model of Weyl quantization.**

3.1.1. *Weyl algebra and Fock space.* Let us recall the construction of the Fock module for a finite dimensional dg symplectic vector space (V, ω, d), where ω is the symplectic pairing on V, and d is the differential which is skew self-adjoint with respect to ω. Let

$$\mathcal{W}(V) := \prod_{n\geq 0} (V^*)^{\otimes n}[[\hbar]]/\sim$$

be the (formal) Weyl algebra of V, which is the pro-free dg algebra generated by the linear dual V^* and a formal parameter $\hbar$, subject to the relation

$$[a, b] \sim \hbar\omega^{-1}(a, b), \quad \forall a, b \in V^*.$$

Here $\omega^{-1} \in \wedge^2 V$ is the inverse of ω, and $[a, b] := a \otimes b \mp b \otimes a$ is the graded commutator in the tensor algebra generated by V^*. Let V_+ be a Lagrangian subcomplex of V, and $Ann(V_+) \subset V^*$ be the annihilator of V_+. Then the Fock module $\mathcal{F}ock(V_+)$ is defined to be the quotient

$$\mathcal{F}ock(V_+) := \mathcal{W}(V)/\mathcal{W}(V)Ann(V_+).$$

Since V_+ is preserved by the differential, $\mathcal{F}ock(V_+)$ naturally inherits a dg structure from d. We will denote it by $\hat{d}$.

Let us choose a complementary linear Lagrangian subspace $V_- \subset V$ such that

$$V = V_+ \oplus V_-.$$

V_- may not be preserved by the differential. It allows us to formally identify

$$V \cong T^*(V_+)$$

Let

$$\mathcal{O}(V_+) = \prod_{n\geq 0} \mathrm{Sym}^n(V_+^*)$$

be the space of formal functions on the graded vector space V_+. V_- defines a splitting of the map $V^* \to V_+^*$, hence a morphism

$$\begin{tikzcd} \mathcal{O}(V_+)[[\hbar]] \arrow[r] \arrow[dr, "\cong"] & \mathcal{W}(V) \arrow[d] \\ & \mathcal{F}ock(V_+) \end{tikzcd}$$

which identifies the Fock module with the algebra $\mathcal{O}(V_+)[[\hbar]]$. The differential $\hat{d}$ can be described as follows. Let $\pi_+ : V \to V_+$ be the projection corresponding to the splitting $V = V_+ \oplus V_-$. Consider $(d \otimes 1)\omega^{-1}$, which is an element of $V \otimes V$. Let P be the projection

$$P = \pi_+ \otimes \pi_+ \left((d \otimes 1)\omega^{-1} \right) \in V_+ \otimes V_+$$

and it is easy to see that $P \in \mathrm{Sym}^2(V_+)$. Let $\partial_P : \mathcal{O}(V_+) \to \mathcal{O}(V_+)$ be the operator of contracting with P

$$\partial_P : \mathrm{Sym}^n(V_+^*) \to \mathrm{Sym}^{n-2}(V_+^*).$$

Lemma 3.1. *Under the isomorphism $\mathcal{F}ock(V_+) \cong \mathcal{O}(V_+)[[\hbar]]$, $\hat{d}$ takes the form*

$$\hat{d} = d + \hbar\partial_P$$

where d here is the induced differential on $\mathcal{O}(V_+)$ from d on V_+.

∂_P will be called a BV operator. It induces a bracket on $\mathcal{O}(V_+)$ by

$$\{\Phi_1, \Phi_2\}_P := \partial_P(\Phi_1\Phi_2) - (\partial_P\Phi_1)\Phi_2 - (-1)^{|\Phi_1|}\Phi_1\partial_P\Phi_2, \quad \Phi_i \in \mathcal{O}(V_+).$$

Here $|\Phi|$ is the cohomology degree of Φ. We will also need a slightly larger Fock space given by

$$\mathcal{F}ock^+(V_+) := \prod_{k=0}^{\infty}\Big(\bigoplus_{\substack{m\geq 0, n\in\mathbb{Z}\\ m+2n=k}} \mathrm{Sym}^m(V_+^*)\hbar^n\Big),$$

i.e. we allow negative powers of $\hbar$ in an appropriate topology.

3.1.2. *Langrangian and quantization.* In the classical geometry, we are interested in a Langragian submanifold $\mathcal{L}$ of V. Under the isomorphism

$$V \cong T^*(V_+),$$

$\mathcal{L}$ can be represented (locally) as a graph $\mathcal{L} = \mathrm{Graph}(dF_0)$. We impose a symmetry condition that d is tangent to $\mathcal{L}$, where we treat d as defining a nilpotent vector field on V. This can be viewed as an infinitesimal gauge symmetry.

Lemma 3.2. *d being tangent to $\mathcal{L}$ is equivalent to the following equation for F_0*

$$dF_0 + \frac{1}{2}\{F_0, F_0\}_P = 0.$$

This is called the *classical master equation*. It says that $d + \{F_0, -\}_P$ defines a new nilpotent vector field on V_+. Geometrically, let

$$\pi_+|_{\mathcal{L}} : \mathcal{L} \to V_+.$$

Then $d + \{F_0, -\}_P = (\pi_+|_{\mathcal{L}})_*(d)$ is the push-forward of the vector field d on $\mathcal{L}$.

In the quantum theory, we are interested in a vector $|F\rangle \in \mathcal{F}ock^+(V_+)$ satisfying the "gauge invariance condition": $d|F\rangle = 0$. To relate $|F\rangle$ to $\mathcal{L}$ in the $\hbar \to 0$ classical limit, we consider $|F\rangle$ of the form represented by $e^{F/\hbar}$

$$|F\rangle \leftrightarrow e^{F/\hbar}, \quad F = \sum_{g\geq 0}\hbar^g F_g \in \mathcal{O}(V_+)[[\hbar]].$$

By Lemma 3.1, the gauge invariance becomes $(d + \hbar\partial_P)e^{F/\hbar} = 0$, or equivalently

$$(d + \hbar\partial_P)F + \frac{1}{2}\{F, F\}_P = 0. \tag{3.1}$$

This is called the *quantum master equation*.

In summary of our toy model, the quantization scheme quantizes the Lagrangian $\mathcal{L}$ to a state $|F\rangle$ in the Fock space. Equivalently, it quantizes F_0 which satisfies the classical master equation to $F = F_0 + \hbar F_1 + \cdots$ which satisfies the quantum master equation.

3.2. **Symlectic geometry and BCOV theory.** Following Givental's symplectic formulation [18, 19] of Gromov-Witten theory in the A-model and the parallel Barannikov's work [1, 3] in the B-model, our dg symplectic vector space is (note that our degree assignment in this article differs from that in [11])

$$S(X) = \mathrm{PV}(X)((z)),$$

with differential $Q = \bar{\partial} + z\partial$ and symplectic pairing ω by Definition 2.2.

In [6], Bershadsky, Cecotti, Ooguri and Vafa introduced a gauge theory for polyvector fields on Calabi-Yau three-folds. This is further extended to arbitrary Calabi-Yau manifolds in [11]. The space of fields of the BCOV theory is

$$S_+(X) \equiv \mathrm{PV}(X)[[z]]$$

which is a linear isotropic subspace of $S(X)$. The classical action functional of the BCOV theory can be constructed from the following Lagrangian (the embedding is in the sense of formal scheme via functor of points on Artinian rings [11])

$$\mathcal{L}_X = \left\{ z(e^{\mu/z} - 1) | \mu \in S_+(X) \right\} \subset S(X).$$

This can be viewed as the lifting of that in Proposition 2.1 to the cochain level. The geometry of $\mathcal{L}_X$ can be described by the following

Lemma 3.3 ([11]). *$\mathcal{L}_X$ is a formal Lagrangian submanifold of $\mathcal{S}(X)$, preserved by the differential $Q = \bar{\partial} + z\partial$. Moreover, $\mathcal{L}_X + z$ is a Lagrangian cone preserved by the infinitesimal symplectomorphism of $S(X)$ given by multiplying by z^{-1}.*

Remark 3.1. $\mathcal{L}_X + z$ is called the dilaton shift of $\mathcal{L}_X$ [18].

Consider the splitting

$$S(X) = S_+(X) \oplus S_-(X) \tag{3.2}$$

where recall $S_-(X) = z^{-1}\,\mathrm{PV}(X)[z^{-1}]$. It allows us to formally identify

$$S(X) \cong T^*(S_+(X)).$$

The generating functional $\mathbf{F}_{\mathcal{L}_X}$ is a formal function on $S_+(X)$ such that

$$\mathcal{L}_X = \mathrm{Graph}(d\mathbf{F}_{\mathcal{L}_X}).$$

The explicit formula is worked out in [11]

Proposition 3.1 ([11])**.** $\mathbf{F}_{\mathcal{L}_X}(\mu) = \mathrm{Tr}\,\langle e^{\mu}\rangle_0$, *where*

$$\langle - \rangle_0 : \mathrm{Sym}(\mathrm{PV}(X)[[z]]) \to \mathrm{PV}(X)$$

is given by intersection of ψ-classes over the moduli space of marked rational curves

$$\left\langle \alpha_1 z^{k_1}, \cdots, \alpha_n z^{k_n} \right\rangle_0 := \alpha_1 \cdots \alpha_n \int_{\overline{M}_{0,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} = \binom{n-3}{k_1, \cdots, k_n} \alpha_1 \cdots \alpha_n$$

Definition 3.1 ([11])**.** The classical BCOV interaction is defined to be the formal local functional on $S_+(X)$ given by $\mathbf{F}_{\mathcal{L}_X}$.

Remark 3.2. Our definition of BCOV interaction extends the original Kodaira-Spencer interaction in [6] by turning on the "gravitational descendants" z. It is also equivalent to that used by Losev-Shadrin-Shneiberg [30] in the discussion of finite dimensional toy models of Hodge field theory.

We can transfer the geometry of the Lagrangian $\mathcal{L}_X$ into properties of $\mathbf{F}_{\mathcal{L}_X}$.

Proposition 3.2 ([11])**.** $\mathbf{F}_{\mathcal{L}_X}$ *satisfies the* classical master equation

$$Q\mathbf{F}_{\mathcal{L}_X} + \frac{1}{2}\{\mathbf{F}_{\mathcal{L}_X}, \mathbf{F}_{\mathcal{L}_X}\} = 0$$

where Q is the induced derivation on the functionals of $S_+(X)$, and $\{-,-\}$ is the Poisson bracket on local functionals induced from the distribution representing the operator ∂ (see Remark 3.3).

This is equivalent to that $\mathcal{L}_X$ is preserved by Q (See Lemma 3.2 for an explanation in the toy model). The classical master equation implies that $Q + \{\mathbf{F}_{\mathcal{L}_X}, -\}$ is a nilpotent operator acting on local functionals. In physics terminology, it generates the gauge symmetry, and defines the gauge theory in the Batalin-Vilkovisky formalism.

3.3. **Givental's formalism via renormalization.** The dg symplectic vector space related to the BCOV theory is $(S(X), \omega, Q)$. If we run the machine to construct the Fock space as in the previous section, we immediately run into trouble: $\mathrm{PV}(X)$ is infinite dimensional! This is a well-known phenomenon in quantum field theory, which is related to the difficulty of ultra-violet divergence. The standard way of solving this is to use the renormalization technique. We will follow the approach developed in [8].

3.3.1. *Functionals on the fields.* Let $S_+(X)^{\otimes n}$ be the completed projective tensor product of n copies of $S_+(X)$. It can be viewed as the space of smooth polyvector fields on X^n with a formal variable z for each factor. Let

$$\mathcal{O}^{(n)}(S_+(X)) = \mathrm{Hom}\left(S_+(X)^{\otimes n}, \mathbb{C}\right)_{S_n}$$

denote the space of continuous linear maps (distributions), and the subscript S_n denotes taking S_n coinvariants. $\mathcal{O}^{(n)}(S_+(X))$ will be the space of homogeneous degree n functionals on the space of fields $S_+(X)$, playing the role of $\mathrm{Sym}^n(V^*)$ in our toy model. We will also let

$$\mathcal{O}^{(n)}_{loc}(S_+(X)) \subset \mathcal{O}^{(n)}(S_+(X))$$

be the subspace of local functionals, i.e. those of the form given by the integration of a lagrangian density

$$\int_X \mathcal{L}(\mu), \quad \mu \in S_+(X).$$

Definition 3.2. The algebra of functionals $\mathcal{O}(S_+(X))$ on $S_+(X)$ is defined to be

$$\mathcal{O}(S_+(X)) = \prod_{n\geq 0} \mathcal{O}^{(n)}(S_+(X))$$

and the space of local functionals is defined to be the subspace

$$\mathcal{O}_{loc}(S_+(X)) = \prod_{n\geq 0} \mathcal{O}^{(n)}_{loc}(S_+(X))$$

3.3.2. *Effective Fock Space.* Let g be a Kähler metric on X. Let

$$K^g_L \in \mathrm{PV}(X) \otimes \mathrm{PV}(X), \quad L > 0$$

be the heat kernel for the operator $e^{-L\left[\bar\partial, \bar\partial^*\right]}$, where $\bar\partial^*$ is the adjoint of $\bar\partial$ with respect to the metric g and $\left[\bar\partial, \bar\partial^*\right] = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ is the Laplacian acting on $\mathrm{PV}(X)$. It is a smooth polyvector field on $X \times X$ defined by the equation

$$\left(e^{-L\left[\bar\partial, \bar\partial^*\right]}\alpha\right)(x) = \int_X \left(K^g_L(x,y)\alpha(y) \vdash \Omega_X(y)\right) \wedge \Omega_X(y)$$

where we have chosen coordinates (x, y) on $X \times X$, and we integrate over the second copy of X using the trace map.

Definition 3.3. The effective inverse $\omega^{-1}_{g,L}$ for the symplectic form ω is defined to be the kernel

$$\omega^{-1}_{g,L} = \sum_{k\in\mathbb{Z}} K^g_L(-z)^k \otimes z^{-k-1} \in S(X) \otimes S(X), \quad L > 0.$$

Note that $\lim_{L\to 0} K_L^g$ is the delta-function distribution, which is no longer a smooth polyvector field, hence not an element of $S(X)\otimes S(X)$. $\omega_{g,L}^{-1}$ can be viewed as the regularization of ω^{-1} in the infinite dimensional setting.

Let $S(X)^*$ be the continuous linear dual of $S(X)$ (distributions on $S(X)$ with extra care on the z-adic topology. See [11] for more details).

Definition 3.4. The effective Weyl algebra $\mathcal{W}(S(X), g, L)$ is the quotient of the completed tensor algebra

$$\left(\prod_{n\geq 0}\left(S(X)^*\right)^{\otimes n}\right)\otimes \mathbb{C}[[\hbar]]$$

by the topological closure of the two-sided ideal generated by

$$[a,b]-\hbar\left\langle \omega_{g,L}^{-1}, a\otimes b\right\rangle, \quad L>0$$

for $a, b\in S(X)^*$. Here $\langle,\rangle$ is the natural pairing between $S(X)$ and its dual.

Similarly, the Fock space can also be defined using the regularized kernel $\omega_{g,L}^{-1}$.

Definition 3.5. The effective Fock space $\mathcal{F}ock\left(S_+(X), g, L\right)$ is the quotient of $\mathcal{W}(\mathcal{S}(X))$ by the left ideal generated topologically by the subspace

$$Ann(S_+(X), g, L)\subset S(X)^*.$$

Similar to the finite dimentional case, the splitting $S(X) = S_+(X)\oplus S_-(X)$ gives the identification

$$\mathcal{F}ock\left(S_+(X), g, L\right)\cong \mathcal{O}(S_+(X))[[\hbar]].$$

We refer to [11] for detailed discussions.

3.3.3. *Effective BV formalism.* We would like to understand the quantized operator $\hat{Q}_L$ for Q acting on the Fock space represented by the above identification. This is completely similar to the toy model. Let

$$(\partial\otimes 1)K_L^g\in \mathrm{Sym}^2(\mathrm{PV}(X))$$

be the kernel for the operator $\partial e^{-L[\bar\partial,\bar\partial^*]}$. It can be viewed as the projection of $(Q\otimes 1)\omega_{L,g}^{-1}\in \mathrm{Sym}^2(S(X))$ to $\mathrm{Sym}^2(S_+(X))$.

Definition 3.6. We define the effective BV operator

$$\Delta_L : \mathcal{O}(S_+(X))\to \mathcal{O}(S_+(X))$$

as the operator of contracting with the smooth kernel $(\partial\otimes 1)K_L^g$.

Since $\Delta_L : \mathcal{O}^{(n)}(S_+(X)) \to \mathcal{O}^{(n-2)}(S_+(X))$, it could be viewed as an order two differential operator on the infinite dimensional vector space $S_+(X)$. Note that Δ_L has odd cohomology degree, and $(\Delta_L)^2 = 0$. It defines a Batalin-Vilkovisky structure on $\mathcal{O}(S_+(X))$, with the Batalin-Vilkovisky bracket defined by

$$\{S_1, S_2\}_L = \Delta_L (S_1 S_2) - (\Delta_L S_1) S_2 - (-1)^{|S_1|} S_1 (\Delta_L S_2), \quad L > 0.$$

Remark 3.3. If $S_1, S_2 \in \mathcal{O}_{loc}(\mathcal{E}(X))$, then $\lim_{L\to 0} \{S_1, S_2\}_L$ is well-defined, which is precisely the Poisson bracket in Proposition 3.2.

Proposition 3.3 ([11]). *Under the isomorphism $\mathcal{F}ock\,(\mathcal{S}_+(X), g, L) \cong \mathcal{O}(S_+(X))[[\hbar]]$, the induced differential $\hat{Q}_L$ is $\hat{Q}_L = Q + \hbar\Delta_L$.*

The proof is similar to Lemma 3.1.

3.3.4. *Renormalization group flow and homotopy equivalence.* We need to specify a choice of the metric g and a positive number $L > 0$ to construct the Fock space $\mathcal{F}ock\,(S_+(X), g, L)$. However, we are in a bit better situation. The general machinery of renormalization theory in [8] allows us to show that the effective Fock spaces are independent of the choice of g and L up to homotopy. This is discussed in detail in [11]. We will discuss here the homotopy between different choices of the scale L, which is related to the renormalization group flow in quantum field theory.

Definition 3.7. The effective propagator is defined to be the smooth kernel

$$P_\epsilon^L = \int_\epsilon^L du (\bar{\partial}^* \partial \otimes 1) K_u^g \in \mathrm{Sym}^2(\mathrm{PV}(X)), \quad L > \epsilon > 0 \tag{3.3}$$

representing the operator $\bar{\partial}^* \partial e^{-L[\bar{\partial}, \bar{\partial}^*]}$.

Lemma 3.4. *As an operator on $\mathcal{O}(S_+(X))[[\hbar]]$,*

$$\hat{Q}_L = e^{\hbar \partial_{P_\epsilon^L}} \hat{Q}_\epsilon e^{-\hbar \partial_{P_\epsilon^L}}$$

where $\partial_{P_\epsilon^L} : \mathcal{O}(\mathcal{E}(X)) \to \mathcal{O}(\mathcal{E}(X))$ is the contraction by the smooth kernel P_ϵ^L.

It follows from this lemma that $e^{\hbar \partial_{P_\epsilon^L}}$ defines the homotopy

$$e^{\hbar \partial_{P_\epsilon^L}} : (\mathcal{O}(S_+(X))[[\hbar]], Q + \hbar\Delta_\epsilon) \to (\mathcal{O}(S_+(X))[[\hbar]], Q + \hbar\Delta_L)$$

between Fock spaces defined at scales ϵ and L. It defines a flow on the space of functionals on the fields, which is called the renormalization group flow in [8] following the physics terminology.

Proposition 3.4 ([11])**.** *The cohomology $H^*(\mathcal{F}ock(S_+(X), g, L), \hat{Q}_L)$ is independent of g and L. There are canonical isomorphisms*

$$H^*(\mathcal{F}ock(S_+(X), g, L), \hat{Q}_L) \cong \mathcal{F}ock\left(H^*(S_+(X), Q)\right)$$

where $\mathcal{F}ock\left(H^*(S_+(X), Q)\right)$ is the Fock space for the Lagrangian subspace $H^*(S_+(X), Q)$ of the symplectic space $(H^*(S(X), Q), \omega)$

Remark 3.4. $\mathcal{F}ock\left(H^*(S_+(X), Q)\right)$ is the mirror of the Fock space of de Rham cohomology classes for Gromov-Witten theory discussed in [7].

3.4. **Quantum BCOV theory.**

3.4.1. *Perturbative quantization.*

Definition 3.8 ([11])**.** A perturbative quantization of BCOV theory on X is given by a family of functionals

$$\mathbf{F}[L] = \sum_{g \geq 0} \hbar^g \mathbf{F}_g[L] \in \mathcal{O}(S_+(X))[[\hbar]]$$

for each $L \in \mathbb{R}_{>0}$, satisfying the following properties

(1) The renormalization group flow equation

$$\mathbf{F}[L] = W\left(P_\epsilon^L, \mathbf{F}[\epsilon]\right)$$

for all $L > \epsilon > 0$. Here $W\left(P_\epsilon^L, \mathbf{F}[\epsilon]\right)$ is the connected Feynman graph integrals (connected graphs) with propagator P_ϵ^L (3.3) and vertices $\mathbf{F}[\epsilon]$. This is equivalent to

$$e^{\mathbf{F}[L]/\hbar} = e^{\hbar \frac{\partial}{\partial P_\epsilon^L}} e^{\mathbf{F}[\epsilon]/\hbar}$$

(2) The quantum master equation holds

$$Q\mathbf{F}[L] + \hbar\Delta_L \mathbf{F}[L] + \frac{1}{2}\{\mathbf{F}[L], \mathbf{F}[L]\}_L = 0, \ \ \forall L > 0.$$

This is equivalent to

$$\left(Q + \hbar\Delta_L\right) e^{\mathbf{F}[L]/\hbar} = 0$$

(3) The locality axiom, as in [8]. This says that $\mathbf{F}[L]$ has a small L asymptotic expansion in terms of local functionals.

(4) The classical limit condition

$$\lim_{L \to 0} \lim_{\hbar \to 0} \mathbf{F}[L] \equiv \lim_{L \to 0} \mathbf{F}_0[L] = \mathbf{F}_{\mathcal{L}_X}.$$

(5) Degree axiom and Hodge weight axiom (see [11]).

3.4.2. *Higher genus B-model.* Given a quantization $\{\mathbf{F}[L]\}_{L>0}$ of the BCOV theory, we obtain a state $\left[e^{\mathbf{F}[L]/\hbar}\right]$ in the Fock space $\mathcal{F}ock^+\left(H^*(S_+(X))\right)$ by Proposition 3.4. We will denote it by $Z_{\mathbf{F}}$. Let us choose an opposite filtration $\mathcal{L}$ (Definition 2.3), which induces isomorphisms

$$H^*(S(X), Q) \cong H^*(X, \wedge^* T_X)((z)), \quad H^*(S_+(X), Q) \cong H^*(X, \wedge^* T_X)[[z]].$$

In particular, it induces a natural identification

$$\Phi_{\mathcal{L}} : \mathcal{F}ock(H^*(S(X))) \stackrel{\cong}{\to} \mathcal{O}(H^*(X, \wedge^* T_X)[[z]])[[\hbar]].$$

Definition 3.9. Let $\mathbf{F}$ be a quantization of the BCOV theory on X, and $\mathcal{L}$ be an opposite filtration of $H^*(S_+(X), Q)$. Let $\alpha_1, \cdots, \alpha_n \in H^*(X, \wedge^* T_X)$. The *correlation functions* associated to $\mathbf{F}, \mathcal{L}$ is defined to be

$$\mathbf{F}_X^{B,\mathcal{L}}\left(z^{k_1}\alpha_1, \cdots, z^{k_n}\alpha_n\right) := \left(\frac{\partial}{\partial z^{k_1}\alpha_1} \cdots \frac{\partial}{\partial z^{k_n}\alpha_n}\right) \hbar \log \Phi_{\mathcal{L}}\left(Z_{\mathbf{F}}\right)(0) \in \mathbb{C}[[\hbar]].$$

Here the superscript "B" refers to the B-model. We can further decompose $\mathbf{F}_X^{B,\mathcal{L}} = \sum\limits_{g \geq 0} \hbar^g \mathbf{F}_{g,X}^{B,\mathcal{L}}$. Then $\mathbf{F}_{g,X}^{B,\mathcal{L}}$ will be the candidate for the higher genus B-model invariants on X. It is conjectured in [11] that there exists a canonical quantization $\mathbf{F}$ (up to homotopy) of the BCOV theory on X which is mirror to the Gromov-Witten theory on the mirror Calabi-Yau manifold. This proves to be the case for X being an elliptic curve [24, 26].

3.4.3. *The opposite filtrations.* There are two natural opposite filtrations of $H^*(\mathcal{S}(X), Q)$.

The first one is given by the complex conjugate splitting of the Hodge filtration, which we denote by $\mathcal{L}_{\bar{X}}$. In this case the correlation function $\mathbf{F}_X^{B,\mathcal{L}_{\bar{X}}}$ can be realized explicitly as follows. Consider the limit

$$\mathbf{F}[\infty] = \lim_{L \to \infty} \mathbf{F}[L]$$

which is well-defined since X is compact, hence P_L^∞ is smooth. The quantum master equation at $L = \infty$ says that

$$Q\mathbf{F}[\infty] = 0$$

as $\lim\limits_{L \to \infty} \Delta_L = 0$. It follows that $\mathbf{F}[\infty]$ descends to a functional on $H^*(S_+(X), Q)$

$$\mathbf{F}[\infty] \in H^*(\mathcal{O}(S_+(X))[[\hbar]], Q) \cong \mathcal{O}(H^*(S_+(X), Q))[[\hbar]].$$

The choice of the Kähler metric induces isomorphisms

$$H^*(\mathcal{S}(X), Q) \cong H^*(X, \wedge^* T_X)((z)), \quad H^*(\mathcal{S}_+(X), Q) \cong H^*(X, \wedge^* T_X)[[z]]$$

via Hodge theory, hence defining an opposite filtration which is precisely $\mathcal{L}_{\bar{X}}$. Then

$$\mathbf{F}_X^{B,\mathcal{L}_{\bar{X}}} = \mathbf{F}[\infty].$$

The second choice is relevant for mirror symmetry, which is defined near a large complex limit in the moduli space of complex structures on X. Near any such large complex limit point, there is an associated monodromy weight filtration $\mathcal{W}$ which splitts the Hodge filtration. Then the correlation function

$$\mathbf{F}_{g,n,X}^{B,\mathcal{W}} : \mathrm{Sym}^n \left(H^*(X, \wedge^* T_X)[[z]]\right) \to \mathbb{C}$$

will be the mirror of the descendant Gromov-Witten invariants

$$\langle - \rangle_{g,n,X^\vee}^{GW} : \mathrm{Sym}^n \left(H^*(X^\vee, \mathbb{C})[[z]]\right) \to \mathbb{C}$$

on the mirror Calabi-Yau $X^\vee$ under the mirror map.

Note that $F_X^{B,\mathcal{L}_{\bar{X}}}$ doesn't vary holomorphically due to the complex conjugate splitting $\mathcal{L}_{\bar{X}}$. This is the famous holomorphic anomaly discovered in [6]. Given a large complex limit point, the natural way to retain holomorphicity is to consider $\mathbf{F}_{g,X}^{B,\mathcal{W}}$, which is usually denoted in physics literature by

$$\mathbf{F}_{g,X}^{B,\mathcal{W}} \equiv \lim_{\bar{\tau}\to\infty} \mathbf{F}_X^{B,\mathcal{L}_{\bar{X}}}$$

as the "$\bar{\tau} \to \infty$-limit" [6] near the large complex limit.

3.4.4. *Higher genus mirror symmetry.* The mirror symmetry for elliptic curves is easy to describe. Let E represent an elliptic curve. In the A-model, we have the moduli of (complexified) Kähler class $[\omega] \in H^2(E, \mathbb{C})$ parametrized by the symplectic volume

$$q = \mathrm{Tr}\, \omega$$

where the trace map in the A-model is given by the integration $\mathrm{Tr} = \int_E$. The mirror in the B-model is the elliptic curve $\mathrm{E}_\tau = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$, with complex structure τ related to q by the mirror map

$$q = e^{2\pi i \tau}.$$

Let

$$\Phi_\tau : \bigoplus_{i,j} H^i(E, \wedge^j T_E^*)[-i-j] \to \bigoplus_{i,j} H^i(\mathrm{E}_\tau, \wedge^j T_{\mathrm{E}_\tau})[-i-j]$$

be the unique isomorphism of commutative bigraded algebras which is compatible with the trace on both sides. This is z-linearly extended to an isomorphism

$$\Phi_\tau : H^*(E, \mathbb{C})[[z]] \to H^*(E_\tau, \wedge^* T_{E_\tau})[[z]].$$

The canonical quantization of BCOV theory was analyzed in [24], and the explicit solution was presented in [26] via vertex algebra techniques. This leads to the establishment of higher genus mirror symmetry on elliptic curves.

Theorem 3.1. [24] *For all $\alpha_1, \cdots, \alpha_n \in H^*(E, \mathbb{C})[[z]]$, the A-model descendant Gromov-Witten invariants on E can be identified with the B-model BCOV correlation functions*

$$\sum_d q^d \langle \alpha_1, \cdots, \alpha_n \rangle_{g,n,d}^{GW(E)} = \lim_{\bar{\tau}\to\infty} \mathbf{F}_{E_\tau}^{B,\mathcal{L}_{E_\tau}} (\Phi_\tau(\alpha_1), \cdots, \Phi_\tau(\alpha_n))$$

where the large complex limit is taken to be $\mathrm{Im}\,\tau \to \infty$ *on the upper half plane* $\mathbb{H}$.

It is proved in [24, 25] that the correlation functions for $\mathbf{F}_{E_\tau}^{B,\mathcal{L}_{E_\tau}}$, before taking the $\bar{\tau} \to \infty$ limit, are almost holomorphic modular forms exhibiting mild anti-holomorphic dependence on $\bar{\tau}$. On the other hand, the correlation functions of Gromov-Witten theory are given by quasi-modular forms [31]. In this example, the $\bar{\tau} \to \infty$ limit is the well-known identification between almost holomorphic modular forms and quasi-modular forms [20].

REFERENCES

[1] S. Barannikov, *Extended moduli spaces and mirror symmetry in dimensions $n > 3$*, ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–University of California, Berkeley.

[2] ______, *Non-commutative periods and mirror symmetry in higher dimensions*, Comm. Math. Phys. **228** (2002), no. 2, 281–325.

[3] ______, *Quantum periods. I. Semi-infinite variations of Hodge structures*, Internat. Math. Res. Notices **23** (2001), 1243–1264.

[4] ______, *Semi-infinite Hodge structures and mirror symmetry for projective spaces*, arxiv: math.AG/0010157.

[5] S. Barannikov and M. Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields*, Internat. Math. Res. Notices **4** (1998), 201–215.

[6] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Comm. Math. Phys. **165** (1994), no. 2, 311–427. MR1301851 (95f:32029)

[7] T. Coates and A. B. Givental, *Quantum Riemann-Roch, Lefschetz and Serre*, Ann. of Math. (2) **165** (2007), no. 1, 15–53.

[8] K. Costello, *Renormalization and effective field theory*, Mathematical Surveys and Monographs, vol. 170, American Mathematical Society, Providence, RI, 2011. MR2778558

[9] ______, *The partition function of a topological field theory*, J. Topol. **2** (2009), no. 4, 779–822.

[10] ______, *Topological conformal field theories and Calabi-Yau categories*, Adv. Math. **210** (2007), no. 1, 165–214.

[11] K. Costello and S. Li, *Quantum BCOV theory on Calabi-Yau manifolds and the higher genus B-model*. arXiv:1201.4501 [math.QA].

[12] A. Douai and C. Sabbah, *Gauss-Manin systems, Brieskorn lattices and Frobenius structures. I*, Proceedings of the International Conference in Honor of Frédéric Pham (Nice, 2002), 2003, pp. 1055–1116.

[13] ———, *Gauss-Manin systems, Brieskorn lattices and Frobenius structures. II*, Frobenius manifolds, Aspects Math., E36, Vieweg, Wiesbaden, 2004, pp. 1–18.

[14] B. Dubrovin, *Geometry of* 2*D topological field theories*, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math., vol. 1620, Springer, Berlin, 1996, pp. 120–348.

[15] H. Fan, T. Jarvis, and Y. Ruan, *The Witten equation, mirror symmetry, and quantum singularity theory*, Ann. of Math. (2) **178** (2013), no. 1, 1–106.

[16] A. B. Givental, *A tutorial on quantum cohomology*, Symplectic geometry and topology (Park City, UT, 1997), IAS/Park City Math. Ser., vol. 7, Amer. Math. Soc., Providence, RI, 1999, pp. 231–264.

[17] ———, *A mirror theorem for toric complete intersections*, Topological field theory, primitive forms and related topics (Kyoto, 1996), Progr. Math., vol. 160, Birkhäuser Boston, Boston, MA, 1998, pp. 141–175.

[18] ———, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, Mosc. Math. J. **1** (2001), no. 4, 551–568, 645. Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary.

[19] ———, *Symplectic geometry of Frobenius structures*, Frobenius manifolds, Aspects Math., E36, Vieweg, Wiesbaden, 2004, pp. 91–112.

[20] M. Kaneko and D. Zagier, *A generalized Jacobi theta function and quasimodular forms*, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 165–172.

[21] C. Li, S. Li, and K. Saito, *Primitive forms via polyvector fields*, arxiv: math.AG/1311.1659.

[22] C. Li, S. Li, K. Saito, and Y. Shen, *Mirror symmetry for exceptional unimodular singularities*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 4, 1189–1229.

[23] S. Li, *Calabi-Yau geometry and higher genus mirror symmetry* (2011). Thesis, Harvard.

[24] ———, *BCOV theory on the elliptic curve and higher genus mirror symmetry*. arXiv:1112.4063 [math.QA].

[25] ———, *Feynman graph integrals and almost modular forms*, Comm. in Number Theory and Physics **6** (2012).

[26] ———, *Vertex algebras and quantum master equation*. arXiv:1612.01292 [math.QA].

[27] W. He, S. Li, Y. Shen, and R. Webb, *Landau-Ginzburg Mirror Symmetry Conjecture*, arxiv: math.AG/1503.01757.

[28] M. Kontsevich and Y. Soibelman, *Notes on A_∞-algebras, A_∞-categories and non-commutative geometry*, Homological mirror symmetry, Lecture Notes in Phys., vol. 757, Springer, Berlin, 2009, pp. 153–219. MR2596638

[29] B.H. Lian, K.F Liu, and S.T. Yau, *Mirror principle. I*, Asian J. Math. **1** (1997), no. 4, 729–763.

[30] A. Losev, S. Shadrin, and I. Shneiberg, *Tautological relations in Hodge field theory*, Nuclear Phys. B **786** (2007), no. 3, 267–296.

[31] A. Okounkov and R. Pandharipande, *Virasoro constraints for target curves*, Invent. Math. **163** (2006), no. 1, 47–108.

[32] K. Saito, *Period mapping associated to a primitive form*, Publ. Res. Inst. Math. Sci. **19** (1983), no. 3, 1231–1264.

[33] ———, *The higher residue pairings $K_F^{(k)}$ for a family of hypersurface singular points*, Singularities, Part 2 (Arcata, Calif., 1981), 1983, pp. 441–463.

[34] ———, *From primitive form to mirror symmetry*, arxiv: math.AG/1408.4208.

[35] M. Saito, *On the structure of Brieskorn lattice*, Ann. Inst. Fourier (Grenoble) **39** (1989), no. 1, 27–72.

[36] ———, *On the structure of Brieskorn lattices, II*, arxiv: math.AG/1312.6629.

[37] A. Takahashi, *Primitive forms, Topological LG models coupled to Gravity and Mirror Symmetry*, arxiv: math.AG/9802059.

[38] B. Zwiebach, *Closed string field theory: quantum action and the Batalin-Vilkovisky master equation*, Nuclear Phys. B **390** (1993), no. 1, 33–152.

YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, CHINA
sili@mail.tsinghua.edu.cn