Calabi-Yau Geometry and Higher Genus Mirror Symmetry

A dissertation presented

by

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Abstract

We study closed string mirror symmetry on compact Calabi-Yau manifolds at higher genus. String theory predicts the existence of two sets of geometric invariants, from the A-model and the B-model on Calabi-Yau manifolds, each indexed by a non-negative integer called *genus*. The A-model has been mathematically established at all genera by the Gromov-Witten theory, but little is known in mathematics for B-model beyond genus zero.

We develop a mathematical theory of higher genus B-model from perturbative quantization techniques of gauge theory. The relevant gauge theory is the *Kodaira-Spencer gauge theory*, which is originally discovered by Bershadsky-Cecotti-Ooguri-Vafa as the closed string field theory of B-twisted topological string on Calabi-Yau three-folds. We generalize this to Calabi-Yau manifolds of arbitrary dimensions including also gravitational descendants, which we call BCOV theory. We give the geometric description of the perturbative quantization of BCOV theory in terms of deformation-obstruction theory. The vanishing of the relevant obstruction classes will enable us to construct the higher genus B-model. We carry out this construction on the elliptic curve and establish the corresponding higher genus B-model. Furthermore, we show that the B-model invariants constructed from BCOV theory on the elliptic curve can be identified with descendant Gromov-Witten invariants on the mirror elliptic curve. This gives the first compact Calabi-Yau example where mirror symmetry can be established at all genera.

Si Li

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1. INTRODUCTION

Mirror symmetry originated from string theory as a duality between superconformal field theories (SCFT). The natural geometric background involved is the Calabi-Yau manifold, and SCFTs can be realized from twisting the so-called σ -models on Calabi-Yau manifolds in two different ways [Wit88, Wit92]: the A-model and the B-model. The physics statement of mirror symmetry says that the A-model on a Calabi-Yau manifold X is equivalent to the B-model on a different Calabi-Yau manifold \check{X} , which is called the *mirror*.

The mathematical interests on mirror symmetry started from the work [CdlOGP91], where a remarkable mathematical prediction was extracted from the physics statement of mirror symmetry: the counting of rational curves on the Quintic 3-fold is equivalent to the period integrals on the mirror Quintic 3-fold. Motivated by this example, people have conjectured that such phenomenon holds for general mirror Calabi-Yau manifolds. The counting of rational curves is refered to as the genus 0 A-model, which has now been mathematically established [RT94, LT98] as *Gromov-Witten theory*. The period integral is related to the variation of Hodge structure, and this is refered to as the genus 0 B-model. Mirror conjecture at genus 0 has been proved by Givental [Giv98] and Lian-Liu-Yau [LLY97] for a large class of Calabi-Yau manifolds inside toric varieties. In the last twenty years, mirror symmetry has lead to numerous deep connections between various branches of mathematics and has been making a huge influence on both mathematics and physics.

The fundamental mathematical question is to understand mirror symmetry at higher genus. In the A-model, the Gromov-Witten theory has been established for curves of arbitrary genus, and the problem of counting higher genus curves on Calabi-Yau manifolds has a solid mathematical foundation. However, little is known for the higher genus B-model. One mathematical approach to the higher genus B-model given by Kevin Costello [Cos09] is categorical, from the viewpoint of Kontsevich's homological mirror symmetry [Kon95b]. The B-model partition function is proposed through the Calabi-Yau A-infinity category of coherent sheaves and a classification of certain 2-dimensional topological field theories. Unfortunately, the computation from categorical aspects is extremely difficult that only results for zero dimensional space, i.e. a point, have been obtained. On the other hand, in the breakthrough work [BCOV94] on topological string theory, Bershadsky-Cecotti-Ooguri-Vafa proposed a closed string field theory interpretation of the B-model, and suggested that the B-model partition function could be constructed from a quantum field theory, which is called the Kodaira-Spencer gauge theory of gravity in [BCOV94]. The solution space of the classical equations of motion in Kodaira-Spencer gauge theory describes the moduli space of deformations of complex structures on the underlying Calabi-Yau manifold, from which we can recover the well-known geometry of the genus 0 B-model. We will call this quantum field theory as BCOV theory. Following this philosophy, a non-trivial prediction has been made in [BCOV94], which says that the genus one partition function in the B-model on Calabi-Yau three-fold is given by certain holomorphic Ray-Singer torsion and it could be identified with the genus one Gromov-Witten invariants on the mirror Calabi-Yau three-fold. This is recently confirmed by Zinger in [Zin93].

The main purpose of this thesis is to understand BCOV theory from the mathematical point of view. In physics, the main difficulty in understanding a quantum gauge theory lies in the appearance of singularities and gauge anomalies arising from the path integral quantization, and this is where the celebrated idea of renormalization comes into playing a significant role. One mathematical approach for perturbative renormalization of quantum field theories based on Wilson's effective action philosophy is developed by Kevin Costello in [Cos11]. We will develop the general framework of constructing higher genus B-model from BCOV theory using the techniques of perturbative renormalization theory. We carry out the construction in details for the case of one-dimensional Calabi-Yaus, i.e., elliptic curves, and prove that the corresponding B-model partition function is identical to the A-model partition function constructed from Gromov-Witten theory on the mirror elliptic curves. This is the first example of compact Calabi-Yau manifolds where mirror symmetry is established at all genera. The thesis is based mainly on the work [CL, Li].

We will give a brief description of the main results in this introduction. In section 1.1, we collect some basics facts on Gromov-Witten theory and the A-model. In section 1.2, we describe the geometry of B-model and the BCOV theory. In section 1.3, we state the main result for the higher genus mirror symmetry on the elliptic curve. 1.1. The A-model and Gromov-Witten theory. Let X be a smooth projective algebraic variety with complexified Kähler form ω_X , where $\operatorname{Re} \omega_X$ is a Kähler form, and $\operatorname{Im} \omega_X \in \operatorname{H}^2(X, \mathbb{R})/\operatorname{H}^2(X, \mathbb{Z})$. The Gromov-Witten theory on X concerns the moduli space

$$\overline{M}_{g,n,\beta}(X)$$

parametrizing Kontsevich's stable maps [Kon95a] f from connected, genus g, nodal curve C to X, with n distinct smooth marked points, such that

$$f_*[C] = \beta \in \mathrm{H}_2(X, \mathbb{Z})$$

This moduli space is equipped with evaluation maps

$$ev_i: \overline{M}_{g,n,\beta}(X) \to X$$

 $[f, (C; p_1, \cdots, p_n)] \to ev_i ([f, (C; p_1, \cdots, p_n)]) = f(p_i)$

The cotangent line to the i^{th} marked point is a line bundle on $\overline{M}_{g,n,\beta}(X)$, whose first Chern class will be denoted by $\psi_i \in \mathrm{H}^2(\overline{M}_{g,n,\beta}(X))$. The Gromov-Witten invariants of X are defined by

$$\begin{array}{lll} \langle -\rangle : \operatorname{Sym}^{n}_{\mathbb{C}} \left(\operatorname{H}^{*}(X)[[t]] \right) & \to & \mathbb{C} \\ \\ \left\langle t^{k_{1}}\alpha_{1}, \cdots, t^{k_{n}}\alpha_{n} \right\rangle^{X}_{g,n,\beta} & = & \int_{\left[\overline{M}_{g,n,\beta}(X)\right]^{vir}} \psi^{k_{1}}_{1} ev^{*}_{1}\alpha_{1} \cdots \psi^{k_{n}}_{n} ev^{*}_{n}\alpha_{n} \end{array}$$

where $\left[\overline{M}_{g,n,\beta}(X)\right]^{vir}$ is the virtual fundamental class [LT98, BF97] of $\overline{M}_{g,n,\beta}(X)$, which is a homology class of dimension

(1.1)
$$(3 - \dim X)(2g - 2) + 2\int_{\beta} c_1(X) + 2n$$

Definition 1.1. X is a Calabi-Yau variety if its anti-canonical bundle is trivial.

From now on we will focus on Calabi-Yau varieties. From (1.1), we see that in the Calabi-Yau case, the dimension of the virtual fundamental class doesn't depend on β , since $c_1(X) = 0$.

Definition 1.2. The genus g A-model partition function $\mathbf{F}_{g,n,X;q}^{A}[-]$ with n inputs is defined to be the multi-linear map

$$\begin{aligned} \mathbf{F}_{g,n,X;q}^{A} &: \operatorname{Sym}_{\mathbb{C}}^{n} \left(\operatorname{H}^{*}(X,\mathbb{C})[[t]] \right) &\to \mathbb{C} \\ \mathbf{F}_{g,n,X;q}^{A} \left[t^{k_{1}} \alpha_{1}, \cdots, t^{k_{n}} \alpha_{n} \right] &= \sum_{\beta \in \operatorname{H}_{2}(X,\mathbb{Z})} q^{\int_{\beta} \omega_{X}} \left\langle t^{k_{1}} \alpha_{1}, \cdots, t^{k_{n}} \alpha_{n} \right\rangle_{g,n,\beta}^{X} \end{aligned}$$

where q is a formal variable.

The A-model partition function satisfies the following basic properties

(1) Degree Axiom. $\mathbf{F}_{g,n,X;q}^{A}\left[t^{k_{1}}\alpha_{1},\cdots,t^{k_{n}}\alpha_{n}\right]$ is non-zero only for

$$\sum_{i=1}^{n} (\deg \alpha_i + 2k_i) = (2g - 2) (3 - \dim X) + 2n$$

Moreover, we have the Hodge decomposition $\mathrm{H}^{n}(X, \mathbb{C}) = \bigoplus_{p+q=n} \mathrm{H}^{p,q}$. If we define the Hodge weight of $t^{k} \alpha \in t^{k} \mathrm{H}^{p,q}$ by $\mathrm{HW}(t^{k} \alpha) = k+p-1$, then the reality condition implies the Hodge weight condition

$$\sum_{i=1}^{n} \operatorname{HW}(\alpha_{i}) = (g-1) \left(3 - \dim X\right)$$

(2) String equation. $\mathbf{F}^{A}_{g,n,X;q}$ satisfies the string equation

$$\mathbf{F}_{g,n+1,X;q}^{A}\left[1,t^{k_{1}}\alpha_{1},\cdots,t^{k_{n}}\alpha_{n}\right] = \sum_{i=1}^{n} \mathbf{F}_{g,n,X;q}^{A}\left[t^{k_{1}}\alpha_{1},\cdots,t^{k_{i}-1}\alpha_{i},\cdots,t^{k_{n}}\alpha_{n}\right]$$

(3) Dilaton equation. $\mathbf{F}_{g,n,X;q}^{A}$ satisfies the dilaton equation

$$\mathbf{F}_{g,n,X;q}^{A}\left[t,t^{k_{1}}\alpha_{1},\cdots,t^{k_{n}}\alpha_{n}\right] = (2g-2+n)\mathbf{F}_{g,n,X;q}^{A}\left[t^{k_{1}}\alpha_{1},\cdots,t^{k_{n}}\alpha_{n}\right]$$

The parameter q can be viewed as the Kähler moduli. Since the Gromov-Witten invariants are invariant under complex deformations, $\mathbf{F}_{g,n,X;q}^A$ only depends on the Kähler moduli, but not on the complex moduli of X. This is the special property characterizing the A-model.

A special role is played by Calabi-Yau 3-folds where the original mirror symmetry is established. In the case of dimension 3,

$$\dim\left[\overline{M}_{g,n,\beta}(X)\right]^{vir} = 2n$$

$$\begin{aligned} & \mathrm{H}^*(X)^{\otimes 3} \quad \to \quad \mathbb{C} \\ & \alpha \otimes \beta \otimes \gamma \quad \to \quad \mathbf{F}^A_{0,3,X;q}\left[\alpha,\beta,\gamma\right] \end{aligned}$$

If $\beta = 0$, we know that the Gromov-Witten invariants are reduced to the classical intersection product

$$\langle \alpha, \beta, \gamma \rangle_{0,3,\beta=0} = \int_X \alpha \wedge \beta \wedge \gamma$$

Therefore, the A-model Yukawa coupling

$$\mathbf{F}_{0,3,X;q}^{A}\left[\alpha,\beta,\gamma\right] = \int_{X} \alpha \wedge \beta \wedge \gamma + \sum_{\beta \neq 0} q^{\int_{\beta} \omega_{X}} \int_{\left[\overline{M}_{0,3,\beta}(X)\right]^{vir}} ev_{1}^{*} \alpha \wedge ev_{2}^{*} \beta \wedge ev_{3}^{*} \gamma$$

can be viewed as a quantum deformation of the classical intersection product. Moreover, it gives a q-deformation of the classical ring structure of $H^*(X, \mathbb{C})$, which is called the *quantum cohomology ring*.

1.2. The B-model and BCOV theory. The geometry of B-model concerns the moduli space of complex structures of Calabi-Yau manifolds. Let \check{X}_{τ} be a Calabi-Yau 3-fold with nowhere vanishing holomorphic volume form $\Omega_{\check{X}_{\tau}}$. Let $T_{\check{X}_{\tau}}$ be the holomorphic tangent bundle. Here τ parametrizes the complex structures of \check{X} .

Definition 1.4. The B-model Yukawa coupling is defined to be

$$\begin{aligned} \mathrm{H}^{*}(\check{X}_{\tau},\wedge^{*}T_{\check{X}_{\tau}})^{\otimes 3} &\to & \mathbb{C} \\ \mu_{1}\otimes\mu_{2}\otimes\mu_{3} &\to & F^{B}_{0,3,\check{X}_{\tau}}\left[\mu_{1},\mu_{2},\mu_{3}\right] = \int_{\check{X}_{\tau}}\left(\mu_{1}\wedge\mu_{2}\wedge\mu_{3}\vdash\Omega_{\check{X}_{\tau}}\right)\wedge\Omega_{\check{X}} \end{aligned}$$

where \vdash is the natural contraction between tensors in $\wedge^* T_{\check{X}}$ and $\wedge^* T_{\check{X}}^*$.

Generally speaking, string theory predicts that we should also have the B-model correlation functions

$$\mathbf{F}^{B}_{g,n,\check{X}_{\tau}}: \operatorname{Sym}^{n}_{\mathbb{C}}\left(\operatorname{H}^{*}(\check{X}_{\tau}, \wedge^{*}T_{\check{X}_{\tau}})[[t]]\right) \to \mathbb{C}$$

However, little is known for genus g > 0 and the inclusion of gravitational descendants t is even more mysterious.

Motivated by the physics idea of [BCOV94], we will approach the B-model correlation functions from the renormalization of BCOV theory using the techniques developed by [Cos11]. Let $\Omega_{\check{X}_{\tau}}$ be a fixed nowhere vanishing holomorphic volume form. The existence of $\Omega_{\check{X}_{\tau}}$ is guaranteed by the Calabi-Yau condition. Let

$$\mathscr{E}_{\check{X}_{\tau}} = \mathrm{PV}_{\check{X}_{\tau}}^{*,*}[[t]]$$

be the space of fields of BCOV theory, where $\mathrm{PV}_{\check{X}_{\tau}}^{*,*}$ is the space of polyvector fields, see (2.1). We define the classical BCOV action as a functional on $\mathscr{E}_{\check{X}_{\tau}}$ by

$$S^{BCOV} = \sum_{n \ge 3} S_n^{BCOV}$$

where

$$S_n^{BCOV} : \operatorname{Sym}\left(\mathscr{E}_{\check{X}_{\tau}}^{\otimes n}\right) \to \mathbb{C}$$
$$t^{k_1}\mu_1 \otimes \cdots \otimes t^{k_n}\mu_n \to \int_{\overline{M}_{0,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \int_{\check{X}_{\tau}} \left(\mu_1 \cdots \mu_n \vdash \Omega_{\check{X}_{\tau}}\right) \wedge \Omega_{\check{X}_{\tau}}$$

where $\int_{\overline{M}_{0,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} = \binom{n-3}{k_1, \cdots, k_n}$ is the ψ -class integration. Let

$$Q = \partial - t\partial : \mathscr{E}_{\check{X}_{\tau}} \to \mathscr{E}_{\check{X}_{\tau}}$$

be the differential, and we refer to (4.1) and the corresponding section for the detailed explanation. Q induces a derivation on the space of functionals on $\mathscr{E}_{X_{\tau}}$, which we still denote by Q. Let $\{-.-\}$ be the Poisson bracket on local functionals defined by definition 4.4. Then S^{BCOV} satisfies the following *classical master equation* (see Lemma 4.6)

(1.2)
$$QS^{BCOV} + \frac{1}{2} \left\{ S^{BCOV}, S^{BCOV} \right\} = 0$$

The physics meaning of classical master equation is that S^{BCOV} is endowed with a gauge symmetry. S^{BCOV} generalizes the original Kodaira-Spencer gauge action on Calabi-Yau 3folds [BCOV94] to arbitrary dimensions, and remarkably, it also includes the gravitational descendants t. Let K_L be the Heat kernel of the operator $e^{-L\Delta}$ and ∂K_L be the smooth kernel of $\partial e^{-L\Delta}$, for L > 0. We define the regularized propagator P_{ϵ}^L by the smooth kernel of the operator $-\int_{\epsilon}^L \bar{\partial}^* \partial e^{-u\Delta} du$, for $\epsilon, L > 0$. Both ∂K_L and P_{ϵ}^L define operators on the space of functionals

$$rac{\partial}{\partial P_{\epsilon}^{L}}, \quad \Delta_{L} = rac{\partial}{\partial \left(\partial K_{L}
ight)}$$

via contraction, see Definition 3.23 and Definition 4.7.

We would like to construct the quantization of the BCOV theory on \check{X}_{τ} , which is given by a family of functionals on $\mathscr{E}_{\check{X}_{\tau}}$ valued in $\mathbb{C}[[\hbar]]$ parametrized by L > 0

$$\mathbf{F}[L] = \sum_{g \ge 0} \hbar^g \mathbf{F}_g[L]$$

which satisfies the renormalization group flow equation

$$e^{\mathbf{F}[L]/\hbar} = e^{\hbar \frac{\partial}{\partial P_{\epsilon}^{L}}} e^{\mathbf{F}[\epsilon]/\hbar}, \ \, \forall \epsilon, L > 0$$

the classical limit condition: $\mathbf{F}[L]$ has a small L asymptotic expansion in terms of local functionals as $L \to 0$ and

$$\lim_{L \to 0} \mathbf{F}_0[L] = S^{BCOV}$$

the quantum master equation

$$(Q + \hbar \Delta_L) e^{\mathbf{F}[L]/\hbar} = 0, \quad \forall L > 0$$

and certain other properties such as the string equations and dilaton equations in this context. All of these will be discussed in details in section 4.

Once we have constructed the quantization $\mathbf{F}[L]$, we can let $L \to \infty$. Since $\lim_{L\to\infty} K_L$ becomes the Harmonic projection, we see that $\lim_{L\to\infty} \partial K_L = 0$. The quantum master equation at $L = \infty$ says

$$Q\mathbf{F}[\infty] = 0$$

This implies that $\mathbf{F}[\infty]$ induces a well-defined functional on the *Q*-cohomology of $\mathscr{E}_{\check{X}_{\tau}}$. We will write

$$\mathbf{F}[\infty] = \sum_{g \ge 0} \hbar^g \mathbf{F}^B_{g, \check{X}_\tau}$$

Using the isomorphism (see Lemma 4.10)

$$\mathrm{H}^*(\mathscr{E}_{\check{X}_{\tau}}, Q) \cong \mathrm{H}^*(\check{X}_{\tau}, \wedge^* T_{\check{X}_{\tau}})[[t]]$$

and decomposing $\mathbf{F}_{g,\check{X}}^B$ into number of inputs, we can define the genus g B-model correlation functions by

$$\mathbf{F}^{B}_{g,n,\check{X}}: \operatorname{Sym}^{n}_{\mathbb{C}} \left(\operatorname{H}^{*}(\check{X}_{\tau}, \wedge^{*}T_{\check{X}_{\tau}})[[t]] \right) \to \mathbb{C}$$

Therefore the problem of constructing higher genus B-model is reduced to the construction of the quantization $\mathbf{F}[L]$. The general formalism of [Cos11] tells us that the quantization is controlled by certain L_{∞} algebraic structure on the space of local functionals on $\mathscr{E}_{X_{\tau}}$. There's an obstruction class for constructing $\mathbf{F}_g[L]$ at each genus g > 0, and it's natural is conjecture that all the obstruction classes vanish for BCOV theory. For X being one-dimensional, i.e., the elliptic curve, we will show that this is indeed the case.

To establish mirror symmetry at higher genus, we need to compare the A-model correlation function $\mathbf{F}_{g,n,X;q}^A$ with the B-model correlation function $\mathbf{F}_{g,n,\check{X}_{\tau}}^B$. In general, $\mathbf{F}_{g,n,\check{X}_{\tau}}^B$ doesn't depend holomorphically on τ , and there's so-called holomorphic anomalies discovered by [BCOV94]. It's predicted by [BCOV94] that we should be able to make sense of the limit $\lim_{\bar{\tau}\to\infty}\mathbf{F}_{g,n,\check{X}_{\tau}}^B$ around the large complex limit of \check{X}_{τ} . The higher genus mirror conjecture can be stated as the identification of

$$F^A_{g,n,X;q} \longleftrightarrow \lim_{\bar{\tau} \to \infty} \mathbf{F}^B_{g,n,\check{X}_{\tau}}$$

under certain identification of cohomology classes

$$\mathrm{H}^*(X, \wedge^* T^*_X) \longleftrightarrow \mathrm{H}^*(\check{X}_\tau, \wedge^* T_{\check{X}_\tau})$$

and the mirror map between Kähler moduli and complex moduli

$$q \longleftrightarrow \tau$$

1.3. Main results. Let $\check{X}_{\tau} = \check{E}_{\tau}$ be the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$, where τ lies in the upper-half plane viewed as the complex moduli of \check{E} .

Theorem 1.5 ([CL]). There exists a unique quantization $\mathbf{F}^{\check{E}_{\tau}}[L]$ of BCOV theory on \check{E}_{τ} satisfying the dilaton equation. Morever, $\mathbf{F}^{\check{E}_{\tau}}[L]$ satisfies the Virasoro equations.

Section 5 is devoted to explain and prove this theorem.

Since we know that the A-model Gromov-Witten invariants on the elliptic curve also satisfies the Virasoro equations [OP06b], the proof of mirror symmetry can be reduced to the so-called stationary sectors [OP06a]. More precisely, let E the dual elliptic curve of \check{E} and $\omega \in H^2(E,\mathbb{Z})$ be the dual class of a point. The stationary sector of Gromov-Witten invariants are defined for descendants of ω

$$\left\langle t^{k_1}\omega,\cdots,t^{k_n}\omega\right\rangle_{g,d,E} = \int_{[\overline{M}_{g,n}(E,d)]^{vir}}\prod_{i=1}^n\psi_i^{k_i}ev_i^*(\omega)$$

On the other hand, we let $\check{\omega} \in \mathrm{H}^1(\check{E}_{\tau}, T_{\check{E}_{\tau}})$ be the class such that $\mathrm{Tr}(\check{\omega}) = 1$.

Theorem 1.6 ([Li]). For any genus $g \ge 0$, n > 0, and non-negative integers k_1, \dots, k_n , (1) $\mathbf{F}_g^{\check{E}_{\tau}}[\infty][t^{k_1}\check{\omega}, \dots, t^{k_n}\check{\omega}]$ is an almost holomorphic modular form of weight

$$\sum_{i=1}^{n} (k_i + 2) = 2g - 2 + 2n$$

It follows that the limit $\lim_{\bar{\tau}\to\infty} \mathbf{F}_g^{\check{E}_{\tau}}[\infty][t^{k_1}\check{\omega},\cdots,t^{k_n}\check{\omega}]$ makes sense and is a quasimodular form of the same weight.

(2) The higher genus mirror symmetry holds on elliptic curves in the following sense

$$\sum_{d\geq 0} q^d \left\langle t^{k_1}\omega, \cdots, t^{k_n}\omega \right\rangle_{g,d,E} = \lim_{\bar{\tau}\to\infty} \mathbf{F}_g^{E_{\tau}}[\infty][t^{k_1}\check{\omega}, \cdots, t^{k_n}\check{\omega}]$$

under the identification $q = \exp(2\pi i \tau)$.

Section 6 is denoted to prove this theorem.

2. Classical Geometry of Calabi-Yau Moduli Space

In this section, we discuss some basics on the classical geometry of the moduli space of Calabi-Yau manifolds, with the purpose of motivating the Kodaira-Spencer gauge theory. We will also set up our notations that will be used throughout this thesis.

2.1. **Polyvector fields.** In this subsection we describe the Batalin-Vilkovisky structure of polyvector fields on the Calabi-Yau manifolds.

2.1.1. DGA structure. Let X be a compact Calabi-Yau manifold of dimension d. Let

(2.1)
$$\operatorname{PV}_{X}^{*,*} = \bigoplus_{0 \le i,j \le d} \operatorname{PV}_{X}^{i,j} \qquad \operatorname{PV}_{X}^{i,j} = \mathcal{A}^{0,j}(X, \wedge^{i}T_{X})$$

denote the space of polyvector fields on X. Here T_X is the holomorphic tangent bundle of X, and $\mathcal{A}^{0,j}(X, \wedge^i T_X)$ is the space of smooth (0, j)-forms valued in $\wedge^i T_X$. $\mathrm{PV}_X^{*,*}$ is a differential bi-graded commutative algebra; the differential is the operator

$$\bar{\partial} : \mathrm{PV}^{i,j}(X) \to \mathrm{PV}^{i,j+1}(X).$$

and the algebra structure arises from wedging polyvector fields. The degree of elements of $PV_X^{i,j}$ is i + j. Explicitly, let $\{z^i\}$ be local holomorphic coordinates on X. Let I = $\{i_1, i_2, \dots, i_k\}$ be an ordered subset of $\{1, 2, \dots, d\}$, with |I| = k. We will use the following notations

$$dz^{I} = dz^{i_{1}} \wedge dz^{i_{2}} \wedge \dots \wedge dz^{i_{k}}, \quad \frac{\partial}{\partial z^{I}} = \frac{\partial}{\partial z^{i_{1}}} \wedge \frac{\partial}{\partial z^{i_{2}}} \wedge \dots \wedge \frac{\partial}{\partial z^{i_{k}}}$$

and similarly for $d\bar{z}^I$ and $\frac{\partial}{\partial \bar{z}^I}$. Given $\alpha \in \mathrm{PV}_X^{i,j}, \beta \in \mathrm{PV}_X^{k,l}$, we write in local coordinates

$$\alpha = \sum_{|I|=i,|J|=j} \alpha_{\bar{J}}^{I} d\bar{z}^{J} \otimes \frac{\partial}{\partial z^{I}}, \ \beta = \sum_{|K|=k,|L|=l} \beta_{\bar{L}}^{K} d\bar{z}^{L} \otimes \frac{\partial}{\partial z^{K}}$$

then

(2.2)
$$\bar{\partial}\alpha = \sum_{|I|=i,|J|=j} \bar{\partial}\alpha_{\bar{J}}^{I} \wedge d\bar{z}^{J} \otimes \frac{\partial}{\partial z^{I}}$$

and the product structure is given by

(2.3)
$$\alpha\beta \equiv \alpha \wedge \beta = \sum_{\substack{|I|=i, |J|=j\\|K|=k, |L|=l}} (-1)^{il} \alpha_{\bar{J}}^{I} \beta_{\bar{L}}^{K} d\bar{z}^{J} \wedge d\bar{z}^{L} \otimes \frac{\partial}{\partial z^{I}} \wedge \frac{\partial}{\partial z^{K}}$$

The graded-commutativity says that

(2.4)
$$\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha$$

where $|\alpha|, |\beta|$ denote the degree of α, β respectively.

2.1.2. *Batalin-Vilkovisky structure*. Calabi-Yau condition implies that there exists a nowhere vanishing holomorphic volume form

$$\Omega_X \in \Omega^{d,0}(X)$$

which is unique up to a multiplication by a constant. Let us fix a choice of Ω_X . It induces an isomorphism between the space of polyvector fields and differential forms

(2.5)
$$\mathrm{PV}_X^{i,j} \stackrel{\vdash \Omega_X}{\cong} \mathcal{A}_X^{d-i,j}$$

$$(2.6) \qquad \qquad \alpha \rightarrow \quad \alpha \vdash \Omega_X$$

where \vdash is the contraction map, which is defined in local coordinates on the basis

(2.7)
$$\frac{\partial}{\partial z^{I}} \vdash dz^{J} = \begin{cases} (-1)^{|I|(|I|-1)/2} dz^{K} & \text{if } dz^{J} = dz^{I} \wedge dz^{K}, I \cap K = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

The holomorphic de Rham differential ∂ on differential forms defines an operator on polyvector fields via the above isomorphism, which we still denote by ∂

$$\partial: \mathrm{PV}_X^{i,j} \to \mathrm{PV}_X^{i-1,j}$$

i.e.

(2.8)
$$(\partial \alpha) \vdash \Omega_X \equiv \partial(\alpha \vdash \Omega_X), \ \alpha \in \mathrm{PV}_X^{i,j}$$

Obviously, the definition of ∂ doesn't depend on the choice of Ω_X .

Given $\alpha \in \mathrm{PV}_X^{*,*}$, the multiplication

$$\alpha \wedge : \mathrm{PV}_X^{*,*} \to \mathrm{PV}_X^{*,*}$$

defines an operator acting on polyvector fields which has the same degree as α .

Lemma 2.1. For any $\alpha, \beta, \gamma \in PV_X^{*,*}$,

$$[[[\partial, \alpha], \beta], \gamma] = 0$$

viewed as an operator acting on $\mathrm{PV}_X^{*,*}$. Here $[\cdot, \cdot]$ is the graded commutator.

Proof. This follows from direct calculation in local coordinates.

It follows from the lemma that the operator $[[\partial, \alpha], \beta]$ is equivalent to multiplying by a polyvector fields, which defines the bracket

(2.9)
$$\{\alpha,\beta\} = [[\partial,\alpha],\beta] \in \mathrm{PV}_X^{*,*}$$

The bracket used here differs from the Schouten-Nijenhuis bracket by a sign. More precisely, if we let $\{,\}_{sn}$ denote the Schouten-Nijenhuis bracket, then

(2.10)
$$\{\alpha, \beta\}_{sn} = -(-1)^{|\alpha|} \{\alpha, \beta\}$$

In particular, if both $\alpha, \beta \in PV_X^{1,0}$, then $\{\alpha, \beta\}$ is just the ordinary Lie-bracket on vector fields.

Lemma 2.2. The following properties hold

(1) Graded symmetry

$$\{\alpha,\beta\} = (-1)^{\deg(\alpha)\deg(\beta)}\{\beta,\alpha\}$$

(2) Leibniz relation

$$\{\alpha, \beta \wedge \gamma\} = \{\alpha, \beta\} \wedge \gamma + (-1)^{|\beta||\gamma|} \{\alpha, \gamma\} \wedge \beta$$

(3) Graded Jacobi Identity

$$\{\{\alpha,\beta\},\gamma\} = -(-1)^{|\alpha|}\{\alpha,\{\beta,\gamma\}\} - (-1)^{(|\alpha|+1)|\beta|}\{\beta,\{\alpha,\gamma\}\}$$

(4) Batalin-Vilkoviski identity (Todorov-Tian's lemma)

$$\partial(\alpha \wedge \beta) = (\partial \alpha) \wedge \beta + (-1)^{\alpha} \alpha \wedge \partial \beta + \{\alpha, \beta\}$$

Proof. Since $[\alpha,\beta]=0,$ it follows from the Jacobi identity that

$$[[\partial, \alpha], \beta] = [\partial, [\alpha, \beta]] - (-1)^{|\alpha|} [\alpha, [\partial, \beta]] = (-1)^{|\alpha||\beta|} [[\partial, \beta], \alpha]$$

This proves (1).

$$\begin{aligned} \{\alpha, \beta \wedge \gamma\} &= [[\partial, \alpha], \beta \wedge \gamma] \\ &= -(-1)^{|\alpha|} [\alpha, [\partial, \beta \wedge \gamma]] \\ &= -(-1)^{|\alpha|} [\alpha, [\partial, \beta] \gamma + (-1)^{|\beta||\gamma|} [\partial, \gamma] \beta] \\ &= -(-1)^{|\alpha|} [\alpha, [\partial, \beta]] \gamma - (-1)^{|\alpha| + |\beta||\gamma|} [\alpha, [\partial, \gamma]] \beta \\ &= \{\alpha, \beta\} \wedge \gamma + (-1)^{|\beta||\gamma|} \{\alpha, \gamma\} \wedge \beta \end{aligned}$$

This proves (2).

$$\begin{aligned} \{\{\alpha,\beta\},\gamma\} &= [[\partial,[[\partial,\alpha],\beta]],\gamma] \\ &= (-1)^{|\alpha|+|\beta|}[[[\partial,\alpha],\beta],[\partial,\gamma]] \\ &= (-1)^{|\alpha|+|\beta|}[[\partial,\alpha],[\beta,[\partial,\gamma]]] - (-1)^{|\alpha||\beta|+|\alpha|}[\beta,[[\partial,\alpha],[\partial,\gamma]]] \\ &= (-1)^{|\alpha|+|\beta|}[[\partial,\alpha],[\beta,[\partial,\gamma]]] - (-1)^{|\alpha||\beta|+|\alpha|}[\beta,[\partial,[\alpha,[\partial,\gamma]]]] \\ &= -(-1)^{|\alpha|}\{\alpha,\{\beta,\gamma\}\} - (-1)^{(|\alpha|+1)|\beta|}\{\beta,\{\alpha,\gamma\}\} \end{aligned}$$

where on the fourth line we have used the fact that $[\partial, [\partial, \cdot]] = 0$. This proves (3).

To prove (4), we identify $\partial(\alpha \wedge \beta)$ with the action of the operator $[\partial, \alpha \wedge \beta]$ on 1 since $\partial(1) = 0$. Therefore

$$\partial(\alpha \wedge \beta) = [\partial, \alpha \wedge \beta] \cdot 1$$

$$= [\partial, \alpha]\beta \cdot 1 + (-1)^{|\alpha|} \alpha \wedge \partial\beta$$

$$= [[\partial, \alpha], \beta] + (-1)^{|\beta|(|\alpha|+1)}\beta \wedge \partial\alpha + (-1)^{|\alpha|}\alpha \wedge \partial\beta$$

$$= (\partial\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \partial\beta + \{\alpha, \beta\}$$

Remark 2.3. The Batalin-Vilkovisky identity has the natural generalization

$$\partial(\alpha_1 \wedge \dots \wedge \alpha_n) = \sum_i \pm (\partial \alpha_i) \alpha_1 \wedge \dots \hat{\alpha_i} \dots \wedge \alpha_n + \sum_{i \neq j} \pm \{\alpha_i, \alpha_j\} \alpha_1 \wedge \dots \hat{\alpha_i} \dots \hat{\alpha_j} \dots \wedge \alpha_n$$

where the signs are given by permuting the α 's. The proof is similar.

Definition 2.4. A **Batalin-Vilkovisky algebra** (BV algebra) is given by the tuple $(\mathcal{A}, \cdot, \Delta, \{,\})$ where

- (1) \mathcal{A} is a graded vector space.
- (2) $\cdot : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is associative and graded commutative.
- (3) $\Delta : \mathcal{A} \to \mathcal{A}$ is an odd differential of degree (-1).
- (4) $\{,\}: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \text{ is a bilinear operation such that for all } \alpha, \beta, \gamma \in \mathcal{A}$

$$\Delta(\alpha \cdot \beta) = (\Delta \alpha) \cdot \beta + (-1)^{|\alpha|} \alpha \cdot \Delta \beta + (-1)^{|\alpha|} \{\alpha, \beta\}$$
$$\{\alpha, \beta \cdot \gamma\} = \{\alpha, \beta\} \cdot \gamma + (-1)^{(|\alpha|+1)|\beta|} \beta \cdot \{\alpha, \gamma\}$$

Corollary 2.5. The tuple $(PV_X^{*,*}, \land, \partial, -\{,\}_{sn})$ is a BV algebra. Here $\{,\}_{sn}$ is the Schouten-Nijenhuis bracket (see (2.10)).

2.1.3. The trace map. Define a map

$$\operatorname{Tr}: \operatorname{PV}_X^{*,*} \to \mathbb{C}$$

by

(2.11)
$$\operatorname{Tr}(\alpha) = \begin{cases} \int_X (\alpha \vdash \Omega_X) \land \Omega_X & \text{if } \alpha \in \mathrm{PV}_X^{d,d} \\ 0 & \text{if } \alpha \notin \mathrm{PV}_X^{d,d} \end{cases}$$

The pairing

$$\mathrm{PV}^{i,j}(X) \otimes \mathrm{PV}^{d-i,d-j}(X) \to \mathbb{C}$$
$$\alpha \otimes \beta \to \mathrm{Tr}(\alpha\beta)$$

is non-degenerate, i.e., it has no kernel.

Lemma 2.6. The operator $\overline{\partial}$ is skew self-adjoint for the pairing $\text{Tr}(\alpha\beta)$, and the operator ∂ is self-adjoint for this pairing.

Proof. The fact that $\bar{\partial}$ is skew self adjoint follows immediately from the fact that $\bar{\partial}$ is a derivation for the algebra structure on $\mathrm{PV}_X^{*,*}$. For ∂ , First we know that

$$\operatorname{Tr} \left((\partial \alpha) \beta \right) = \int_X \left((\partial \alpha) \beta \vdash \Omega_X \right) \wedge \Omega_X$$
$$= \pm \int_X \left(\partial \alpha \vdash \Omega \right) \wedge \left(\beta \vdash \Omega_X \right)$$
$$= \pm \int_X \left(\alpha \vdash \Omega \right) \wedge \left(\partial \beta \vdash \Omega_X \right)$$
$$= \pm \int_X \left(\alpha \wedge \partial \beta \vdash \Omega \right) \wedge \Omega_X$$
$$= \pm \operatorname{Tr} \left(\alpha \partial \beta \right)$$

for some sign \pm . To determine this sign, we choose local holomorphic coordinates z^1, \dots, z^d such that $\Omega_X = dz^1 \wedge \dots \wedge dz^d$, then it's easy to see that

$$\partial = \sum_{i} \frac{\partial}{\partial z^{i}} \frac{\partial}{\partial (\partial_{z^{i}})}$$

where $\frac{\partial}{\partial(\partial_{z^i})}$ is the odd derivation generated by

$$\frac{\partial}{\partial(\partial_{z^i})}\partial_{z^j} = \delta^j_i$$

Therefore the sign can be determined by

$$\operatorname{Tr}\left((\partial\alpha)\beta\right) = -\sum_{i} \operatorname{Tr}\left(\left(\frac{\partial}{\partial(\partial_{z^{i}})}\alpha\right)\frac{\partial}{\partial z^{i}}\beta\right) = (-1)^{|\alpha|} \operatorname{Tr}\left(\alpha\partial\beta\right)$$

2.2. Deformation theory of Calabi-Yau manifolds.

2.2.1. Local deformation of complex structures. Let X be a compact Calabi-Yau manifold of dimension d with a fixed Kähler metric. We assume that $H^0(X, T_X) = 0$. The complex structure of X is equivalent to the decomposition of the complexified cotangent bundles

$$\Omega^1_X \otimes_{\mathbb{R}} \mathbb{C} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$

into types (1,0) and (0,1) with additional integrability conditions. We consider a nearby deformation of the complex structure, which can be viewed as deforming the above decomposition into new types of (1,0) and (0,1) forms. This can be described as follows: let $\mu \in PV_X^{1,1}$ be a smooth polyvector field, and z^1, \dots, z^d are local holomorphic coordinates on X such that locally

$$\mu = \sum_{i=1}^{d} \mu_{\bar{j}}^{i} d\bar{z}^{j} \otimes \frac{\partial}{\partial z^{i}}$$

If $\|\mu\|$ is sufficiently small ($\|\cdot\|$ is the norm with respect to the fixed metric), then we obtain a new almost complex structure J_{μ} by requiring that the new (1,0)-form is spanned locally by

(2.12)
$$\theta^i = dz^i + \mu \vdash dz^i, \quad 1 \le i \le d$$

The integrability condition says that $d\theta^i$ is of type (2,0) + (1,1) in the new decomposition, which is equivalent to

(2.13)
$$\bar{\partial}\mu + \frac{1}{2}\{\mu,\mu\} = 0$$

Let Def_X be the local universal deformation space of X, which is a germ of the *Te-ichmüller* space of X at the given complex structure of X. Def_X represents the following deformation functor

$$\mathcal{D}ef_X$$
: Artin local \mathbb{C} -algebra \rightarrow sets

such that for a given Artin local \mathbb{C} -algebra A with maximal ideal m_A ,

$$\mathcal{D}ef_X(A) = \left\{ \mu \in \mathrm{PV}_X^{1,1} \otimes m_A \left| \,\bar{\partial}\mu + \frac{1}{2} \{\mu, \mu\} = 0 \right\} / \sim \right\}$$

The equivalence relation is given by the gauge action of $PV^{1,0} \otimes m_A$ on $PV^{1,1} \otimes m_A$

$$\mu \to e^{\operatorname{ad} \alpha}(\mu) + \frac{\operatorname{Id} - e^{\operatorname{ad} \alpha}}{\operatorname{ad} \alpha} \bar{\partial} \alpha$$

for $\alpha \in \mathrm{PV}^{1,0} \otimes m_A, \mu \in \mathrm{PV}^{1,1} \otimes m_A$. Here ad α is the adjoint action $\{\alpha, \cdot\}$. This equivalence can be viewed as generated by diffeomorphisms. From the general theory of deformation of complex structures, the tangent space of the germ Def_X is given by

$$\mathcal{D}ef_X(\mathbb{C}[\epsilon]/\epsilon^2) = \mathrm{H}^1(X, T_X)$$

and $\mathrm{H}^2(X, T_X)$ serves as an obstruction space for the deformation functor $\mathcal{D}ef_X$.

In the case of Calabi-Yau manifolds, the deformation functor $\mathcal{D}ef_X$ is unobstructed and Def_X is the germ of a smooth manifold. To see this, let

$$\mu_1 \in \mathrm{H}^{0,1}_{\bar{\partial}}(X, T_X)$$

be a harmonic element with respect to the Kähler metric. μ_1 represents a tangent vector of Def_X . We need to prove the existence of

$$\mu_t \in \mathrm{PV}_X^{1,1} \otimes \mathbb{C}[[t]]$$

such that

$$\bar{\partial}\mu_t + \frac{1}{2} \{\mu_t, \mu_t\} = 0, \quad \mu_t \equiv t\mu_1 \mod t^2$$

Then an argument of Kuranishi shows that the formal power series is convergent given t sufficiently small. It follows that every first order deformation can be realized, i.e., $\mathcal{D}ef_X$ is unobstructed.

We follow the approach of Todorov [Tod89] to construct μ_t . By Todorov-Tian's lemma as in Lemma 2.2, the bracket $\{,\}$ preserves the subspace ker $\partial \subset PV_X^{*,*}$

$$(2.14) \qquad \{,\}: \ker \partial \otimes \ker \partial \to \operatorname{im} \partial \subset \ker \partial$$

If we write

$$\mu_t = \sum_{i \ge 1} t^i \mu_i$$

then $\mu_i, i \geq 2$, can be solved recursive by imposing the following equation

(2.15)
$$\mu_t - t\mu_1 = -\frac{1}{2}\bar{\partial}^* G\{\mu_t, \mu_t\}$$

Here $G = \frac{1}{\Delta}$ is the Green operator with respect to the Kähler metric. Explicitly,

$$\mu_i = -\frac{1}{2} \sum_{k=1}^{i-1} \bar{\partial}^* G\{\mu_k, \mu_{i-k}\}, \quad \forall i \ge 2$$

We show that such constructed μ indeed solves Eqn (2.13) and satisfies

$$\partial^* \mu_t = \partial \mu_t = 0$$

First, observe that the recursive relation and (2.14) implies that

$$\mu_i \in \operatorname{im} \partial, \quad \forall i \ge 2$$

Apply $\bar{\partial}$ to (2.15), we find

$$\bar{\partial}\mu_t = -\frac{1}{2}\bar{\partial}\bar{\partial}^* G\{\mu_t, \mu_t\} = -\frac{1}{2}\{\mu_t, \mu_t\} + \frac{1}{2}\bar{\partial}^* G\bar{\partial}\{\mu_t, \mu_t\}$$

where we have used the fact that $\{\mu_t, \mu_t\} \in \operatorname{im} \partial$ has no harmonic part. Therefore

$$\bar{\partial}\{\mu_t, \mu_t\} = 2\{\bar{\partial}\mu_t, \mu_t\} = -\{\{\mu_t, \mu_t\}, \mu_t\} + \{\bar{\partial}^*G\bar{\partial}\{\mu_t, \mu_t\}, \mu_t\}$$

Jacobi Identity implies that $\{\{\mu_t, \mu_t\}, \mu_t\} = 0$, hence

$$\bar{\partial}\{\mu_t,\mu_t\} = \{\bar{\partial}^* G \bar{\partial}\{\mu_t,\mu_t\},\mu_t\}$$

This recursive relation and the initial condition $\bar{\partial}\{\mu_1,\mu_1\}=0$ implies that

$$\bar{\partial}\{\mu_t,\mu_t\}=0$$

which in turn implies

$$\bar{\partial}\mu_t = -\frac{1}{2}\{\mu_t, \mu_t\}$$

The deformation of the holomorphic volume form can be also described by this solution. Let Ω_0 be the holomorphic volume form on the undeformed X. Consider

(2.16)
$$\Omega_t = e^{\mu_t} \vdash \Omega_0$$

From (2.12), we see that Ω_t is of type (d,0) in the new complex structure given by μ_t . Since

$$d\Omega_t = \left(\bar{\partial}e^{\mu_t} + \partial e^{\mu_t}\right) \vdash \Omega_0 = \left(\bar{\partial}\mu_t + \partial\mu_t + \frac{1}{2}\{\mu_t, \mu_t\}\right)e^{\mu_t} \vdash \Omega_0 = 0$$

where we have used the BV identity in Remark 2.3. It follows that Ω_t is in fact holomorphic in the new complex structure.

If follows from the unobstructedness of $\mathcal{D}ef_X$ that we can choose a linear coordinate $\{t^i\}$ of $\mathrm{H}^1(X, T_X)$ as local coordinates of $\mathrm{D}ef_X$. This is called the **canonical coordinate**, which is unique up to linear transformations. The corresponding holomorphic family of top holomorphic forms (2.16) is called the **canonical family** of holomorphic volume forms.

2.2.2. Extended deformation space and the Formality Theorem. The smoothness theorem for Calabi-Yau manifolds is extended in [BK98] to a bigger deformation space whose tangent space includes all

$$\bigoplus_{i,j} \mathrm{H}^i(X, \wedge^j T_X)$$

We give a brief discussion here for the purpose of later discussion on the higher genus B-model.

The deformation functor $\mathcal{D}ef_X$ is the restriction of the moduli functor associated with the DGLA

$$\left(\mathrm{PV}_X^{1,*}, \bar{\partial}, \{,\}\right)$$

to Artin algebras of degree 0. We can consider the full DGLA

$$\left(\mathrm{PV}_X^{*,*}, \bar{\partial}, \{,\}\right)$$

and let $\mathcal{D}ef_X^{ext}$ be the associated moduli functor. The corresponding moduli space is called the *extended deformation space* of X. There're two closely related DGLA's. The first one is

$$(\ker \partial, \bar{\partial}, \{,\})$$

where ker $\partial \subset \mathrm{PV}_X^{*,*}$ as before is the space of polyvector fields annihilated by ∂ , and Todorov-Tian's lemma implies that $\{,\}$ is a well-defined Lie bracket on ker ∂ .

The second one is

$$(\mathbb{H}, 0, 0)$$

where $\mathbb{H} \subset \mathrm{PV}_X^{*,*}$ denotes the space of harmonic elements. We associate the trivial differential and Lie bracket. Consider the following diagram



where j is the natural embedding, and π is the projection to the harmonic part. j is obviously a map of DGLA. By Hodge theory, we have the isomorphism of cohomology groups

$$\mathrm{H}_{\bar{\partial}}(\ker \partial) \cong \mathrm{H}_{\bar{\partial}}(\mathrm{PV}_X^{*,*}) \cong \mathbb{H}$$

Therefore j is in fact a quasi-isomorphism of DGLA's. On the other hand, since the bracket of two elements in ker ∂ is in fact ∂ -exact, the projection map π is also a quasi-isomorphism of DGLA's. Therefore we come to the *Formality Theorem*

Proposition 2.7 ([BK98]). The DGLA ($PV_X^{*,*}, \bar{\partial}, \{,\}$) is L_{∞} quasi-isomorphic to the DGLA ($\mathbb{H}, 0, 0$)

By Proposition A.12, the moduli functor $\mathcal{D}ef_X^{ext}$ is smooth with tangent space \mathbb{H} . This gives a conceptual generalization of Todorov-Tian's smoothness theorem on Calabi-Yau manifolds.

2.3. Special geometry and tt^* Equations. In this section, we restrict X to be Calabi-Yau 3-fold and for simiplicity we assume that $h^{1,0}(X) = h^{2,0}(X) = 0$. We review the special geometry and tt^* equation on the moduli space used in [BCOV94] to describe the holomorphic anomaly equation of higher genus topological string invariants and motivate the appearance of Kodaira-Spencer field theory.

2.3.1. Weil-Petersson metric. Let \mathcal{M} be the moduli stack of complex structures of X or the Teichmüller space if we avoid the orbifold structure, with the universal family π : $\mathfrak{X} \to \mathcal{M}$. Todorov-Tian's smoothness theorem implies that \mathcal{M} is smooth of dimension $\dim \mathrm{H}^1(X, T_X) = h^{2,1}(X)$. We denote by \mathcal{H}^3

$$\mathcal{H}^3 = R^3 \pi_*(\mathbb{C})$$

the vector bundle on \mathcal{M} of the middle cohomology of the fiber, with flat holomorphic structure given by the Gauss-Manin connection. We will use ∇^{GM} to denote the (1,0) component of the Gauss-Manin connection and $\bar{\nabla}^{GM}$ the (0,1) component. Let $F^p\mathcal{H}^3$ be the Hodge filtration, and

$$\mathcal{H}^{p,3-p} = F^p \mathcal{H}^3 / F^{p+1} \mathcal{H}^3$$

is the Hodge bundle of type (p, 3 - p). There's a canonical smooth identification of vector bundles

(2.17)
$$\mathcal{H}^3 = \mathcal{H}^{3,0} \oplus \mathcal{H}^{2,1} \oplus \mathcal{H}^{1,2} \oplus \mathcal{H}^{0,3}$$

We will use \mathcal{L} to denote the line bundle on \mathcal{M}

$$\mathcal{L} = \mathcal{H}^{3,0}$$

which is in fact a holomorphic subbundle of \mathcal{H}^3 by Griffiths transversality. \mathcal{L} is called the *vacuum line bundle* in the physics literature. For a given point $[X] \in \mathcal{M}, \mathcal{L}_{[X]}$ is the space of holomorphic volume forms on X. The following notation will be used throughout this section

(2.18)
$$\langle \alpha, \beta \rangle = \sqrt{-1} \int_X \alpha \wedge \beta$$

 \langle,\rangle induces a natural metric on \mathcal{L}

$$\langle,\rangle:\mathcal{L}\otimes\bar{\mathcal{L}}$$
 \rightarrow \mathbb{C}

$$\Omega\otimes\bar\Omega \quad \to \quad \sqrt{-1}\int_X\Omega\wedge\bar\Omega$$

The curvature form gives a Kähler metric on \mathcal{M} , which is called the *Weil-Petersson metric*. The induced connection will be denoted by $\nabla^{\mathcal{L}}$ for the (1,0)-component and $\bar{\nabla}^{\mathcal{L}}$ for the (0,1)-component.

Explicitly, let us choose local holomorphic coordinates $\{t^i\}$ on \mathcal{M} and Ω_t be a local holomorphic section of \mathcal{L} . The Kähler potential $K(t, \bar{t})$ is given by

$$e^{-K(t,\bar{t})} = \sqrt{-1} \int_X \Omega_t \wedge \bar{\Omega}_t$$

Then the Weil-Petersson metric is given by

$$(2.19) G_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K$$

where $\partial_i = \frac{\partial}{\partial t^i}$ and $\bar{\partial}_{\bar{j}} = \frac{\partial}{\partial \bar{t}^j}$. Griffiths transversality implies that

$$\nabla_i^{GM} \Omega_t = f_i \Omega_t + \Xi_i$$

where f_i is a local function on \mathcal{M} , and Ξ_i is a local section of $\mathcal{H}^{2,1}$. Both sides are viewed as sections of \mathcal{H}^3 . Therefore

$$\begin{aligned} G_{i\bar{j}} &= -\partial_i \bar{\partial}_{\bar{j}} \log \langle \Omega_t, \bar{\Omega}_t \rangle \\ &= -\frac{\left\langle \nabla_i^{GM} \Omega_t, \overline{\nabla_j^{GM}} \Omega_t \right\rangle}{\left\langle \Omega_t, \bar{\Omega}_t \right\rangle} + \frac{\left\langle \nabla_i^{GM} \Omega_t, \bar{\Omega}_t \right\rangle \left\langle \Omega_t, \overline{\nabla_j^{GM}} \Omega_t \right\rangle}{\left\langle \Omega_t, \bar{\Omega}_t \right\rangle^2} \\ &= -\frac{\left\langle \Xi_i, \bar{\Xi}_{\bar{j}} \right\rangle}{\left\langle \Omega_t, \bar{\Omega}_t \right\rangle} \end{aligned}$$

which shows that $G_{i\bar{j}}$ is indeed positive definite.

2.3.2. tt^* geometry. The bracket \langle, \rangle defines a metric on $\mathcal{H}^{3,0}$ as above, and also defines a metric on $\mathcal{H}^{2,1}$ by

$$\alpha\otimes\beta\to-\sqrt{-1}\int\alpha\wedge\bar\beta$$

where α, β are local sections of $\mathcal{H}^{2,1}$. It defines a Hermitian metric and compatible connection on $\mathcal{H}^{3,0} \oplus \mathcal{H}^{2,1}$, and their complex conjugates on $\mathcal{H}^{1,2} \oplus \mathcal{H}^{0,3} = \overline{\mathcal{H}^{2,1}} \oplus \overline{\mathcal{H}^{3,0}}$. Using

the smooth identification (2.17), we obtain a Hermitian metric on \mathcal{H}^3 , which is called the tt^* -metric, together with a connection called the tt^* -connection, which we denote by $D + \overline{D}$. Here D refers to the (1,0) component and \overline{D} the (0,1)-component. The relation between tt^* -connection and Gauss-Manin connection can be seen as follows: The Kodaira-Spencer map gives a homomorphism

$$T_{\mathcal{M}} \to \bigoplus_{p} \operatorname{Hom}(\mathcal{H}^{p,3-p}, \mathcal{H}^{p-1,4-p}) \subset \operatorname{End}(\mathcal{H}^{3})$$

from which we get a section of the bundle $\Omega^{1,0}_{\mathcal{M}}(\operatorname{End}(\mathcal{H}^3))$, denoted by C. Its complex conjugate can be identified with a section of $\Omega^{0,1}_{\mathcal{M}}(\operatorname{End}(\mathcal{H}^3))$, which is denoted by \overline{C} .

Let's choose local holomorphic coordinates $\{t^i\}$ of \mathcal{M} . We denote by D_i the covariant derivative along $\frac{\partial}{\partial t^i}$ with respect to the tt^* -connection, and $C_i = C(\frac{\partial}{\partial t^i})$. Similarly we have $\overline{D}_{\overline{i}}$ and $\overline{C}_{\overline{i}}$. Let e_0 be a local holomorphic basis of $\mathcal{H}^{3,0}$. Then $\{e_i = C_i e_0\}$ forms a local holomorphic basis of $\mathcal{H}^{2,1}$. The basis of $\overline{\mathcal{H}^{2,1}}$ and $\overline{\mathcal{H}^{3,0}}$ are given by the complex conjugates $\overline{e}_i, \overline{e}_0$. The tt^* metric is given by

$$g_{0\bar{0}} = \langle e_0, \bar{e}_0 \rangle, \quad g_{i\bar{j}} = - \langle e_i, \bar{e}_j \rangle$$

and the tt^* -connection on the above basis reads

$$\begin{aligned} D_{i}e_{0} &= (g^{00}\partial_{i}g_{0\bar{0}})e_{0} \quad D_{i}e_{j} &= (g^{\bar{m}k}\partial_{i}g_{j\bar{m}})e_{k} \quad D_{i}\bar{e}_{\bar{j}} &= 0 \\ D_{\bar{i}}e_{0} &= 0 \qquad D_{\bar{i}}e_{j} &= 0 \\ D_{\bar{i}}e_{0} &= 0 \qquad D_{\bar{i}}\bar{e}_{\bar{j}} &= (g^{\bar{k}m}\bar{\partial}_{\bar{i}}g_{m\bar{j}})\bar{e}_{\bar{k}} \quad D_{\bar{i}}\bar{e}_{\bar{0}} &= (g^{\bar{0}0}\bar{\partial}_{\bar{i}}g_{0\bar{0}})\bar{e}_{\bar{0}} \end{aligned}$$

Proposition 2.8. The Gauss-Manin connection and the tt^{*}-connection satisfy the following equations

(2.20)
$$\nabla^{GM} = D + C, \quad \bar{\nabla}^{GM} = \bar{D} + \bar{C}$$

Proof. We check on the above local basis. $D_i e_0$ is the projection of $\nabla_i^{GM} e_0$ to the $\mathcal{H}^{3,0}$ component. It follows from Griffiths transversality that

$$\nabla_i^{GM} e_0 = D_i e_0 + C_i e_0$$

Similarly, to check that $\nabla_i^{GM} e_j = D_i e_j + C_i e_j$, we only need to check that $\nabla_i^{GM} e_j$ has no $\mathcal{H}^{3,0}$ component. This follows from

$$\int \nabla_i^{GM} e_j \wedge \bar{e}_0 = -\int e_j \wedge \nabla_i^{GM} \bar{e}_0 = 0$$

On \bar{e}_j , since

$$\bar{e}_j = \bar{C}_j \bar{e}_0 = \bar{\nabla}_i^{GM} \bar{e}_0 - \bar{D}_i \bar{e}_0$$

and $\bar{D}_i \bar{e}_0$ lies in $\mathcal{H}^{0,3}$

$$\nabla_i^{GM}\bar{e}_j = \nabla_i^{GM}\bar{\nabla}_j^{GM}\bar{e}_0 - \nabla_i^{GM}\bar{D}_j\bar{e}_0 = \bar{\nabla}_j^{GM}\nabla_i^{GM}\bar{e}_0 - \nabla_i^{GM}\bar{D}_j\bar{e}_0 = -\nabla_i^{GM}\bar{D}_j\bar{e}_0$$

also lies in $\mathcal{H}^{0,3}$. It follows from $D_i \bar{e}_j = 0$ that

$$\nabla_i^{GM} \bar{e}_j = C_i \bar{e}_j = D_i \bar{e}_j + C_i \bar{e}_j$$

Finally,

$$\nabla_i^{GM} \bar{e}_0 = 0 = D_i \bar{e}_0 + C_i \bar{e}_i$$

Proposition 2.9. The following identities hold

$$\begin{split} [D_i, D_j] &= [\bar{D}_{\bar{i}}, \bar{D}_{\bar{j}}] = 0, & [D_i, \bar{C}_{\bar{j}}] = [\bar{D}_{\bar{i}}, C_j] = 0\\ [D_i, C_j] &= [D_j, C_i], & [\bar{D}_{\bar{i}}, \bar{C}_{\bar{j}}] = [\bar{D}_{\bar{j}}, \bar{C}_{\bar{i}}]\\ [D_i, \bar{D}_{\bar{j}}] &= -[C_i, \bar{C}_{\bar{j}}] \end{split}$$

This set of equations is called the tt^* -equations [CV91].

Proof. Since the curvature of the tt^* -connection is of type (1,1), $[D_i, D_j] = [\bar{D}_{\bar{i}}, \bar{D}_{\bar{j}}] = 0$. $[D_i, \bar{C}_{\bar{j}}] = 0$ follows from the fact that the Kodaira-Spencer map is holomorphic. All the other equations follow from $\nabla^{GM} = D + C, \bar{\nabla}^{GM} = \bar{D} + \bar{C}$ and that the Gauss-Manin connection is flat.

 tt^* equations imply that we actually have a family of flat connections on \mathcal{H}^3 :

$$\nabla^a = D + aC, \quad \bar{\nabla}^a = \bar{D} + a^{-1}\bar{C}$$

where $a \in \mathbb{C}^*$. When a = 1, we get back to the Gauss-Manin connection.

2.3.3. Special geometry. The tt^* equations give very restrictive constraints on the curvature of the Weil-Petersson metric. We keep the same notations as in the previous section for the choice of local coordinates and local basis. Since e_0 is a local holomorphic section of \mathcal{L} , the Kähler potential is in fact given by the tt^* -metric

$$e^{-K} = \langle e_0, \bar{e}_0 \rangle = g_{0\bar{0}}$$

and the Weil-Petersson metric is related to the tt^* -metric by

$$G_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K = \frac{g_{i\bar{j}}}{g_{0\bar{0}}}$$

Let $\Gamma_{ij}^k = G^{k\bar{m}}\partial_i G_{j\bar{m}}$ be the connection with respect to the Weil-Petersson metric $G_{i\bar{j}}$, and $R_{ki\bar{j}}^l = -\bar{\partial}_{\bar{j}}\Gamma_{ik}^l$ be the curvature. We also make the action of C_i on basis explicit

$$\begin{aligned} C_{i}e_{0} &= C_{i0}^{j}e_{j} = e_{i} \quad C_{i}e_{j} = C_{ij}^{\bar{k}}\bar{e}_{\bar{k}} \quad C_{i}\bar{e}_{\bar{j}} = C_{i\bar{j}}^{\bar{0}}\bar{e}_{0} \quad C_{i}\bar{e}_{\bar{0}} = 0 \\ C_{\bar{i}}e_{0} &= 0 \quad C_{\bar{i}}e_{j} = C_{ij}^{0}e_{0} \quad C_{\bar{i}}\bar{e}_{\bar{j}} = \bar{C}_{\bar{k}j}^{k}e_{k} \quad C_{\bar{i}}\bar{e}_{0} = \bar{C}_{\bar{i}0}^{\bar{k}}\bar{e}_{\bar{k}} = \bar{e}_{\bar{i}} \end{aligned}$$

Apply the tt^* -equations to the basis, we find

$$\begin{split} [D_{i}, D_{\bar{j}}]e_{0} &= -D_{\bar{j}}(g^{00}\partial_{i}g_{0\bar{0}}e_{0}) = G_{i\bar{j}}e_{0} \\ [C_{i}, C_{\bar{j}}]e_{0} &= -C_{\bar{j}}e_{i} = -G_{i\bar{j}}e_{0} \\ [D_{i}, D_{\bar{j}}]e_{k} &= -D_{\bar{j}}(g^{\bar{m}l}\partial_{i}g_{k\bar{m}}e_{l}) = -\partial_{\bar{j}}(g^{\bar{m}l}\partial_{i}g_{k\bar{m}})e_{l} \\ &= -\partial_{\bar{j}}(G^{\bar{m}l}\partial_{i}G_{k\bar{m}} - \delta_{j}^{l}\partial_{i}K)e_{l} \\ &= (R_{ki\bar{j}}^{l} + G_{i\bar{j}}\delta_{k}^{l})e_{l} \\ [C_{i}, C_{\bar{j}}]e_{k} &= C_{i}(G_{k\bar{j}}e_{0}) - C_{\bar{j}}C_{i\bar{k}}^{\bar{m}}\bar{e}_{\bar{m}} \\ &= G_{k\bar{j}}e_{i} - C_{i\bar{k}}^{\bar{m}}\bar{C}_{\bar{j}\bar{m}}^{l}e_{l} \end{split}$$

which shows that

U

$$(2.21) R^l_{ki\bar{j}} = -G_{i\bar{j}}\delta^l_k - G_{k\bar{j}}\delta^l_i + C^{\bar{m}}_{ik}\bar{C}^l_{\bar{j}\bar{m}}$$

This is called the *special geometry relation*. The following quantity

$$C_{ijk} = g_{i\bar{m}}C^{\bar{m}}_{jk} = -\sqrt{-1}\int \nabla^{GM}_i e_0 \wedge \nabla^{GM}_j \nabla^{GM}_k e_0 = \sqrt{-1}\int e_0 \wedge \nabla^{GM}_i \nabla^{GM}_j \nabla^{GM}_k e_0$$

is called the Yukawa coupling, which plays an important role in mirror symmetry.

Remark 2.10. The relation $G_{i\bar{j}} = \frac{g_{i\bar{j}}}{g_{0\bar{0}}}$, together with the natural identification

$$\mathcal{H}^3 = \mathcal{L} \oplus \mathcal{L} \otimes T_{\mathcal{M}} \oplus \overline{\mathcal{L} \otimes T_{\mathcal{M}}} \oplus \overline{\mathcal{L}}$$

implies that the tt^* -connection is the same as the induced connection from the connection on \mathcal{L} by the metric e^{-K} and the connection on $T_{\mathcal{M}}$ by the Weil-Petersson metric. As an example of the application, it implies that

$$\nabla_i^{GM} e_j - \Gamma_{ij}^k e_k + \partial_i K e_j = \nabla_i^{GM} e_j - D_i e_j = C_i e_j$$

is the projection of $\nabla_i^{GM} \nabla_j^{GM} e_0$ to the $\mathcal{H}^{1,2}$ component. Here we have identified e_i with $e_0 \otimes \frac{\partial}{\partial t^i}$ under the natural isomorphism $\mathcal{H}^{2,1} \cong \mathcal{L} \otimes T_{\mathcal{M}}$.

2.4. Yukawa coupling and prepotential.

2.4.1. Integrability of Yukawa coupling. Recall that the Yukawa coupling on the moduli space \mathcal{M} of Calabi-Yau manifolds is the holomorphic section of the bundle $\mathcal{L}^{-2} \otimes \text{Sym}^3(T^*_{\mathcal{M}})$ locally given by

$$C_{ijk} = \sqrt{-1} \int \Omega \wedge \nabla_i \nabla_j \nabla_k \Omega$$

where Ω is a local holomorphic section of \mathcal{L} . ∇ is the covariant derivative induced by the connection $\nabla^{\mathcal{L}}$ on \mathcal{L} , the Weil-Petersson connection on $T_{\mathcal{M}}$ and the Gauss-Manin connection on \mathcal{H}^3 .

Lemma 2.11.

$$\nabla_i C_{jkl} = \nabla_j C_{ikl}$$

and

Proof. By the type reason, we have

$$C_{jkl} = \sqrt{-1} \int \nabla_j \nabla_k \Omega \wedge \nabla_l \Omega$$

It follows that

$$\nabla_i C_{jkl} = \sqrt{-1} \int \nabla_i \nabla_j \nabla_k \Omega \wedge \nabla_l \Omega + \sqrt{-1} \int \nabla_j \nabla_k \Omega \wedge \nabla_i \nabla_l \Omega$$

By Remark 2.10, we see that $\nabla_j \nabla_k \Omega$ is of pure type (1,2). Therefore the second term vanishes and the lemma follows.

It follows from the lemma that there exists a local section F_0 of \mathcal{L}^{-2} such that

$$C_{ijk} = \nabla_i \nabla_j \nabla_k \mathbf{F}_0$$

 \mathbf{F}_0 is called the *prepotential*.

2.4.2. Prepotential in canonical coordinates. Using canonical coordinates, we can have an explicit formula for F_0 as shown in [BCOV94]. Let $[X] \in \mathcal{M}$ and Ω_0 be a holomorphic top form on X. Let $\{\mu_i\}$ be a harmonic basis of $\mathrm{H}^1(X, T_X)$, and $\{t^i\}$ be the linear coordinates with respect to the basis. $\{t^i\}$ serves as a local coordinates for the local deformation space of X around $[X] \in \mathcal{M}$. Let

$$\mu_t = \sum_i \mu_i t^i + \tilde{\mu}_t = \sum_i \mu_i t^i + \sum_{|I| \ge 2} \mu_I t^I \in PV_X^{1,1}[[t]]$$

which is recursively solved by (2.15)

$$\tilde{\mu}_t = -\frac{1}{2}\bar{\partial}^* G\left\{\mu_t, \mu_t\right\} = -\frac{1}{2}\bar{\partial}^* G\partial\left(\mu_t \mu_t\right)$$

We will choose the local holomorphic section Ω_t of \mathcal{L} to be the canonical family

$$\Omega_t = e^{\mu_t} \vdash \Omega_0$$

The Kähler potential is given by

$$e^{-K} = \sqrt{-1} \int \Omega_t \wedge \bar{\Omega}_t$$

The reason for the name "canonical" is by the following lemma

Lemma 2.12. Let Γ_{ij}^k be the connection of the Weil-Petersson metric in local coordinates $\{t^i\}$, then all the holomorphic derivatives of Γ_{ij}^k and K at t = 0 vanish

(2.22)
$$\partial_{i_1}\partial_{i_2}\cdots\partial_{i_n}\Gamma^k_{i_j}(0) = 0, \quad \partial_{i_1}\partial_{i_2}\cdots\partial_{i_n}K(0) = 0$$

for all $\{i_1, \cdots, i_n\}$.

Proof. Since

$$K = -\log\left(\sqrt{-1}\int\Omega_0\wedge\bar{\Omega}_0\right) - \log\left(\frac{\int\Omega_t\wedge\Omega_t}{\int\Omega_0\wedge\bar{\Omega}_0}\right)$$

where the second term is a power series in t such that each term has at least one \bar{t}^i . Therefore

$$\partial_{i_1}\partial_{i_2}\cdots\partial_{i_n}K(0)=0$$

Similarly,

$$\Gamma_{ij}^k = G^{k\bar{m}} \partial_i G_{j\bar{m}} = G^{k\bar{m}} \partial_i \partial_j \bar{\partial}_{\bar{m}} K$$

and the same reason applies to Γ_{ij}^k .

]	lt follo	ows f	from	the	lemr	ma	that	if	∇_i	is	the	holor	norphic	c cc	ovaria	ant	derivativ	ve	on	vector
bui	ndles d	const	truct	ed fr	om ,	L a	nd T	M	wit	h	the i	induc	ed conr	nect	tion,	the	n			

$$\nabla_i|_{t=0} = \partial_{t^i}$$

We are looking for the prepotential \mathbf{F}_0 , which is locally a smooth function under the trivialization of \mathcal{L} by the canonical family Ω_t . The structure equation $C_{ijk} = \nabla_i \nabla_j \nabla_j \mathbf{F}_0$ is equivalent to

$$\nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_n} C_{ijk}|_{t=0} = \nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_n} \nabla_i \nabla_j \nabla_k \mathbf{F}_0|_{t=0}$$

If we choose the canonical coordinates and the canonical family, then it's equivalent to the equation

$$C_{ijk} = \partial_i \partial_j \partial_k \mathbf{F}_0$$

The Yukawa coupling reads

$$C_{ijk} = \sqrt{-1} \int \Omega_t \wedge \partial_i \partial_j \partial_k \Omega_t$$

= $\sqrt{-1} \int (e^{\mu_t} \vdash \Omega_0) \wedge (\partial_i \partial_j \partial_k e^{\mu_t} \vdash \Omega_0)$
= $\sqrt{-1} \int \Omega_0 \wedge (e^{-\mu_t} \partial_i \partial_j \partial_k e^{\mu_t}) \vdash \Omega_0$
= $\sqrt{-1} \int \Omega_0 \wedge (\partial_i \mu_t \partial_j \mu_t \partial_k \mu_t) \vdash \Omega_0$
= $-\sqrt{-1} \operatorname{Tr} (\partial_i \mu_t \partial_j \mu_t \partial_k \mu_t)$

where Tr is the trace operator (2.11) with respect to Ω_0 .

Proposition 2.13. [BCOV94] Let $\tilde{\mu}_t = \partial \psi_t$, where $\psi_t = \frac{1}{2} \bar{\partial}^* G(\mu_t^2)$. Then the prepotential can be chosen to be

(2.23)
$$\sqrt{-1}\mathbf{F}_0 = -\frac{1}{2}\operatorname{Tr}\left(\partial\psi_t\bar{\partial}\psi_t\right) + \frac{1}{6}\operatorname{Tr}\left(\mu_t^3\right)$$

Proof. First, we have

$$\partial_i \partial_j \partial_k \frac{1}{2} \operatorname{Tr} \left(\partial \psi_t \ \bar{\partial} \psi_t \right)$$

= $\operatorname{Tr} \left(\partial_i \partial_j \partial_k \partial \psi_t \wedge \bar{\partial} \psi_t \right) + \operatorname{Tr} \left(\partial_j \partial_k \partial \psi_t \wedge \partial_i \bar{\partial} \psi_t + (i \leftrightarrow j) + (i \leftrightarrow k) \right)$

Since $\bar{\partial}^* \mu_t = 0$ and $\partial_i \partial_j \mu_t$ has no harmonic component, we have

$$\partial_{i}\partial_{j}\partial_{k}\frac{1}{2}\operatorname{Tr}\left(\partial\psi_{t}\ \bar{\partial}\psi_{t}\right)$$

$$=\operatorname{Tr}\left(\partial_{i}\partial_{j}\partial_{k}\mu_{t}\wedge\frac{1}{2}\mu_{t}^{2}\right)+\operatorname{Tr}\left(\partial_{j}\partial_{k}\mu_{t}\wedge\partial_{i}\left(\frac{1}{2}\mu_{t}^{2}\right)+(i\leftrightarrow j)+(i\leftrightarrow k)\right)$$

$$=\partial_{i}\partial_{j}\partial_{k}\frac{1}{6}\operatorname{Tr}\left(\mu_{t}^{3}\right)-\operatorname{Tr}\left(\partial_{i}\mu_{t}\partial_{j}\mu_{t}\partial_{k}\mu_{t}\right)$$

By the construction of μ_t , we know that $\bar{\partial}\mu_t$ is ∂ -exact. It's instructive to write the formula (2.24) as

(2.24)
$$\sqrt{-1}\mathbf{F}_{0} = \frac{1}{2}\operatorname{Tr}\left(\mu_{t} \wedge \frac{1}{\partial}\bar{\partial}\mu_{t}\right) + \frac{1}{6}\operatorname{Tr}\left(\mu_{t}^{3}\right)$$
where $\frac{1}{\partial} \bar{\partial} \mu_t$ is any element whose image under ∂ is $\bar{\partial} \mu_t$. Since $\partial \mu_t = 0$, it doesn't depend on the choice in the above formula. This is the form adopted in [BCOV94] to describe the Kodaira-Spencer gauge theory.

Remark 2.14. All the above formulae apply to the extended deformation space of Calabi-Yau manifolds, as described in section 2.2.2. In fact, the extended deformation space carries a natural Frobenius structure and \mathbf{F}_0 is the corresponding potential function [BK98].

2.5. Kodaira-Spencer gauge theory. We will discuss the classical geometry of the Kodaira-Spencer gauge theory, which is proposed in [BCOV94] to describe the closed string field theory on the B-side. The quantization of Kodaira-Spencer gauge theory is the main content of this thesis. To simplify the notations, we work on Calabi-Yau three-fold and adopt the original approach with the purpose of motivating the discussion in Chapter 4.

2.5.1. Fields. Let X be a Calabi-Yau three-fold with fixed Kähler metric and holomorphic top form Ω_X . The classical field content of Kodaria-Spencer gauge theory is given by

Fields : ker
$$\partial \cap \mathrm{PV}^{1,1}_X$$

From Hodge theory, we can further decompose it into

$$\mathbb{H}^{1,1} \oplus \operatorname{im} \partial \cap \operatorname{PV}^{1,1}_X$$

where $\mathbb{H}^{1,1}$ is the space of Harmonic elements of $\mathrm{PV}_X^{1,1}$. We will use x + A to represent a general field, where $x \in \mathbb{H}, A \in \mathrm{im} \partial \cap \mathrm{PV}_X^{1,1}$. A will be a dynamical field, while x will only be a background field [BCOV94].

2.5.2. Kodaira-Spencer action. The Kodaira-Spencer action is given by

(2.25)
$$KS[A, x] = \frac{1}{2} \operatorname{Tr} \left(A \wedge \frac{1}{\partial} \bar{\partial} A \right) + \frac{1}{6} \operatorname{Tr} \left(x + A \right)^3$$

where for $\frac{1}{\partial}\bar{\partial}A$ we choose an arbitrary element of $\mathrm{PV}_X^{2,2}$ whose image under ∂ is $\bar{\partial}A$, and the value of the action doesn't depend on the choice since A is ∂ -exact.

Let's fix x and consider the variation with respect to A

$$\delta_{\epsilon} A = \partial \epsilon, \ \epsilon \in \mathrm{PV}_X^{2,1}$$

The variation of Kodaira-Spencer action is given by

$$\delta_{\epsilon} KS[A, x] = \operatorname{Tr} \left(A \wedge \frac{1}{\partial} \bar{\partial} \delta_{\epsilon} A \right) + \frac{1}{2} \operatorname{Tr} \left((x+A)^2 \delta_{\epsilon} A \right)$$
$$= -\operatorname{Tr} \left(A \wedge \bar{\partial} \epsilon \right) + \frac{1}{2} \operatorname{Tr} \left((x+A)^2 \partial \epsilon \right)$$
$$= \operatorname{Tr} \left(\bar{\partial} A \wedge \epsilon \right) + \frac{1}{2} \operatorname{Tr} \left(\partial \left((x+A)^2 \right) \wedge \epsilon \right)$$
$$= \operatorname{Tr} \left(\left(\bar{\partial} A + \frac{1}{2} \left\{ x+A, x+A \right\} \right) \wedge \epsilon \right)$$

Therefore the equation of motion for the critical point is

$$\bar{\partial}A + \frac{1}{2}\left\{x + A, x + A\right\} = 0$$

which describes precisely the deformation of the complex structure along the tangent vector x.

2.5.3. Gauge symmetry. The Kodaira-Spencer action is invariant under the diffeomorphism group preserving Ω_X , whose infinitesimal generator is given by

$$\ker \partial \cap \mathrm{PV}^{1,0}_X$$

The infinitesimal action on A is given by the formula

$$\delta_{\alpha}A = \bar{\partial}\alpha + \{x + A, \alpha\}, \text{ where } \epsilon \in \ker \partial \cap \mathrm{PV}_X^{1,0}$$

and we can directly check that

$$\delta_{\alpha} KS[A, x]$$

$$= \operatorname{Tr}\left(A \wedge \frac{1}{\partial} \bar{\partial} \{x + A, \alpha\}\right) + \frac{1}{2} \operatorname{Tr}\left((x + A)^{2} \wedge \left(\bar{\partial}\alpha + \{x + A, \alpha\}\right)\right)$$

$$= -\operatorname{Tr}\left(A \wedge \bar{\partial}\left((x + A)\alpha\right)\right) - \frac{1}{2} \operatorname{Tr}\left(\bar{\partial}\left((x + A)^{2}\right)\right) \alpha + \frac{1}{2} \operatorname{Tr}\left((x + A)^{2} \partial\left((x + A)\alpha\right)\right)$$

$$= \frac{1}{2} \operatorname{Tr}\left(\left(\partial(x + A)^{2}\right)(x + A)\alpha\right)$$

$$= \frac{1}{6} \operatorname{Tr}\left(\partial(x + A)^{3}\alpha\right) = \frac{1}{6} \operatorname{Tr}\left((x + A)^{3} \partial\alpha\right) = 0$$

If we choose the gauge-fixing condition

$$\bar{\partial}^* A = 0$$

then for each fixed small background field x, there's a unique critical point of the Kodaira-Spencer action solving

$$\bar{\partial}A(x) + \frac{1}{2} \left\{ x + A(x), x + A(x) \right\} = 0, \quad \bar{\partial}^*A = \partial A = 0$$

The critical value KS[x, A(x)] becomes a function on x, which is precisely the prepotential (see (2.24)).

3. Perturbative Quantization of Gauge Theory

In this section, we give a quick overview of the algebraic techniques of perturbative renormalization of gauge theory in the Batalin-Vilkovisky formalism developed by Kevin Costello in [Cos11]. Such techniques will be intensively used for the BCOV theory on Calabi-Yau manifolds in the later sections. In section 3.1 and 3.2, we motivate by discussing the finite dimensional model for perturbative theory and Batalin-Vilkovisky geometry, which can be viewed as the toy model of quantum gauge field theory. In section 3.3 and 3.4, we discuss the framework of perturbative renormalization of quantum field theory, and review the obstruction theory for renormalization with gauge symmetry in the Batalin-Vilkovisky formalism. In section 3.5, we prove a result on the absent of ultraviolet divergence for a certain type of complex one dimensional field theory that will be used in constructing the quantization of BCOV theory on the elliptic curve in section 5.

3.1. Feynman Diagrams. Let $V = \mathbb{R}^N$ be a linear space, with linear coordinates $\{x^i\}_{1 \le i \le N}$. We would like to consider the following integration

$$\int_{V} d^{N} x \exp \frac{1}{\hbar} \left(-\frac{1}{2}Q(x,x) + \lambda I(x+a) \right)$$

as a function of $\{a^i\}$. Here \hbar is a positive real number, $Q(x,x) = \sum_{i,j} Q_{ij} x^i x^j$ is a nondegenerate positive definite quadratic form, and I(x) is a polynomial function whose lowest degree component is at least cubic. The integration is not convergent in general, and we understand it as a formal power series in λ

(3.1)
$$Z_{\lambda}(a) = \sum_{m=0}^{\infty} \frac{\lambda^m}{\hbar^m m!} \int_V d^N x I(x+a)^m \exp \frac{1}{\hbar} \left(-\frac{1}{2} Q(x,x) \right)$$

To compute each term in the summation, we consider the following auxiliary integral

$$Z[J] = \int_{V} d^{N}x \exp\left(-\frac{1}{2\hbar}Q(x,x) + \sum_{i} x^{i}J_{i}\right)$$
$$= e^{\frac{\hbar}{2}Q^{-1}(J,J)} \int_{V} d^{N}x \exp\left(-\frac{1}{2\hbar}Q(x_{J},x_{J})\right)$$
$$= \mathcal{N}e^{\frac{\hbar}{2}Q^{-1}(J,J)}$$

where $Q^{-1}(J,J) = \sum_{i,j} (Q^{-1})^{ij} J_i J_j$, $x_J^i = x^i - \hbar \sum_j (Q^{-1})^{ij} J_j$, $(Q^{-1})^{ij}$ is the inverse matrix of Q_{ij} , and \mathcal{N} is the normalization factor

$$\mathcal{N} = Z[0] = \int_{V} d^{N} x \exp\left(-\frac{1}{2\hbar}Q(x,x)\right)$$

Therefore

$$\int_{V} d^{N}x I(x+a)^{m} \exp \frac{1}{\hbar} \left(-\frac{1}{2}Q(x,x) \right)$$
$$= \int_{V} d^{N}x e^{\frac{1}{\hbar} \left(-\frac{1}{2}Q(x,x) \right)} \left(e^{\sum_{i} x^{i} \frac{\partial}{\partial a^{i}}} \left(I(a)^{m} \right) \right)$$
$$= \mathcal{N}e^{\frac{\hbar}{2}Q^{-1} \left(\frac{\partial}{\partial a}, \frac{\partial}{\partial a} \right)} \left(I(a)^{m} \right)$$

Proposition 3.1. As a formal power series in λ , we have

(3.2)
$$Z_{\lambda}(a) = \mathcal{N}e^{\frac{\hbar}{2}Q^{-1}\left(\frac{\partial}{\partial a},\frac{\partial}{\partial a}\right)}\exp\left(\lambda I(a)/\hbar\right)$$

where $Q^{-1}\left(\frac{\partial}{\partial a}, \frac{\partial}{\partial a}\right) = \sum_{i,j} \left(Q^{-1}\right)^{ij} \frac{\partial}{\partial a^i} \frac{\partial}{\partial a^j}.$

Now we give a graph interpretation of the above formula. Let

$$I(x) = \sum_{k \ge 3} I^{(k)}(x), \quad I^{(k)}(x) = \frac{1}{k!} \sum_{i_1, \cdots, i_k} I^{(k)}_{i_1 \cdots i_k} x^{i_1} \cdots x^{i_k}$$

where $I^{(k)}$ is zero for k sufficiently large. Let Γ be a graph with tails such that each vertex is at least trivalent. We define the weight of the graph as a function of a

$$W_{\Gamma}(Q^{-1},\lambda I)(a)$$

as follows. For a vertex with valency $k \geq 3$, we decorate the edges connecting to this vertex by indices $i_1, \dots, i_k \in \{1, 2, \dots, N\}$, and put the value $\frac{\lambda}{\hbar}I_{i_1,\dots,i_k}$ on this vertex. For each edge connecting v_1, v_2 , with decoration i on v_1 and j on v_2 , we put the value $\hbar (Q^{-1})^{ij}$. For each tail connecting to a vertex, with decoration i, we put the value a^i . Then $W_{\Gamma}(Q^{-1}, I)(a)$ is defined to be the product of all the values associated to the vertices, edges, and tails, and take the summation over the indices of the decoration.

Proposition 3.2.

(3.3)
$$Z_{\lambda}(a) = \mathcal{N} \sum_{\Gamma} \frac{W_{\Gamma}(Q^{-1}, \lambda I)(a)}{|\operatorname{Aut}(\Gamma)|}$$

where the summation is over all possible graphs as above, and $Aut(\Gamma)$ is the automorphism group of Γ as a graph with tails.

Proof. This follows directly from (3.2). The only tricky thing is the factor $|\operatorname{Aut}(\Gamma)|$. We leave the details to the reader.

Proposition 3.3. Let $\mathbf{F}_{\lambda}(a) = \hbar \log \left(\frac{Z_{\lambda}(a)}{N}\right)$, then

(3.4)
$$\mathbf{F}_{\lambda}(a) = \hbar \sum_{\Gamma \text{ connected}} \frac{W_{\Gamma}(Q^{-1}, \lambda I)(a)}{|\operatorname{Aut}(\Gamma)|}$$

where the summation is over all connected graphs.

 \mathbf{F}_{λ} is called the *free energy* in physics. Formula (3.3) and (3.4) are called the *Feynman* diagram expansions. We can furthermore decompose $\mathbf{F}_{\lambda}(a)$ in terms of powers of \hbar . Let Γ be a connected diagram, $V(\Gamma)$ be the set of vertices, $E(\Gamma)$ be the set of internal edges, and $T(\Gamma)$ be the set of tails. Since each vertex contributes \hbar^{-1} and each edge contributes \hbar , we see that $W_{\Gamma}(Q^{-1}, \lambda V)$ contains the power of \hbar by

$$\hbar^{-|V(\Gamma)|+|E(\Gamma)|} = \hbar^{l(\Gamma)-1}$$

where $l(\Gamma)$ is the number of loops of Γ . Therefore we have the following expansion

(3.5)
$$\mathbf{F}_{\lambda}(a) = \sum_{g \ge 0} \hbar^{g} \mathbf{F}_{\lambda,g}(a)$$

where

(3.6)
$$\mathbf{F}_{\lambda,g}(a) = \hbar \sum_{\substack{\Gamma^g \text{ connected}\\g-\text{loops}}} \frac{W_{\Gamma^g}(Q^{-1},\lambda I)(a)}{|\operatorname{Aut}(\Gamma_g)|}$$

The summation is over all connected g-loop diagrams.

Remark 3.4. It's easy to see that for each homogenous degree of a, the summation in (3.6) is actually a finite sum, hence always convergent. This shows that the free energy with the

fixed loop number is in fact a well-defined formal power series of a. In fact, we can also allow V to have non-negative powers of \hbar . Precisely, let

$$\mathcal{I}^{\hbar} = \left\{ I \subset \mathbb{C}[[a^{i}, \hbar]] \middle| I \text{ is at least cubic in } a^{i} \text{ modulo } \hbar \right\}$$

Then Feynman diagram expansion actually gives a well-defined map

$$\begin{array}{lcl} \mathcal{I}^{\hbar} & \to & \mathcal{I}^{\hbar} \\ & I & \to & W(Q^{-1}, I) = \hbar \sum_{\Gamma \text{ connected}} \frac{W_{\Gamma^g}(Q^{-1}, I)}{|\mathrm{Aut}(\Gamma)|} \end{array}$$

This will be the key formula for the renormalization group flow equation in quantum field theory.

3.2. Batalin-Vilkovisky geometry.

3.2.1. *Odd symplectic geometry.* We would like to add fermions and also gauge symmetry to the previous discussion. The super-geometry will play an important role in this case.

Definition 3.5. A supermanifold of dimension (n, m) is defined to be a superspace (M, \mathcal{O}_M) where M is a topological space, \mathcal{O}_M is a sheaf of graded commutative ring such that locally it's isomorphic to $C^{\infty}(\mathbb{R}^n) \otimes \wedge^* \mathbb{R}^m$. Let J_M be the subsheaf of nilpotent elements of \mathcal{O}_M , then $(M_{red}, \mathcal{O}_{red}) = (M, \mathcal{O}_M/J_M)$ defines a topological manifold, which is called the reduced manifold of (M, \mathcal{O}_M) .

Let (M, \mathcal{O}_M) be a smooth super-manifold of dimension (n, m). Let U be a local open subset of M, and we choose coordinates $\{x^i, \xi^\alpha\}$ which are even and odd elements of \mathcal{O}_U respectively. Every elements f of \mathcal{O}_U can be uniquely written in the form

$$f = \sum_{I} \xi^{I} f_{I}(x^{i})$$

where $I \subset \{1, \dots, m\}$ runs over the index set, $\xi^I = \prod_{i \in I} \xi^i$, and $f_I(x^i)$ is a smooth function of $\{x^i\}$. Let $\mathcal{O}_{U,c}$ be the space of compactly supported elements in \mathcal{O}_U . There's a well-defined integral on U

$$\int_M d^n x d^m \xi : \mathcal{O}_{U,c} \quad \to \quad \mathbb{C}$$

$$f \rightarrow \int_{U} d^{n}x d^{m}\xi f \equiv \int_{U_{red}} d^{n}x f_{12\cdots n}(x^{i})$$

which is called the *Berezin integral*. If we choose another local coordinates $\{y^i, \eta^{\alpha}\}$, and let

$$J(x,\xi;y,\eta) = \begin{pmatrix} \frac{\partial x^i}{\partial y^j} & \frac{\partial x^i}{\partial \eta^\beta} \\ -\frac{\partial \xi^\alpha}{\partial y^k} & \frac{\partial \xi^\alpha}{\partial \eta^\beta} \end{pmatrix}$$

be the Jacobian matrix. Note that there's an extra sign here which is compatible with the chain rule $J(x,\xi;z,\theta) = J(x,\xi;y,\eta)J(y,\eta;z,\theta)$ due to the anti-commutativity of the odd variables. The Berezin's formula [Ber87] says that

(3.7)
$$\int_{U} d^{n}x d^{m}\xi f = \int_{U} d^{n}y d^{m}\eta \operatorname{Ber}(J(x,\xi;y,\eta))f$$

where Ber refers to the Berezinian (or the super-determinant) defined as follows: let A be the matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11}, A_{22} are even, A_{12}, A_{21} are odd, and A_{22} is invertible, then

(3.8)
$$\operatorname{Ber} A = \det \left(A_{11} - A_{12} A_{22}^{-1} A_{21} \right) \det \left(A_{22} \right)^{-1}$$

The Berezinian Ber has the same multiplicative property as the determinant

Example 3.6. Consider the super-manifold $\mathbb{R}^{2|2}$ and two sets of coordinates (x^1, x^2, ξ^1, ξ^2) , $(y^1, y^2, \eta^1, \eta^2)$, with coordinate transformation

$$x^{1} = y^{1} + \eta^{1}\eta^{2} \quad x^{2} = y^{2}$$

$$\xi^{1} = e^{y^{1}}\eta^{1} \qquad \xi^{2} = \eta^{2}$$

The Jacobian matrix is given by

$$J(x,\xi;y,\eta) = \begin{pmatrix} 1 & 0 & \eta^2 & -\eta^1 \\ 0 & 1 & 0 & 0 \\ -e^{y^1}\eta^1 & 0 & e^{y^1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\operatorname{Ber}(J(x,\xi;y,\eta))=e^{-y^1}(1-\eta^1\eta^2).$ Let

$$f(x;\xi) = f(x) + f_1(x)\xi^1 + f_2(x)\xi^2 + f_{12}\xi^1\xi^2$$

Then Berezin's formula in this case is simply

$$\int d^2x f_{12}(x^1, x^2) = \int d^2y \left(e^{-y^1} \partial_{y^1} f(y^1, y^2) - e^{-y^1} f(y^1, y^2) + f_{12}(y^1, y^2) \right)$$

which is just an integration by parts assuming f has compact support.

Definition 3.7. The *Berezin bundle* Ber(M) of a supermanifold M is the locally free \mathcal{O}_M sheaf of rank one defined as follows: for each local chart U with coordinates (x^i, ξ^{α}) , we associate the basis $D_U(x^i, \xi^{\alpha})$ such that the transition function between two charts $\{U, (x^i, \xi^{\alpha})\}$ and $\{V, (y^i, \eta^{\alpha})\}$ is given by

$$D_U(x^i,\xi^\alpha)|_{U\cap V} = J(x^i,\xi^\alpha;y^i,\eta^\alpha)D_V(y^i,\eta^\alpha)|_{U\cap V}$$

A smooth section of Ber(M) is called a *Berezinian density*.

It follows from Berezin's formula (3.7) that the Berezin integral is a well-defined map

$$\int_M : \Gamma_c(M, \operatorname{Ber}(M)) \to \mathbb{C}$$

where $\Gamma_c(M, \text{Ber}(M))$ is the space of compactly supported smooth sections of Ber(M).

Example 3.8. Let X be a smooth manifold, and $M = T_X[1]$ be the super-manifold of the shifted tangent bundle of X, i.e., the fibers of the tangent bundle will have odd degree. If we choose local coordinates $\{x^i\}$ on a small open subset U of X, then it induces a canonical local coordinate system on M

$$\{x^i, \theta^i\}$$

where we can view θ^i as dx^i which forms a basis of odd functions on the shifted tangent bundle. Since dx^i and θ^i transform in the same way under coordinate transformations, the Berezin bundle Ber(M) is in fact a trivial bundle. Under the natural identification

$$egin{array}{rcl} \mathcal{O}_M &\simeq& \Omega^*_X \ \Phi &
ightarrow & \omega_\Phi \end{array}$$

of the functions on M with the differential forms on X, it's easy to see that the Berezin integral becomes the ordinary integral of differential forms

$$\int_M \Phi = \int_X \omega_\Phi$$

Definition 3.9. Given a super-manifold (M, \mathcal{O}_M) , the *tangent sheaf* \mathcal{T}_M is defined to be the sheaf of graded derivations of the graded commutative algebra \mathcal{O}_M

$$\mathcal{T}_M = \mathrm{Der}_{\mathbb{C}}\left(\mathcal{O}_M\right)$$

which has a natural graded Lie algebra structure. The sheaf of *p*-forms Ω^p_M is defined to be

$$\Omega_M^p = \mathcal{H}om_{\mathcal{O}_M}\left(\operatorname{Sym}_{\mathcal{O}_M}^p\left(\mathcal{T}_M[1]\right), \mathcal{O}_M\right)$$

where [1] is the shifting operator which shifts the grading by 1. There's the natural de Rham differential

$$d:\Omega^p_M\to\Omega^{p+1}_M$$

Locally, if we have coordinates $\{x^i, \xi^\alpha\}$, then $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \xi^\alpha}\}$ form a local basis of \mathcal{T}_X , and $\{dx^i, d\xi^\alpha\}$ form a local basis of one-form Ω^1_X . Note that dx^i is of odd degree and $d\xi^\alpha$ is of even degree respectively. The differential d is given by

$$d = \sum_{i} dx^{i} \frac{\partial}{\partial x^{i}} + \sum_{\alpha} d\xi^{\alpha} \frac{\partial}{\partial \xi^{\alpha}}$$

Definition 3.10. An *odd symplectic manifold*, or *P*-manifold, $(M, \mathcal{O}_M, \omega)$ is a supermanifold with an odd closed two-form ω which gives a non-degenerate pairing on the tangent sheaf. Let $(M, \mathcal{O}_M, \omega)$ be an odd symplectic manifold. For any function $f \in \mathcal{O}_M$, we can define its Hamiltonian vector field V_f by

$$\iota\left(V_f\right)\omega = (-1)^{|f|+1}df$$

The Lie bracket on vector fields induces a Poisson bracket on functions via

$$\{f,g\} = V_f(g) = \iota(V_f)\iota(V_g)\omega$$

which satisfies

$$\begin{split} \{f,g\} &= (-1)^{|f||g|} \{g,f\} \\ V_{\{f,g\}} &= (-1)^{|f|+1} [V_f,V_g] \\ V_{fg} &= (-1)^{|f|} f V_g + (-1)^{|f||g|+|g|} g V_f \end{split}$$

Suppose that we have in addition a no-where vanishing Berezinian density μ on M. It defines a measure on functions with compact support by

$$f \to \int_{\mu} f \equiv \int_{M} \mu f$$

The divergence of a vector field is defined via

(3.10)
$$\int_{\mu} (\operatorname{div}_{\mu} X) f = -\int_{\mu} X(f)$$

which satisfies the equation

(3.11)
$$\operatorname{div}_{\mu}(fX) = f \operatorname{div}_{\mu}(X) + (-1)^{|X||f|} X(f)$$

The odd Laplacian operator with respect to density μ is defined to be

$$\Delta_{\mu}(f) = \frac{1}{2} \operatorname{div}_{\mu} V_{f}$$

Locally, suppose we can choose Darboux coordinates [Sch93] $\{x^i, \xi_i\}$, where x^i 's are even and ξ_i 's are odd, such that

$$\omega = \sum_{i} dx^{i} d\xi_{i}, \quad \mu = 1$$

then

(3.12)
$$V_f = \sum_{i} \left(\frac{\partial}{\partial \xi_i} f\right) \frac{\partial}{\partial x^i} + (-1)^{|f|} \sum_{i} \left(\frac{\partial}{\partial x^i} f\right) \frac{\partial}{\partial \xi_i}$$

(3.13)
$$\{f,g\} = V_f(g) = \sum_i \left(\frac{\partial}{\partial\xi_i}f\right) \left(\frac{\partial}{\partial x^i}g\right) + (-1)^{|f|} \sum_i \left(\frac{\partial}{\partial x^i}f\right) \left(\frac{\partial}{\partial\xi_i}g\right)$$

and

(3.14)
$$\Delta_{\mu}f = \sum_{i} \frac{\partial}{\partial\xi_{i}} \frac{\partial}{\partial x^{i}} f$$

Lemma 3.11.

(3.15)
$$\Delta_{\mu}(fg) = (\Delta_{\mu}f)g + (-1)^{|f|}f\Delta_{\mu}g + \{f,g\}$$

Proof.

$$\begin{split} \Delta_{\mu} \left(fg \right) &= \frac{1}{2} \operatorname{div}_{\mu} V_{fg} \\ &= \frac{1}{2} \operatorname{div}_{\mu} \left((-1)^{|f|} fV_{g} + (-1)^{|f||g|+|g|} gV_{f} \right) \\ &= (-1)^{|f|} f\Delta_{\mu}g + (-1)^{|f||g|} \frac{1}{2} V_{g}(f) + (-1)^{|f||g|+|g|} g\Delta_{\mu}f + \frac{1}{2} V_{f}(g) \\ &= (\Delta_{\mu}f) g + (-1)^{|f|} f\Delta_{\mu}g + \{f,g\} \end{split}$$

Definition 3.12. A Batalin-Vilkovisky supermanifold (M, ω, μ) is an odd symplectic supermanifold with Berezinian density μ such that $\Delta_{\mu}^2 = 0$.

It follows from Lemma 3.11 that the sheaf of functions \mathcal{O}_M on a Batalin-Vilkovisky supermanifold is a sheaf of Batalin-Vilkovisky algebra.

The good thing about Batalin-Vilkovisky supermanifold is that it has a natural cohomology theory similar to the smooth manifold case. The counter-part of closed cycles is the closed orientable Lagrangian super-manifold, which is middle dimensional subsupermanifolds where the odd symplectic form restricts to zero. Let L be a Lagrangian super-manifold of M. The odd sympletic form induces an exact sequence of vector bundles

$$0 \to \mathcal{T}_L \to \mathcal{T}_M|_L \to (\mathcal{T}_L[1])^{\vee} \to 0$$

which implies that $\operatorname{Ber}(M)|_L = \operatorname{Ber}(L)^{\otimes 2}$. Therefore we have an induced Berezinian density on L given by $\mu_L = \sqrt{\mu}$.

Proposition 3.13. [Batalin-Vilkovisky, Schwarz [Sch93]] Let Φ be a smooth function with compact support on a Batalin-Vilkovisky manifold M, and L is a Lagrangian super-manifold in M. If $\Delta \Phi = 0$, then $\int_{\mu_L} \Phi|_L$ depends only on the homological class of L. Moreover, if $\Phi = \Delta \Psi$, then $\int_{\mu_L} \Phi|_L = 0$.

Example 3.14. Let X be an orientable smooth manifold, and $M = \mathcal{T}_X^{\vee}[1]$ be the supermanifold of the shifted cotangent bundle of X, i.e., the fibers of the cotangent bundle will have odd degree. If we choose local coordinates $\{x^i\}$ on a small open subset U of X, then it induces a canonical local coordinate system on M

$$\{x^i, \theta_i\}$$

where we can view θ_i as ∂_{x^i} which forms a basis of odd functions on the shifted cotangent bundle. The sheaf of functions on M is therefore identified with the sheaf of polyvector fields on X

$$\mathcal{O}_M \stackrel{P}{\simeq} \wedge^* \mathcal{T}_X$$
$$\Phi = f(x)\theta_{i_1}\cdots\theta_{i_k} \rightarrow P_\Phi = f(x)\partial_{x^{i_1}}\wedge\cdots\wedge\partial_{x^{i_k}}$$

If $\{y^i,\eta_i\}$ is another set of local coordiantes, then the Jacobian is given by

$$J(x^{i},\theta_{i};y^{i},\eta_{i}) = \det^{2}\left(\frac{\partial x^{i}}{\partial y^{j}}\right)$$

and the local basis of Ber(M) transformes as

$$D(x^i, \theta_i) = \det^2\left(\frac{\partial x^i}{\partial y^j}\right) D(y^i, \eta_i)$$

Let Ω be a no-where vanishing top differential forms on X. It follows that $\mu = \Omega^{\otimes 2}$ gives naturally a Berezin density on the super-manifold M. The Berezin integral is then given by

$$\int_{\mu} \Phi = \int_{X} \left(P_{\phi} \vdash \Omega \right) \wedge \Omega$$

where \vdash is the contraction between polyvector fields and differential forms.

There's a canonical odd symplectic form ω , which is given in local coordinates by

$$\omega = \sum_i dx^i d\theta_i$$

Let's compute the induced Batalin-Vilkovisky operator. Locally, we will write $\Omega = \rho(x)dx^1 \wedge \cdots \wedge dx^n$. Using the formula

$$\int_{\mu} (\operatorname{div}_{\mu} V) \Phi = - \int_{\mu} V(\Phi)$$

for any vector field V and compactly supported Φ , we find

$$\operatorname{div}_{\mu} \partial_{x^{i}} = \partial_{x^{i}} \rho(x), \quad \operatorname{div}_{\mu} \partial_{\theta_{i}} = 0$$

It follows that

$$\begin{split} \Delta_{\mu}(\Phi) &= \frac{1}{2} \operatorname{div}_{\mu} V_{\Phi} \\ &= \frac{1}{2} \operatorname{div}_{\mu} \left(\sum_{i} (\partial_{\theta_{i}} \Phi) \partial_{x^{i}} + (-1)^{|\Phi|} \sum_{i} (\partial_{x^{i}} \Phi) \partial_{\theta_{i}} \right) \\ &= \frac{1}{2} \sum_{i} (\partial_{\theta_{i}} \Phi) \partial_{x^{i}} \rho + \sum_{i} \partial_{x^{i}} \partial_{\theta_{i}} \Phi \end{split}$$

This is equivalent to the following formula

$$P_{\Delta_{\mu}\Phi} \vdash \Omega = d \left(P_{\Phi} \vdash \Omega \right)$$

i.e., Δ_{μ} can be identified with the de Rham differential d under the isomorphism

$$\mathcal{O}_M \xrightarrow{P} \wedge^* \mathcal{T}_X \xrightarrow{\vdash \Omega} \Omega^*_X$$

In particular, $\Delta_{\mu}^2 = 0$ is satisfied and we obtain a Batalin-Vilkovisky structure on M.

Now we consider the Lagrangian super-submanifolds of M. A naive one is the underlying reduced manifold X. The density μ induces a density $\mu_X = \sqrt{\mu} = \Omega$ on X, and the Batalin-Vilkovisky integral in Proposition 3.13 is just

$$\int_{\mu_X} \Phi|_X = \int_X \Phi|_X \wedge \Omega = \int_X \left(P_\Phi \vdash \Omega \right) \wedge \Omega$$

More generally, let $C \hookrightarrow X$ be a smooth orientable sub-manifold, $N_{C/X}$ be the normal bundle. Then

$$N_{C/X}^{\vee}[1] \subset \mathcal{T}_X^{\vee}[1]$$

is naturally a Lagrangian super-submanifold of M, which we denote by L_C . If we choose local coordinates x^1, \dots, x^n on $U \subset X$ such that

$$C \cap U = \{x^{k+1} = \dots = x^n = 0\}$$

then L_C is locally described by

$$x^{k+1} = \dots = x^n = 0, \quad \theta_1 = \dots = \theta_k = 0$$

We will identify the sheaves

$$\mathcal{O}_{L_C} \simeq \wedge^* N_{C/X}$$

The induced Berezin density μ_{L_C} can be described by

$$\int_{\mu_{L_C}} \Phi = \int_C (\Phi \vdash \Omega)|_C, \quad \forall \Phi \in \mathcal{O}_{L_C}$$

The Batalin-Vilkovisky integral in Proposition 3.13 is therefore

$$\int_{\mu_{L_C}} \Phi|_{L_C} = \int_C \left(P_\Phi \vdash \Omega \right)|_C, \quad \forall \Phi \in \mathcal{O}_M$$

If $\Delta_{\mu} \Phi = 0$, then $P_{\Phi} \vdash \Omega$ is closed, and the above integral only depends on the homology class of C.

3.2.2. Batalin-Vilkovisky formalism. Now we come back to the finite dimensional integration theory. Let $V = \mathbb{R}^N$ be as before, but we have a Lie-group G acting on V. We will use \mathfrak{g} to denote the Lie algebra of G. Let f be a function on V that is G-invariant. We would like to make sense of the following integration

$$\int_{V/G} e^{f/\hbar}$$

in a homological fashion. Let $\mathcal{O}(V)$ be the space of functions on V. We can naturally identify $\mathcal{O}(V/G)$ as the *G*-invariant subspace $\mathcal{O}(V)^G$. First, we replace $(\mathcal{O}_V)^G$ by the Chevalley-Eilenberg complex $C^*(\mathfrak{g}, \mathcal{O}(V))$, which can be viewed as the space of functions $\mathcal{O}(\mathfrak{g}[1] \oplus V)$ on the super-manifold $\mathfrak{g}[1] \oplus V$. The Chevalley-Eilenberg differential gives an odd derivation of $\mathcal{O}(\mathfrak{g}[1] \oplus V)$, which is called the *BRST* operator. Let X denote this odd vector field, which satisfies

$$[X, X] = 0$$

The next step is to view $\mathfrak{g}[1] \oplus V$ as a Lagrangian supermanifold of its cotangent bundle

$$E = \left(\mathcal{T}_{\mathfrak{g}[1]\oplus V}\right)^{\vee} = \mathfrak{g}[1] \oplus V \oplus V^{\vee}[-1] \oplus \mathfrak{g}^{\vee}[-2]$$

and add a term which deals with the odd variables. Using the canonical odd symplecture structure of E, the odd vector field X gives rise to the Hamiltonian function H_X which vanishes at the origin. The *G*-invariance of f says that X(f) = 0, which implies that $\{f, H_X\} = 0$. The condition [X, X] = 0 implies that $\{H_X, H_X\} = 0$. Let

$$S_0 = f + H_X$$

then S satisfies the following classical master equation

$$(3.16) \qquad \{S_0, S_0\} = 0$$

Therefore we are lead to consider the integral

$$(3.17)\qquad\qquad \int_{L}e^{S_{0}/\hbar}$$

as a candidate for $\int_{V/G} e^{f/\hbar}$. Here *L* is a Lagrangian super-submanifold of *E*, which is usually obtained by a small perturbation of $\mathfrak{g}[1] \oplus V$ in *E*, such that the quadratic part of *S* is non-degenerate along *L*. Therefore we can do perturbation theory using Feynman diagram techniques as before. Such a choice of L is called the *gauge fixing* following physics terminology.

However, we would like that the formula (3.17) is independent of the choice of L, as motivated from the expression $\int_{V/G} e^{f/\hbar}$. Therefore we search for a deformation S_0 by

$$S = S_0 + \sum_{i \ge 1} \hbar^i S_i$$

such that

$$\Delta e^{S/\hbar} = 0$$

Here we have chosen the standard Berezinian density on E. Using Lemma 2.3, it's equivalent to

(3.18)
$$\hbar\Delta S + \frac{1}{2} \{S, S\} = 0$$

which is called the *quantum master equaiton*. Once we have found such S solving the quantum master equation, we can form the integral

$$\int_L e^{S/\hbar}$$

which is now invariant under the small deformation of L. Such formalism is called the *Batalin-Vilkovisky formalism*.

Usually, we can isolate the quadratic part $S_0^{(2)}$ of S_0 by

$$S_0 = S_0^{(2)} + I_0$$

where the lowest degree term of I_0 is at least cubic. $S_0^{(2)}$ plays the role of propagator in our discussion of Feynman diagrams, and I_0 is called the *classical interaction term*. The Hamiltonian vector field of $S_0^{(2)}$ is an odd vector field, which is denoted by Q. The classical master equation and the degree condition implies that

$$\{S_0^{(2)}, S_0^{(2)}\} = 0, \ \{S_0^{(2)}, I_0\} + \frac{1}{2}\{I_0, I_0\} = 0$$

which is equivalent to

(3.19)
$$Q^2 = 0, \quad QI_0 + \frac{1}{2} \{I_0, I_0\} = 0$$

The quantum master equation becomes

(3.20)
$$\operatorname{div} Q = 0, \quad QI + \frac{1}{2}\{I, I\} + \hbar \Delta I = 0$$

where $I = I_0 + \sum_{k \ge 1} \hbar^k S_k$. The condition div Q = 0 says that the vector field Q preservers the measure.

3.3. Effective field theory and renormalization. We will consider the quantum field theory in this subsection. We focus on the case that the fields are geometrically described by sections of a graded vector bundle E on a smooth orientable manifold M

$$Fields: \mathscr{E} = \Gamma(M, E)$$

which is an infinite dimensional vector space. In quantum field theory, we would like to make sense of the following "path integral"

$$\int_{\phi \in \mathscr{E}} [D\phi] e^{S[\phi]/\hbar}$$

where $S[\phi]$ is a functional on E, which is called the *action functional* of the theory. Unfortunately, since \mathscr{E} is not finite dimensional, the integration measure $[D\phi]$ is difficulty to define. However, in many situations, the Feynman diagrams similar to (3.4) still make sense, which can be used as a candidate for the path integral in the perturbative sense. The resulting theory is called the *perturbative field theory*. The difficulty of the infinite dimension goes into the fact that the values of the Feynman diagrams in this case are usually singular (divergent). This is where Wilson's approach of effective field theory and renormalization comes in and plays an important role. We will explain Wilson's effective field theory point of view in this subsection as well as set up our notations following [Cos11].

3.3.1. Functionals and local functionals. Let $\mathscr{E} = \Gamma(M, E)$ be the space of fields, which is a topological vector space in a natural way. If M, N are two smooth manifold and E, F are two graded vector bundles on M, N respectively, the following notation will be used

$$\Gamma(M, E) \hat{\otimes} \Gamma(N, F) = \Gamma(M \times N, E \boxtimes F)$$

which can be viewed as the completed projective tensor product (see [Cos11]).

Definition 3.15. The space of functionals $\mathcal{O}(\mathscr{E})$ on \mathscr{E} is defined to be the graded-commutative algebra

$$\mathcal{O}(\mathscr{E}) = \prod_{n \ge 0} \mathcal{O}^{(n)}(\mathscr{E}) = \prod_{n \ge 0} \operatorname{Hom}\left(\mathscr{E}^{\hat{\otimes}n}, \mathbb{C}\right)_{S_n}$$

where Hom denotes the space of continuous linear maps, S_n acts on $\mathscr{E}^{\hat{\otimes}n}$ via permutation (with signs from the grading), and the subscript S_n denotes taking S_n coinvariants. Elements of $\mathcal{O}^{(n)}(\mathscr{E})$ are said to be of order n. Given $S \in \mathcal{O}(\mathscr{E})$, its component of order n is called the degree n Taylor coefficient, denoted by D_nS .

Given a functional S of order n, we will use the following notation to represent the map

$$S: \mathscr{E}^{\hat{\otimes}n} \to \mathbb{C}$$
$$\alpha_1 \otimes \cdots \otimes \alpha_n \to S[\phi_1, \cdots, \phi_n]$$

The product structure is defined as follows. Let $S_1 \in \mathcal{O}^{(n)}(\mathscr{E}), S_2 \in \mathcal{O}^{(m)}(\mathscr{E})$, then

$$(S_1S_2) [\alpha_1, \cdots, \alpha_{n+m}] = \sum_{\sigma} (-1)^{|\sigma|+|S_2|(|\alpha_{\sigma(1)}|+\cdots+|\alpha_{\sigma(n)}|)} S_1 [\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(n)}] S_2 [\alpha_{\sigma(n+1)}, \cdots, \alpha_{\sigma(n+m)}]$$

where $|\sigma|$ is the parity for permuting $\alpha_1 \cdots \alpha_{n+m}$ to $\alpha_{\sigma(1)} \cdots \alpha_{\sigma(n+m)}$. We will also use $\operatorname{Sym}^n(\mathscr{E})$ for S_n invariant elements of $\mathscr{E}^{\hat{\otimes}n}$, and $\mathcal{O}(\mathscr{E})$ is sometimes written as

$$\prod_{n\geq 0} \operatorname{Hom}\left(\operatorname{Sym}^n(\mathscr{E}), \mathbb{C}\right)$$

S is called *local* if S takes the following form

$$S[\alpha_1, \cdots, \alpha_n] = \int_M D_1(\alpha_1) \cdots D_n(\alpha_n) d$$
 Vol

where D_i 's are arbitrary differential operators from \mathscr{E} to $C^{\infty}(M)$, and we have fixed a nowhere vanishing volume form d Vol to do the integral. The space of local functionals on \mathscr{E} will be denoted by $\mathcal{O}_{loc}(\mathscr{E})$. In other words, S is local if

$$S = \int_M \mathcal{L}$$

where \mathcal{L} is poly-differential map from $\prod_{n\geq 0} \mathscr{E}^{\hat{\otimes}n}$ to the line bundle of top differential forms $\wedge^{top}T^*_M$ on M (see [Cos11] for more precise definition). \mathcal{L} is called the *Lagrangian*.

Given an element $\alpha \in \operatorname{Sym}^{n}(\mathscr{E})$, it defines a contraction map on $\mathcal{O}(\mathscr{E})$ by

$$\left(\frac{\partial}{\partial\alpha}S\right)\left[\cdot\right] \equiv (-1)^{|S||\alpha|}S\left[\alpha,\cdot\right]$$

which can be viewed as order n differential operators in the functional sense. If $\alpha \in \mathscr{E}$, then $\frac{\partial}{\partial \alpha}$ defines a derivation on the space of functionals, i.e.,

$$\frac{\partial}{\partial \alpha} \left(S_1 S_2 \right) = \left(\frac{\partial}{\partial \alpha} S_1 \right) S_2 + (-1)^{|\alpha||S|} S_1 \left(\frac{\partial}{\partial \alpha} S_2 \right)$$

3.3.2. Derivations. The space of derivations on $\mathcal{O}(\mathscr{E})$ is defined to be

$$\operatorname{Der}(\mathcal{O}(\mathscr{E})) = \prod_{n \ge 0} \operatorname{Der}^{(n)}(\mathcal{O}(\mathscr{E})) = \prod_{n \ge 0} \operatorname{Hom}\left(\mathscr{E}^{\hat{\otimes}n}, \mathscr{E}\right)_{S_n}$$

which can be viewed as the space of formal vector fields. The map

$$\operatorname{Der}(\mathcal{O}(\mathscr{E})) \times \mathcal{O}(\mathscr{E}) \to \mathcal{O}(\mathscr{E})$$

can be described as follows. Let $X \in \text{Der}^{(n)}(\mathscr{E}), S \in \mathcal{O}^{(m)}(\mathscr{E})$, then

$$X(S) \in \mathcal{O}^{n+m-1}(\mathscr{E})$$

is given by the explicit formula

$$X(S) [\alpha_1, \cdots, \alpha_{n+m-1}]$$

$$= (-1)^{|X||S|} \sum_{\sigma} \frac{(-1)^{|\sigma|}}{n!(m-1)!} S \left[X \left[\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(n)} \right], \alpha_{\sigma(n+1)}, \cdots, \alpha_{\sigma(n+m-1)} \right]$$

Using this formula, it's easy to see that the graded Leibniz rule is satisfied

$$X(S_1S_2) = X(S_1)S_2 + (-1)^{|S_1||X|}S_1X(S_2)$$

The space of local derivations is the subspace of $Der(\mathcal{O}(\mathscr{E}))$ defined by

$$\operatorname{Der}_{loc}(\mathcal{O}(\mathscr{E})) = \prod_{n \ge 0} \operatorname{PolyDiff}\left(\mathscr{E}^{\hat{\otimes}n}, \mathscr{E}\right)^{S_n}$$

where PolyDiff represents the space of poly-differential operators [Cos11]. Both $Der(\mathcal{O}(\mathscr{E}))$ and $Der_{loc}(\mathcal{O}(\mathscr{E}))$ have a natural graded Lie algebra structure.

3.3.3. Feynman Diagrams. Let P be an element

$$P \in \operatorname{Sym}^2(\mathscr{E}^{\otimes 2})$$

and S be a functional valued in $\mathbb{C}[[\hbar]]$

$$S \in \mathcal{O}(\mathscr{E})[[\hbar]]$$

such that S is at least cubic modulo \hbar . We would like to consider the following functional

$$e^{\hbar\frac{\partial}{\partial P}}e^{S/\hbar}$$

This is usually not well-defined due to the infinite sums. However, its logarithm makes sense, which is denoted by

(3.21)
$$W(P,S) = \hbar \log \left(e^{\hbar \frac{\partial}{\partial P}} e^{S/\hbar} \right) \in \mathcal{O}(\mathscr{E})[[\hbar]]$$

In fact, by Remark 3.4, W(P, S) can be defined by

(3.22)
$$W(P,S) = \hbar \sum_{\Gamma \text{ connected}} \frac{W_{\Gamma}(P,S)}{|\operatorname{Aut}(\Gamma)|}$$

where we are summing over connected diagrams Γ , whose weight $W_{\Gamma}(P, S)$ is computed by putting $\hbar P$ on the edges and S/\hbar on the vertices. The condition of S implies that only non-negative powers of \hbar appears in the above formula, and for each fixed power of \hbar , there's only a finite sum. Therefore it's a well-defined element of $\mathcal{O}(\mathscr{E})[[\hbar]]$. *P* is called the *propagator*, and *S* is called the *vertices*.

3.3.4. *Effective functional and renormalization*. In many examples of quantum field theory, the action will usually look like

$$S = S_2 + I$$

where S_2 is the quadratic part of S taking the form

$$S_2(\phi,\phi) = -\int_M \langle \phi, \Box \phi \rangle$$

I is the interaction part, whose lowest order term is at least cubic. Here \Box is certain Laplacian type operator on \mathscr{E} , and \langle,\rangle is certain inner product on \mathscr{E} . We will use this inner product to identify E with its dual E^{\vee} . Following the philosophy of Feynman diagrams in finite dimensional case, the propagator will be defined by the inverse of \Box , which can be represented by the Green kernel

$$P \in \Gamma(M \times M - \Delta, E \boxtimes E)$$

where Δ is the diagonal of $M \times M$. The problem is that P is not an element of $\text{Sym}^2(\mathscr{E})$, but exhibits a singularity along Δ . Therefore the naive definition

$$e^{W(P,I)/\hbar} {\stackrel{\Delta}{=}} \int_{\mathscr{E}} e^{S/\hbar}$$

would not work, since the weight of the Feynman diagral $W_{\Gamma}(P, I)$ will be divergent.

On the other hand, the propagator P can be re-written as

$$P = \int_0^\infty dt e^{-t\Box}$$

where $e^{-t\Box}$ is the heat kernel of \Box , which is an element of $\operatorname{Sym}^2(\mathscr{E})$ if $t \neq 0$. We can define the *regularized propagator* by the cut-off

$$(3.23) P_{\epsilon}^{L} = \int_{\epsilon}^{L} dt e^{-t\Box}$$

which is now smooth for any $\epsilon, L > 0$. Given a functional $S \in \mathcal{O}(\mathscr{E})$, the Feynman diagram interpretation shows that

$$W(P_{L_1}^{L_3},S) = W(P_{L_1}^{L_2} + P_{L_2}^{L_3},S) = W(P_{L_2}^{L_3},W(P_{L_1}^{L_2},S))$$

for any $0 < L_1 < L_2 < L_3$. This motivates the following definitions

Definition 3.16. A family of functionals $S[T] \in \mathcal{O}(\mathscr{E})[[\hbar]]$ for T > 0, which are at least cubic modulo \hbar , are said to satisfy the *renormalization group flow equation* if

$$(3.24) S[L] = W(P_{\epsilon}^{L}, S[\epsilon])$$

holds for every $\epsilon, L > 0$.

Definition 3.17. A family of functionals $S[T] \in \mathcal{O}(\mathscr{E})$ for T > 0, which are at least cubic modulo \hbar , is said to satisfy the *tree-level renormalization group flow equation* if

(3.25)
$$S[L] = W_{tree} \left(P_{\epsilon}^{L}, S[\epsilon] \right)$$

holds for every $\epsilon, L > 0$. Here for W_{tree} , we mean that we only sum over the tree diagrams in the Feynman diagram expansion.

It's easy to see that if $S[T] = \sum_{g \ge 0} \hbar^g S_g[T]$ satisfy the renormalization group flow equation, then $S_0[L]$ satisfies the tree-level renormalization group flow equation.

In the tree-level, there's no divergence for Feynman graph integrals. In fact, let S be an arbitrary local functional, then the limit

$$S[L] = \lim_{\epsilon \to 0} W_{tree} \left(P_{\epsilon}^{L}, S \right)$$

exists and defines a family of functionals satisfying tree-level renormalization group flow equation such that $\lim_{L\to 0} S[L] = S$. Therefore the ultraviolet divergence is a quantum effect.

Definition 3.18. [Cos11] A system of *effective functionals* on \mathscr{E} is given by a family of functionals $S[T] \in \mathcal{O}(\mathscr{E})[[\hbar]]$ for any T > 0, which are at least cubic modulo \hbar , such that the renormalization group flow equation holds and $\lim_{T\to 0} S[T]$ becomes local in the following

sense: there exists some T-dependent local functional $\Phi_T \in \mathcal{O}_{loc}(\mathscr{E})[[\hbar]]$ for T > 0 such that

(3.26)
$$\lim_{T \to 0} (S[T] - \Phi_T) = 0$$

Let $S \in \mathcal{O}_{loc}(\mathscr{E})[[\hbar]]$ be an even local functional, which is at cubic modulo \hbar . The following functional is well-defined

$$W(P^L_{\epsilon}, S)$$

As we have mentioned, it's singular as $\epsilon \to 0$. This is called Ultraviolet divergence in physics terminology. The following result is widely used in physics literature, and a mathematical proof can be found in [Cos11]

Proposition 3.19. There exists ϵ -dependent local functional

$$S^{CT}(\epsilon) \in \hbar \mathcal{O}_{loc}(\mathscr{E})[[\hbar]]$$

such that the limit

$$\lim_{\epsilon \to 0} W(P^L_{\epsilon}, S + S^{CT}(\epsilon))$$

exists.

Such correction $S^{CT}(\epsilon)$ is called the *counter terms*. It follows that

$$S^{eff}[T] = \lim_{\epsilon \to 0} W(P^L_{\epsilon}, S + S^{CT}(\epsilon))$$

defines a system of effective functionals. If the manifold M is compact, then P_T^{∞} is also a smooth kernel. Therefore

$$S^{eff}[\infty] = \lim_{T \to \infty} S^{eff}[T]$$

is well-defined element of $\mathcal{O}(\mathscr{E})[[\hbar]]$. The following diagram illustrates the procedure

classical action
$$S \xrightarrow{counter terms} S + S^{CT}(\epsilon)$$

 $\downarrow Feynman \ diagram$
effective functional $S^{eff}[T]$

This procedure is called the *perturbative renormalization*. Note that the choice of the counter terms is usually not unique.

Remark 3.20. In some special field theories, the limit $\lim_{\epsilon \to 0} W(P_{\epsilon}^{L}, S)$ exists already. Chern-Simons theory on three dimensional manifold is such an example [AS92, AS94, Kon94]. We will prove in section 3.5 that one-dimensional holomorphic theory is another such example.

3.4. Gauge theory and quantum master equation. We will quickly review the gauge theory and its quantization in Batalin-Vilkovisky formalism following the discussion in [Cos11].

3.4.1. Classical gauge symmetry in Batalin-Vilkovisky formalism. Our starting point for the geometric data of gauge theory in Batalin-Vilkovisky formalism is the following

(1) Fields. The space of fields will be the space of smooth sections of a graded vector bundle E on a smooth orientable manifold M of dimension d, denoted by

$$\mathscr{E} = \Gamma(M, E)$$

(2) Odd symplectic structure. A degree -1, skew-symmetric and fiber-wise non-degenerate pairing of graded vector bundles

$$(3.27) \qquad \langle,\rangle: E\otimes E \to \det_M$$

where $\det_M \equiv \det(T^*_{\mathbb{R}}M)$ is the line bundle of top differential forms on M. We assign the grading on \det_M such that it's concentrated in degree zero, and \langle, \rangle is a morphism of graded vector bundles of degree -1. It induces a natural isomorphism $E \cong E^{\vee} \otimes \det_M[-1]$, which defines a Poisson bracket

$$\{-,-\}: \mathcal{O}_{loc}(\mathscr{E}) \otimes \mathcal{O}(\mathscr{E}) \to \mathcal{O}(\mathscr{E})$$

as follows: let $S_1 \in \mathcal{O}_{loc}(\mathscr{E})$ and $S_2 \in \mathcal{O}(\mathscr{E})$. The locality of S_1 implies that it's given by a lagrangian \mathcal{L} which is a poly-differential map from $\prod_{n\geq 0} \mathscr{E}^{\hat{\otimes}n}$ to \det_M . It specifies uniquely a poly-differential map from $\prod_{n>0} \mathscr{E}^{\hat{\otimes}n-1}$ to $\Gamma(E^{\vee}) \otimes \det_M$. Using the isomorphism of graded vector bundles $E \cong (E)^{\vee} \otimes \det_M[-1]$, it further defines an element of local derivation

$$V_{S_1} \in \operatorname{Der}_{loc}(\mathcal{O}(\mathscr{E}))$$

$$S_1\left[\alpha_1,\cdots,\alpha_n\right] = \left\langle V_{S_1}\left[\alpha_1,\cdots,\alpha_{n-1}\right],\alpha_n\right\rangle, \quad \forall \alpha_1,\cdots,\alpha_n \in \mathscr{E}$$

The Poisson bracket is defined via

$$(3.28) {S_1, S_2} = V_{S_1}(S_2)$$

and it's easy to check that $\{S_1, S_2\} = (-1)^{|S_1||S_2|} \{S_2, S_1\}.$

(3) The differential. An odd linear elliptic differential operator Q : & → & of cohomological degree 1, which is skew self-adjoint with respect to the symplectic pairing, and Q² = 0. It defines the quadratic part of the action

(3.29)
$$S_2(e) = \frac{1}{2} \int \langle Qe, e \rangle, \quad e \in \mathscr{E}$$

Q naturally induces a derivation on the space of functionals which we still denote by Q, and it's precisely the Hamiltonian vector field V_{S_2} .

(4) Gauge fixing operator. An odd linear elliptic differential operator $Q^{GF} : \mathscr{E} \to \mathscr{E}$ of cohomological degree -1, which is self-adjoint with respect to the symplectic pairing, and $(Q^{GF})^2 = 0$. Moreover, the operator

$$H = \left[Q, Q^{GF}\right]$$

is a second order elliptic operator which is a generalized Laplacian.

(5) Classical action. The interaction term of the classical action

$$I^{cl} \in \mathcal{O}_{loc}(\mathscr{E})$$

which satisfies the classical master equation

(3.30)
$$QI^{cl} + \frac{1}{2} \left\{ I^{cl}, I^{cl} \right\} = 0$$

If we write the full action by

$$S^{cl} = S_2 + I^{cl}$$

then the classical master equation is equivalent to

$$(3.31)\qquad\qquad \left\{S^{cl},S^{cl}\right\}=0$$

Remark 3.21. The classical master equation $\{S^{cl}, S^{cl}\} = 0$ defines a gauge symmetry which leaves the action S^{cl} invariant. In fact, the gauge transformation is represented by the Hamiltonian vector field associated to S^{cl} , and the gauge invariance is nothing but

$$V_{S^{cl}}\left(S^{cl}\right) = \left\{S^{cl}, S^{cl}\right\} = 0$$

Example 3.22 (Chern-Simons Theory). The underlying manifold will be a compact 3dimensional Riemannian manifold (M, g). Here g is a chosen metric. Let G be a compact Lie subgroup of U(N), \mathcal{G} be the Lie algebra of G. For simplicity, we will consider the trivial G-bundle P on M. The space of fields for Chern-Simons theory in the Batalin-Vilkovisky formalism is

$$\mathscr{E} = \Omega^*(M, \mathcal{G})[1]$$

where $\Omega^*(M, \mathcal{G})$ is the space of differential forms valued in \mathcal{G} . We have shifted the degree by one such that the degree zero part $\Omega^1(M, \mathcal{G})$ is the space of connections on P, which is the field content for the usual Chern-Simons theory. We will use $|\alpha|$ for the degree of $\alpha \in \mathscr{E}$.

The odd symplectic structure is given by

$$\langle \alpha, \beta \rangle \equiv (-1)^{|\alpha|} \operatorname{Tr} \alpha \wedge \beta, \ \forall \alpha, \beta \in \mathscr{E}$$

where Tr is some normalized Killing form on \mathcal{G} . The sign is chosen such that

$$\langle \alpha, \beta \rangle = -(-1)^{|\alpha||\beta|} \langle \beta, \alpha \rangle$$

The differential Q is given by the de Rham differential

$$Q=d:\mathscr{E}\to\mathscr{E}$$

and the gauge fixing opear tor is given by the adjoint of d with respect to the chosen metric

g

$$Q^{GF}=d^*:\mathscr{E}\to\mathscr{E}$$

Therefore $H = [d, d^*] = dd^* + d^*d$ is the standard Laplacian operator.

Now we describe the classical action S^{CS} . S^{CS} will have non-trivial Taylor coefficients in order 2 and 3

$$S^{CS} = S_2^{CS} + S_3^{CS}$$

where

$$S_2^{CS}\left[\alpha_1,\alpha_2\right] = \int_M \left\langle d\alpha_1,\alpha_2 \right\rangle = -\int_M \operatorname{Tr} \alpha_1 \wedge d\alpha_2, \quad \forall \alpha_1,\alpha_2 \in \mathscr{E}$$

and

$$S_3^{CS}\left[\alpha_1, \alpha_2, \alpha_3\right] = (-1)^{|\alpha_2|} \int_M \operatorname{Tr} \alpha_1 \wedge [\alpha_2, \alpha_3]$$

where the extra sign $(-1)^{|\alpha_2|}$ is to make sure that S_3^{CS} is graded symmetric in our grading convention for \mathscr{E} . Then

$$QS_3^{CS} = 0$$

as it gives rise to total derivative. Also

$$\left\{S_3^{CS},S_3^{CS}\right\}=0$$

which is equivalent to the Jacobi identity. In particular, the classical master equation is satisfied

$$QS_3^{CS} + \frac{1}{2} \left\{ S_3^{CS}, S_3^{CS} \right\} = 0$$

3.4.2. Regularized BV operator. The heat kernel e^{-tH} for t > 0 defines a smooth section of

$$E \boxtimes (E^{\vee} \otimes \det_M)$$

on $M \times M$. Using the isomorphism $E \cong E^{\vee} \otimes \det_M$, we will identify the above bundle with $E \boxtimes E$, and use

$$K_t \in \mathscr{E}^{\otimes 2}$$

to represent the heat kernel e^{-tH} under the above identification. Note that since the symplectic pairing is odd of degree -1, this K_t will have degree one. It's easy to see that

the Poisson bracket for $S_1 \in \mathcal{O}_{loc}(\mathscr{E}), S_2 \in \mathcal{O}(\mathscr{E})$ is just

$$(3.32) \qquad \{S_1, S_2\} = \lim_{t \to 0} \left(\frac{\partial}{\partial K_t} \left(S_1 S_2 \right) - \left(\frac{\partial}{\partial K_t} S_1 \right) S_2 - (-1)^{|S_1|} S_1 \frac{\partial}{\partial K_t} S_2 \right)$$

However, the limit usually doesn't exist if neither S_1 nor S_2 is local.

Definition 3.23. The regularized BV operator Δ_L at L > 0 is the second order operator on $\mathcal{O}(\mathscr{E})$ defined by

$$\Delta_L = \frac{\partial}{\partial K_L}$$

The regularized BV bracket $\{,\}_L$ is defined by

$$\{S_1, S_2\}_L = \Delta_L (S_1 S_2) - (\Delta_L S_1) S_2 - (-1)^{|S_1|} S_1 \Delta_L S_2$$

for any $S_1, S_2 \in \mathcal{O}(\mathscr{E})$.

The oddness of K_L implies that $\Delta_L^2 = 0$, therefore $\{\Delta_L, \{,\}_L\}$ defines a Batalin-Vilkovisky structure on $\mathcal{O}(\mathscr{E})$ for each L > 0.

3.4.3. Regularized propagator. By the form of the quadratic term of the classical action (3.29), we see that the naive propagator would represent the inverse of the operator Q. The gauge fixing operator Q^{GF} allows us to replace it by the operator

$$Q^{GF} \frac{1}{H}$$

whose kernel in fact exists, but exhibits singularities on the diagonal of $M \times M$. By the same philosophy of Wilson's effective functional point of view, we can smooth out this kernel by using certain cut-off as follows

Definition 3.24. The regularized propagator P_{ϵ}^{L} is defined to be the kernel

(3.33)
$$P_{\epsilon}^{L} = \int_{\epsilon}^{L} dt Q^{GF} K_{t} = \int_{\epsilon}^{L} dt Q^{GF} e^{-tH} \in \mathscr{E}^{\hat{\otimes}2}$$

for $\epsilon, L > 0$.

The regularized propagator gives a homotopy between BV operators at different scales. Precisely,

Lemma 3.25.

(3.34)
$$\left[Q, \frac{\partial}{\partial P_{\epsilon}^{L}}\right] = \Delta_{\epsilon} - \Delta_{L}$$

It follows from the lemma that

(3.35)
$$\left[Q, e^{\hbar\partial_{P_{\epsilon}^{L}}}\right] = \left(\hbar\Delta_{\epsilon} - \hbar\Delta_{L}\right) e^{\hbar\partial_{P_{\epsilon}^{L}}}$$

3.4.4. Quantum master equation. Let $\{S[T]\}_{T>0}$ be a system of effective action, which we mean that $S[T] \in \mathcal{O}(\mathscr{E})[[\hbar]]$ for each T > 0, at least cubic modulo \hbar , and satisfy the renormalization group flow equation

$$S[L] = W(P_{\epsilon}^{L}, S[\epsilon])$$

for all $\epsilon, L > 0$.

Definition 3.26. $\{S[T]\}_{T>0}$ is said to satisfy quantum master equation if

(3.36)
$$QS[L] + \frac{1}{2} \{S[L], S[L]\}_L + \hbar \Delta_L S[L] = 0$$

holds for some L > 0.

Note that if (3.36) holds for some L > 0, then it holds for all L > 0. In fact, (3.36) can be written symbolically by

$$(3.37) \qquad \qquad (Q + \hbar \Delta_L) e^{S[L]/\hbar} = 0$$

The renormlization group flow links the quantum master equation at different scales

$$(Q + \hbar\Delta_L) e^{S[L]/\hbar} = (Q + \hbar\Delta_L) \left(e^{\hbar\partial_{P_{\epsilon}^L}} e^{S[\epsilon]/\hbar} \right) = e^{\hbar\partial_{P_{\epsilon}^L}} \left(Q + \hbar\Delta_{\epsilon} \right) e^{S[\epsilon]/\hbar}$$

Definition 3.27. A quantization of the classical action I^{cl} satisfying the classical master equation is given by a system of effective functionals $\{S[T]\}_{T>0}$ which satisfies the renormalization group flow equation, quantum master equation, asymptotically local as $T \to 0$, and the classical limit condition

$$\lim_{T \to 0} S[T] = I^{cl} \text{ modulo } \hbar$$

3.4.5. *Obstruction theory.* By Proposition 3.19, a family of effective functionals which satisfies renormalization group flow equation and classical limit condition always exists, though not unique. But there's usually an obstruction for the quantum master equation.

Assume that the effective family of actions $\{S[L]\}$ satisfies the RG flow

$$e^{S[L]/\hbar} = e^{\hbar P_{\epsilon}^{L}} e^{I[\epsilon]/\hbar}$$

and satisfies the quantum master equation modulo \hbar^n , i.e.

$$QS[L] + \frac{1}{2} \{S[L], S[L]\}_L + \hbar \Delta_L S[L] = O(\hbar^{n+1})$$

We will write

$$S[L] = \sum_{k \ge 0} \hbar^k S_k[L]$$

and the classical limit condition becomes

$$\lim_{L \to 0} S_0[L] = I^{cl}$$

Let

$$O[L] = QS[L] + \frac{1}{2} \{S[L], S[L]\}_L + \hbar \Delta_L S[L]$$

or equivalently

(3.38)
$$O[L]e^{S[L]/\hbar} = \hbar \left(Q + \hbar \Delta_L\right) e^{S[L]/\hbar}$$

The compatibility of renormalization group flow and quantum master equation implies that

(3.39)
$$O[L]e^{S[L]/\hbar} = e^{\hbar P_{\epsilon}^{L}} O[\epsilon]e^{S[\epsilon]/\hbar}$$

By assumption, we can write

$$O[L] = \sum_{k \ge n+1} \hbar^k O_k[L]$$

Equation (3.39) can be rewritten as

(3.40)
$$e^{S[L]/\hbar + \delta O[L]/\hbar^{n+2}} = e^{\hbar P_{\epsilon}^{L}} e^{S[\epsilon]/\hbar + \delta O[\epsilon]/\hbar^{n+2}}$$

where δ is an odd variable of cohomological degree -1, $\delta^2 = 0$. This is equivalent to saying that $S_0[L] + \delta O_{n+1}[L]$ satisfies the tree-level renormalization group flow equation, which implies that

$$O_{n+1} = \lim_{L \to 0} O_{n+1}[L] \in \mathcal{O}_{loc}(\mathscr{E})$$

exists as a local functional.

On the other hand, since $(Q + \hbar \Delta_L)^2 = 0$, we have

$$0 = (Q + \hbar \Delta_L) \left(O_L e^{S[L]/\hbar} \right) = (QO[L] + \{S[L], O[L]\} + \hbar \Delta_L O[L]) e^{S[L]/\hbar}$$

which implies that

(3.41)
$$QO[L] + \{S[L], O[L]\} + \hbar \Delta_L O[L] = 0$$

If we pick up the leading power of \hbar , we find

(3.42)
$$QO_{n+1}[L] + \{S_0[L], O_{n+1}[L]\}_L = 0$$

We can take the limit $L \to 0$ and find

$$QO_{n+1} + \{I^{cl}, O_{n+1}\} = 0$$

i.e., O_{n+1} is $Q + \{I^{cl}, \cdot\}$ closed. Therefore O_{n+1} gives a cohomology class

$$[O_{n+1}] \in H^1(\mathcal{O}_{loc}(\mathscr{E}), Q + \{I^{cl}, \cdot\})$$

If $[O_{n+1}]$ is trivial, which means that there exists a local functional U_{n+1} of cohomological degree 0 such that

$$O_{n+1} = QU_{n+1} + \left\{ I^{cl}, U_{n+1} \right\}$$

Let $U_{n+1}[L]$ be the effective functional such that $S_0[L] + \epsilon U_{n+1}[L]$ satisfies the tree-level renormalization group flow equation for some odd variable $\epsilon, \epsilon^2 = 0$. We modify S[L] by

$$S'_g[L] = \begin{cases} S_g[L] & \text{if } g \neq n+1 \\ \\ S_g[L] - U_g[L] & \text{if } g = n+1 \end{cases}$$

then

$$QS'[L] + \frac{1}{2} \{ S'[L], S'[L] \}_L + \hbar \Delta_L S'[L] = O(\hbar^{n+2})$$

and S'[L] satisfies the renormalization group flow equation up to order \hbar^{n+1} . It's proved in [Cos11] that we can furthermore modify $S'_g[L]$ for g > n+1 such that S'[L] satisfies the renormalization group flow equation.

It follows that the existence of the modification of S[L] to let $O_{n+1}[L]$ vanish is equivalent to the vanishing of the cohomology class $[O_{n+1}]$. Therefore $[O_{n+1}]$ is the obstruction class for extending the quantum master equation to order n + 1.

The complex $(\mathcal{O}_{loc}(\mathscr{E}), Q + \{I^{cl}, \cdot\})$ is called the *deformation-obstruction complex* of the gauge theory in the Batalin-Vilkovisky formalism. The corresponding cohomology groups H^{-1}, H^0, H^1 play the role of automorphism, tangent space and the obstruction space for the quantization of the classical action I^{cl} . The obstruction class is also called *anomaly* in physics literature.

The cohomology of $(\mathcal{O}_{loc}(\mathscr{E}), Q + \{I, \cdot\})$ can be computed using the Jet bundle. See Appendix B for a quick summary for the D-module and jet bundles. Let E be the graded vector bundle where the fields live, and J(E) be the sheaf of jets. Let D_M be the algebra of differential operators on M. Then J(E) can be naturally viewed as a D_M -module. Let

$$J(E)^{\vee} = \operatorname{Hom}_{C_M^{\infty}}(J(E), C_M^{\infty})$$

be the sheaf of continuous linear maps of C_M^{∞} -modules, which has an induced D_M -module structure. The space of local functionals on E is precisely

$$\mathcal{O}_{loc}(\mathscr{E}) = \det_M \bigotimes_{D_M} \prod_{n \ge 0} \operatorname{Sym}_{C_M^{\infty}}^n \left(J(E)^{\vee} \right)$$

If we mod out the constant functional, then we are in a slightly better situation.

Proposition 3.28 ([Cos11]). There's a canonical quasi-isomorphism of cochain complexes

(3.43)
$$\mathcal{O}_{loc}(\mathscr{E})/\mathbb{C} \cong \det_M \bigotimes_{D_M}^{\mathbb{L}} \prod_{n>0} \operatorname{Sym}_{C_M^{\infty}}^n \left(J(E)^{\vee} \right)$$

$$\mathfrak{g} = J(\mathscr{E})[-1]$$

where the L_{∞} structure is given by

$$Q + V_{I^{cl}} \in \operatorname{Der}_{loc}(\mathscr{E})$$

Recall that $V_{I^{cl}}$ is the Hamiltonian vector field associated with I^{cl} . Then $\mathcal{O}_{loc}(E)/\mathbb{C}$ can be expressed in terms of reduced Chevalley-Eilenberg complex

$$\mathcal{O}_{loc}(\mathscr{E})/C = \det_M \bigotimes_{D_M}^{\mathbb{L}} C^*_{red}\left(\mathfrak{g}\right)$$

Here \det_M has a natural right D_M -module structure. By (B.2), we have a quasi-isomorphism of complexes of D_M -modules

$$\det_M \simeq \Omega^*_M[d] \otimes_{C^\infty_M} D_M$$

It follows that the deformation-obstruction complex is quasi-isomorphic to the de Rham comoplex of the D_M -module $C^*_{red}(\mathfrak{g})$

$$\Omega^*_M(C^*_{red}\left(\mathfrak{g}\right))[d]$$

whose cohomology can be computed via spectral sequence.

3.4.6. Independence of gauge fixing condition. In most examples of gauge theory, the gauge fixing operator Q^{GF} is given by the adjoint of Q with respect to a chosen metric. A particular example is the Chern-Simons theory described in the beginning of this section. We will focus our discussion on theories of this type here. The more general set-up is described in [Cos11]. We would like to understand how the theory changes under the change of the gauge fixing condition, i.e., the change of the metric. We will sketch the result of [Cos11] which says that the quantizations at different gauge fixing conditions are homotopy equivalent.

We first describe the simplicial structure of the space of quantizations.

Let Δ^n be the standard *n*-simplex, $\{g_t\}_{t\in\Delta^b}$ be a smooth family of metrics parametrized by Δ^n . This family of metrics leads to a family of operators given by the adjoint of Q with respect to the metric g_t , depending smoothly on $t \in \Delta^b$,

$$\mathscr{E} \to \mathscr{E} \otimes C^{\infty}_{\Delta^n}$$

The $\Omega^*_{\Delta_n}$ -linear extension of the above defines our gauge fixing operator over Δ^n

$$Q^{GF}:\mathscr{E}\otimes\Omega^*_{\Delta_n}\to\mathscr{E}\otimes\Omega^*_{\Delta_n}$$

If

$$\Delta^m \to \Delta^n$$

is a face or degeneracy map, then one can pull a family of gauge fixing operators over Δ^n to Δ^m . In this way, gauge fixing operators form a simplicial set, which we will denote by $\mathcal{GF}(\mathscr{E},Q)$. Since the space of metrics is contractible, this defines a contractible simplicial set.

Given a family of gauge fixing conditions over Δ^n , we consider the following operator acting on $\mathscr{E} \otimes \Omega^*_{\Delta_n}$

$$H = \left[Q + d_t, Q^{GF}\right]$$

where d_t is the de Rham differential on $\Omega^*_{\Delta_n}$. *H* is linear in $\Omega^*_{\Delta_n}$ and we assume that it is a generalized Laplacian. Let

$$K_u \in \mathscr{E} \otimes \mathscr{E} \otimes \Omega^*_{\Delta_n}$$

be the kernel for the $\Omega^*_{\Delta_n}$ -linear operator e^{-uH} , which defines the regularized BV operator

$$\Delta_u = \frac{\partial}{\partial K_u}$$

Similarly, we define the regularized propagator P_{ϵ}^{L} over $\Omega_{\Delta_{n}}^{*}$ by the kernel of the operator

$$\int_{\epsilon}^{L} du Q^{GF} e^{-uH}$$

Definition 3.29. A quantization of the classical action I^{cl} over $\Omega^*_{\Delta_n}$ is given by a family of effective functionals $\{S[L]\}_{L \in 0}$, where $S[L] \in \mathcal{O}(\mathscr{E}) \otimes \Omega^*_{\Delta_n}[[\hbar]]$, which satisfies the

$$e^{S[L]/\hbar} = e^{\hbar \frac{\partial}{\partial P_{\epsilon}^{L}}} e^{S[\epsilon]/\hbar}$$

the quantum master equation over $\Omega^*_{\Delta_n}$

$$(Q+d_t+\hbar\Delta_L)\,e^{S[L]/\hbar}=0$$

and similar classical limit condition and asymptotic local conditions for $L \rightarrow 0$.

This defines a simplicial set of quantizations, which we will denote by $Quan(\mathscr{E}, Q)$. Note that a 0-simplex of $Quan(\mathscr{E}, Q)$ is just given by the quantization at a fixed metric as we have discussed.

From the above construction, we see that there's a canonical map of simplicial set

$$Quan(\mathscr{E}, Q) \to \mathcal{GF}(\mathscr{E}, Q)$$

The proposition in [Cos11] says that this map is in fact a *fibration* of simplical sets. Since the space of metrics is contractible, this implies that any two fibers of the above map is homotopy equivalent. Therefore any choice of metric for quantization will be equivalent. This will be implicitly used in our construction of the BCOV theory on the elliptic curves.

3.5. Feynman graph integral for holomorphic theory on \mathbb{C} . We consider some generalities for the Feynman graph integrals of field theories living on the complex plane \mathbb{C} , where the lagrangian consists of holomorphic derivatives only. Examples of such theories include one dimensional holomorphic Chern-Simons theory [Cos] as well as one dimensional BCOV theory [CL]. We prove that the counter terms can be chosen to be zero and the Feynman graph integrals are finite, i.e., ultraviolet divergence is absent. This will be used to give an explicit local formula for the quantum master equation in the one dimensional BCOV theory in section 5.

Let z be the linear holomorphic coordinate on \mathbb{C} , $\Box = -4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$ be the standard Laplacian operator. The following notations will be used throughout this section

$$H_{\epsilon}^{L}(z,\bar{z}) = \int_{\epsilon}^{L} \frac{dt}{4\pi t} e^{-|z|^{2}/4t}$$
which is the kernel function for the operator $\int_{\epsilon}^{L} dt e^{-t\Box}$. In specific examples, the principle part of the propagator will be the holomorphic derivatives of H_{ϵ}^{L} , as we will see in the example of BCOV theory.

Given an arbitrary connected graph Γ without self-loops, we consider the following Feynman graph integral

$$W_{\Gamma,\{n_e\}}(H^L_{\epsilon},\Phi) \equiv \int \prod_{v \in V(\Gamma)} d^2 z_v \left(\prod_{e \in E(\Gamma)} \partial^{n_e}_{z_e} H^L_{\epsilon}(z_e,\bar{z}_e)\right) \Phi, \text{ where } z_e = z_{l(e)} - z_{r(e)}$$

here $V(\Gamma)$ is the set of vertices, and $E(\Gamma)$ is the set of edges. We choose an arbitrary orientation of the edge, so l(e) and r(e) represents the corresponding two vertices associated to the edge. n_e 's are some non-negative integers associated to each $e \in E$. Φ is a smooth function on $\mathbb{C}^{|V(\Gamma)|}$ with compact support. In the above integral, we view $H_{\epsilon}^{L}(z_e, \bar{z}_e)$ as propagators associated to the edge $e \in E$, and we have only holomorphic derivatives on the propagators.

Theorem 3.30. The following limit exists for the above graph integral

$$\lim_{\epsilon \to 0} W_{\Gamma, \{n_e\}}(H^L_{\epsilon}, \Phi)$$

Proof. Let $V = |V(\Gamma)|$ be the number of vertices and $E = |E(\Gamma)|$ be the number of edges. We index the vertices by

$$v: \{1, 2, \cdots, V\} \to V(\Gamma), \quad V = |V(\Gamma)|$$

and write z_i for $z_{v(i)}$ if there's no confusion. We specify the last vertex by v_{\bullet}

$$v(V) = v_{\bullet}$$

Define the incidence matrix $\{\rho_{v,e}\}_{v\in V(\Gamma),e\in E(\Gamma)}$ by

$$\rho_{v,e} = \begin{cases} 1 & l(e) = v \\ -1 & r(e) = v \\ 0 & \text{otherwise} \end{cases}$$

Define the $(V-1) \times (V-1)$ matrix $M_{\Gamma}(t)$ by

(3.44)
$$M_{\Gamma}(t)_{i,j} = \sum_{e \in E(G)} \rho_{v(i),e} \frac{1}{t_e} \rho_{v(j),e}, \qquad 1 \le i, j \le V - 1$$

where t_e is a variable introduced for each edge coming from the propagator. Consider the following linear change of variables

$$\begin{cases} z_i = y_i + y_V & 1 \le i \le V - 1 \\ z_V = y_V \end{cases}$$

The graph integral can be written as

$$W_{\Gamma,\{n_e\}}(H_{\epsilon}^L,\Phi)$$

$$= \int_{\mathbb{C}} d^2 y_V \int_{\mathbb{C}^{V-1}} \prod_{i=1}^{V-1} d^2 y_i \int_{[\epsilon,L]^E} \prod_{e \in E(\Gamma)} \frac{dt_e}{4\pi t_e} \prod_{e \in E(\Gamma)} \left(\frac{\sum_{i=1}^{V-1} \rho_{v(i),e} \bar{y}_i}{4t_e}\right)^{n_e}$$

$$\exp\left(-\frac{1}{4} \sum_{i,j=1}^{V-1} M_{\Gamma}(t)_{i,j} y_i \bar{y}_j\right) \Phi$$

Using integration by parts, we get

$$W_{\Gamma,\{n_e\}}(H_{\epsilon}^L,\Phi) = \int_{\mathbb{C}} d^2 y_V \int_{\mathbb{C}^{V-1}} \prod_{i=1}^{V-1} d^2 y_i \int_{[\epsilon,L]^E} \prod_{e \in E(\Gamma)} \frac{dt_e}{4\pi t_e} \exp\left(-\frac{1}{4} \sum_{i,j=1}^{V-1} M_{\Gamma}(t)_{i,j} y_i \bar{y}_j\right)$$
$$\prod_{e \in E(\Gamma)} \left(\sum_{j=1}^{V-1} \frac{\sum_{i=1}^{V-1} \rho_{v(i),e} M_{\Gamma}^{-1}(t)_{i,j}}{t_e} \frac{\partial}{\partial y_j}\right)^{n_e} \Phi$$

By Lemma 3.35 below, we see that

$$\left| \prod_{e \in E(\Gamma)} \left(\sum_{j=1}^{V-1} \frac{\sum_{i=1}^{V-1} \rho_{v(i),e} M_{\Gamma}^{-1}(t)_{i,j}}{t_e} \frac{\partial}{\partial y_j} \right)^{n_e} \Phi \right| \le C \left| \tilde{\Phi} \right|$$

where C is a constant which doesn't depend on $\{t_e\}$ and $\{y_i\}$, and $\tilde{\Phi}$ is some smooth function with compact support. To prove that $\lim_{\epsilon \to 0} W_{\Gamma,\{n_e\}}(H^L_{\epsilon}, \Phi)$ exists, we only need to show that

$$\lim_{\epsilon \to 0} \int_{\mathbb{C}^{V-1}} \prod_{i=1}^{V-1} d^2 y_i \int_{[\epsilon,L]^E} \prod_{e \in E(\Gamma)} \frac{dt_e}{4\pi t_e} \exp\left(-\frac{1}{4} \sum_{i,j=1}^{V-1} M_{\Gamma}(t)_{i,j} y_i \bar{y}_j\right)$$
$$= \lim_{\epsilon \to 0} \int_{[\epsilon,L]^E} \prod_{e \in E(\Gamma)} \frac{dt_e}{4\pi t_e} \frac{1}{\det M_{\Gamma}(t)}$$

exists. By Lemma 3.31, we have

$$\lim_{\epsilon \to 0} \int_{[\epsilon,L]^E} \prod_{e \in E(\Gamma)} \frac{dt_e}{4\pi t_e} \frac{1}{\det M_{\Gamma}(t)} = \lim_{\epsilon \to 0} \int_{[\epsilon,L]^E} \prod_{e \in E(\Gamma)} \frac{dt_e}{4\pi} \frac{1}{\sum_{T \in \operatorname{Tree}(\Gamma)} \prod_{e \notin T} t_e}$$

where $\operatorname{Tree}(\Gamma)$ is the set of spanning trees of Γ . Let v(1), v(2) be two vertices of Γ , $\{e_1, \dots, e_k\}$ be the set of edges that connects v(1), v(2). Let $\overline{\Gamma}$ be the graph obtained from Γ by collapsing v(1) and v(2) and all the edges e_1, \dots, e_k into one single vertex. Then $\overline{\Gamma}$ is also a connected graph without self-loops, with $E(\overline{\Gamma}) = E(\Gamma) \setminus \{e_1, \dots, e_k\}$. Obviously, for non-negative t_e 's,

$$\sum_{T \in \operatorname{Tree}(\Gamma)} \prod_{e \notin T} t_e \ge \left(\sum_{i=1}^k t_{e_1} \cdots \hat{t}_{e_i} \cdots t_{e_k} \right) \sum_{T \in \operatorname{Tree}(\bar{\Gamma})} \prod_{e \notin T} t_e$$

Therefore

$$\begin{split} \prod_{e \in E(\Gamma)} \int_{\epsilon}^{L} \frac{dt_{e}}{4\pi} \frac{1}{\sum\limits_{T \in \operatorname{Tree}(\Gamma)} \prod\limits_{e \notin T} t_{e}} &\leq \prod_{i=1}^{k} \int_{\epsilon}^{L} \frac{dt_{i}}{4\pi} \frac{1}{\sum\limits_{i=1}^{k} t_{1} \cdots \hat{t}_{i} \cdots t_{k}} \prod\limits_{e \in E(\bar{\Gamma})} \int_{\epsilon}^{L} \frac{dt_{e}}{4\pi} \frac{1}{\sum\limits_{T \in Tree(\bar{\Gamma})} \prod\limits_{e \notin T} t_{e}} \\ &\leq \prod_{i=1}^{k} \int_{\epsilon}^{L} \frac{dt_{i}}{4\pi} \frac{k}{\prod\limits_{i=1}^{k} t_{i}^{\frac{k-1}{k}}} \prod\limits_{e \in E(\bar{\Gamma})} \int_{\epsilon}^{L} \frac{dt_{e}}{4\pi} \frac{1}{\sum\limits_{T \in Tree(\bar{\Gamma})} \prod\limits_{e \notin T} t_{e}} \\ &\leq C(L) \prod_{e \in E(\bar{\Gamma})} \int_{\epsilon}^{L} \frac{dt_{e}}{4\pi} \frac{1}{\sum\limits_{T \in Tree(\bar{\Gamma})} \prod\limits_{e \notin T} t_{e}} \end{split}$$

where C(L) is a constant that depends only on L. By successive collapsing of vertices, we see that $\lim_{\epsilon \to 0} \int_{[\epsilon,L]^E} \prod_{e \in E(\Gamma)} \frac{dt_e}{4\pi t_e} \frac{1}{\det M_{\Gamma}(t)}$ exists. This proves the lemma.

Lemma 3.31. The determinant of the $(V-1) \times (V-1)$ matrix $\{M_{\Gamma}(t)_{i,j}\}_{1 \leq i,j \leq V-1}$ defined by equation (3.44) is given by

(3.45)
$$\det M_{\Gamma}(t) = \sum_{T \in \operatorname{Tree}(\Gamma)} \prod_{e \in T} \frac{1}{t_e}$$

where $\text{Tree}(\Gamma)$ is the set of spanning trees of the graph Γ .

Proof. See for example [BEK06].

Remark 3.32. A tree $T \subset \Gamma$ is said to be a spanning tree for the connected graph Γ if every vertex of Γ lies in T.

Definition 3.33. Given a connected graph Γ and two disjoint subsets of vertices $V_1, V_2 \subset V(\Gamma), V_1 \cap V_2 = \emptyset$, we define $\operatorname{Cut}(\Gamma; V_1, V_2)$ to be the set of subsets $C \subset E(\Gamma)$ satisfying the following property

- (1) The removing of the edges in C from Γ divides Γ into exactly two connected trees, which we denoted by $\Gamma_1(C), \Gamma_2(C)$, such that $V_1 \subset V(\Gamma_1(C)), V_2 \subset V(\Gamma_2(C))$.
- (2) C doesn't contain any proper subset satisfying property (1).

It's easy to see that each cut $C \in \text{Cut}(\Gamma; V_1, V_2)$ is obtained by adding one more edge to some $\{e \in E(\Gamma) | e \notin T\}$ where T is some spanning tree of Γ .

Lemma 3.34. The inverse of the matrix $M_{\Gamma}(t)$ is given by

$$M_{\Gamma}^{-1}(t)_{i,j} = \frac{1}{\mathcal{P}_{\Gamma}(t)} \sum_{C \in \operatorname{Cut}(\Gamma; \{v(i), v(j)\}, \{v_{\bullet}\})} \prod_{e \in C} t_e$$

where

$$\mathcal{P}_{\Gamma}(t) = \sum_{T \in \text{Tree}(\Gamma)} \prod_{e \notin T} t_e = \det M_{\Gamma}(t) \prod_{e \in E(\Gamma)} t_e$$

Proof. Let

$$A_{i,j} = \frac{1}{\mathcal{P}_{\Gamma}(t)} \sum_{C \in \operatorname{Cut}(\Gamma; \{v(i), v(j)\}, \{v_{\bullet}\})} \prod_{e \in C} t_e$$

For $1 \leq i \leq V - 1$, consider the summation

$$\begin{aligned} \mathcal{P}_{\Gamma}(t) \sum_{j=1}^{V-1} A_{i,j} M_{\Gamma}(t)_{j,i} &= \sum_{j=1}^{V-1} M_{\Gamma}(t)_{j,i} \sum_{C \in \operatorname{Cut}(\Gamma; \{v(i), v(j)\}, \{v_{\bullet}\})} \prod_{e \in C} t_{e} \\ &= \sum_{\substack{C \in \operatorname{Cut}(\Gamma; \{v(i)\}, \{v_{\bullet}\}) \\ v(i) \in V(\Gamma_{1}(C)), v_{\bullet} \in V(\Gamma_{2}(C))}} \prod_{e \in C} t_{e} \sum_{\substack{e' \in E(G) \\ v(j) \in \Gamma_{1}(C)}} \sum_{\substack{1 \le j \le V-1 \\ v(j) \in \Gamma_{1}(C)}} \rho_{v(i), e'} \frac{1}{t_{e'}} \rho_{v(j), e'} \\ &= \sum_{\substack{C \in \operatorname{Cut}(\Gamma; \{v(i)\}, \{v_{\bullet}\}) \\ v(i) \in V(\Gamma_{1}(C)), v_{\bullet} \in V(\Gamma_{2}(C))}} \prod_{e \notin C} t_{e} \sum_{\substack{e' \in E(G) \\ l(e) = v(i), r(e) \in V(\Gamma_{2}) \\ \text{or } r(e) = v(i), l(e) \in V(\Gamma_{2})}} \frac{1}{t_{e'}} \\ &= \sum_{T \in \operatorname{Tree}(\Gamma)} \prod_{e \notin T} t_{e} \end{aligned}$$

where in the last step, we use the fact that given $v \neq v_{\bullet}$ and a spanning tree T of Γ , there's a unique way to remove one edge in T, which is attached to v, to make a cut that separates v and v_{\bullet} . Therefore

$$\sum_{j=1}^{V-1} A_{i,j} M_{\Gamma}(t)_{j,i} = 1, \ 1 \le i \le V-1$$

Similar combinatorial interpretation leads to

$$\sum_{k=1}^{V-1} A_{i,k} M_{\Gamma}(t)_{k,j} = 0, \quad 1 \le i, j \le V_1, i \ne j$$

We leave the details to the reader. It follows that $A_{i,j}$ is the inverse matrix of $M_{\Gamma}(t)_{i,j}$. \Box

Lemma 3.35. The following sum is bounded

$$\left| \frac{\sum_{i=1}^{V-1} \rho_{v(i),e} M_{\Gamma}^{-1}(t)_{i,j}}{t_e} \right| \le 2, \quad \forall e \in E(G), 1 \le j \le V-1$$

Proof.

$$\sum_{i=1}^{V-1} \frac{\rho_{v(i),e}}{t_e} M_{\Gamma}^{-1}(t)_{i,j}$$

$$= \frac{1}{\mathcal{P}_{\Gamma}(t)} \sum_{\substack{C \in \operatorname{Cut}(\Gamma; \{v(j)\}, \{v \bullet\})\\v(j) \in V(\Gamma_1(C)), v_{\bullet} \in V(\Gamma_2(C))}} \prod_{e' \in C} t_{e'} \sum_{\substack{1 \le i \le V-1\\v(i) \in \Gamma_1(C)}} \frac{\rho_{v(i),e}}{t_e}$$

$$= \frac{1}{\mathcal{P}_{\Gamma}(t)} \sum_{C \in \operatorname{Cut}(\Gamma; \{v(j), l(e)\}, \{v_{\bullet}, r(e)\})} \frac{\prod_{e' \in C} t_{e'}}{t_e} - \frac{1}{\mathcal{P}_{\Gamma}(t)} \sum_{C \in \operatorname{Cut}(\Gamma; \{v(j), r(e)\}, \{v_{\bullet}, l(e)\})} \frac{\prod_{e' \in C} t_{e'}}{t_e}$$

Since each cut in the above summation is obtained from removing the edge e from a spanning tree containing e, the lemma follows from fact that $\mathcal{P}_{\Gamma}(t) = \sum_{T \in \text{Tree}(\Gamma)} \prod_{e \notin T} t_e$ represents the sum of the contributions from all such spanning trees.

4. QUANTUM GEOMETRY OF CALABI-YAU MANIFOLDS

In this section we discuss the quantization of BCOV theory on Calabi-Yau manifolds. In section 4.1, we introduce the classical BCOV action which generalizes the original Kodaira-Spencer action on Calabi-Yau three-folds to Calabi-Yau manifolds of arbitrary dimensions, and which also includes the gravitational descendants. In section 4.2, we discuss the general framework of constructing higher genus B-model from the perturbative quantization of the classical BCOV theory.

4.1. Classical BCOV theory. Let X be a Calabi-Yau manifold of dimension d with a fixed holomorphic volume form Ω_X and Kähler metric. We will follow the notations used in section 2: $PV_X^{*,*}$ will be the space of polyvector fields, and Ω_X induces a natural trace map of degree -2d

$$\operatorname{Tr}: \operatorname{PV}_{X}^{*,*} \to \mathbb{C}$$

The original Kodaira-Spencer gauge theory is developed in [BCOV94] to describe the B-twisted closed string field theory on Calabi-Yau three-folds. The space of fields is

$$\ker \partial = \mathbb{H} \oplus \operatorname{im} \partial \subset \mathrm{PV}_X^{*,*}$$

where \mathbb{H} is the subspace of harmonic elements with respect to the chosen metric. The Kodaira-Spencer gauge action is

$$KS[x+\mu] = \frac{1}{2} \operatorname{Tr}\left(\frac{1}{\partial}\bar{\partial}\mu\right)\mu + \frac{1}{6} \operatorname{Tr}(x+\mu)^{3}$$

where $x \in \mathbb{H}, \mu \in \text{im} \partial$. Here we have enlarged the space of fields in section 2.5 to include polyvector fields of all types, which can be viewed as the Batalin-Vilkovisky formalism of the classical gauge action (2.25) [BCOV94]. The equation of motion with respect to the variation of μ is

$$\bar{\partial}(x+\mu) + \frac{1}{2}\{x+\mu, x+\mu\} = 0$$

which describes the extended deformation space of X. However, the sheaf

$$U \to \ker \partial|_U \subset \mathrm{PV}^{*,*}_X$$

is not a sheaf of $C^{\infty}(X)$ modules, i.e., the fields are *non-local*. This non-locality of fields and also the non-locality of the Kodaira-Spencer action lead to the difficulty for its quantization.

To bypass this difficulty and generalize BCOV theory to arbitrary dimensions, we consider the derived version of ker ∂ . The operator

$$\partial: \mathrm{PV}_X^{*,*} \to \mathrm{PV}_X^{*,*}$$

is a cochain map of cohomological degree -1. ∂ can be viewed as a vector field on the infinite dimensional space $PV_X^{*,*}$, while ker ∂ is the fixed locus. The equivariant cohomology construction leads us to consider the complex

$$\mathrm{PV}_X^{*,*}[[t]]$$

with differential $\bar{\partial} - t\partial$. Here t is a formal variable of cohomological degree two. This will be our new space of fields

$$\mathcal{E} = \mathrm{PV}_X^{*,*}[[t]]$$

with a differential

$$(4.2) Q = \bar{\partial} - t\partial$$

The non-locality of the quadratic term in the Kodaira-Spencer action comes from the non-local odd symplectic pairing on im ∂

(4.3)
$$\omega(\alpha,\beta) = \langle \alpha,\beta \rangle \to \operatorname{Tr}\left(\frac{1}{\partial}\alpha\right)\beta$$

and the quadratic term can be written as

$$\frac{1}{2}\left\langle \bar{\partial}\mu,\mu\right\rangle$$

where we are in a very similar situation of section 3.4.1 except for the non-locality. However, all we need for the local odd symplectic pairing in section 3.4.1 is to define a Poisson bracket on the space of functionals. Let's recall how this is done. First, we define a map

$$\mathcal{O}_{loc}(\mathrm{PV}_X^{*,*}) \to \mathrm{Der}_{loc}(\mathrm{PV}_X^{*,*})$$

as follows. Let S be a local functional. If the odd symplectic pairing is local (coming from a fiberwise pairing on vector bundles), then we can always rewrite S in the following form

$$S(\alpha_1, \cdots, \alpha_n) = \omega (V_S(\alpha_1, \cdots, \alpha_{n-1}), \alpha_n)$$

which defines $V_S \in \text{Der}_{loc}(\text{PV}_X^{*,*})$. In the current case, although ω is non-local, the trace pairing Tr is in fact local. Therefore we can write S in terms of

$$S(\alpha_1, \cdots, \alpha_n) = \operatorname{Tr}(W_S(\alpha_1, \cdots, \alpha_{n-1}), \alpha_n)$$

for some $W_S \in \text{Der}_{loc}(\text{PV}_X^{*,*})$, then the expression in (4.3) suggests the following

Definition 4.1. The Hamiltonian vector field $V_S \in \text{Der}_{loc}(\text{PV}_X^{*,*})$ of a local functional $S \in \mathcal{O}_{loc}(\text{PV}_X^{*,*})$ is defined to be the composition

The *Poisson bracket* on the space of functionals is defined as the pairing

$$\{,\}: \mathcal{O}_{loc}(\mathrm{PV}_X^{*,*}) \otimes \mathcal{O}(\mathrm{PV}_X^{*,*}) \to \mathcal{O}(\mathrm{PV}_X^{*,*})$$
$$S_1 \otimes S_2 \to \{S_1, S_2\} = V_{S_1}(S_2)$$

The Poisson bracket defined for functionals on $\mathrm{PV}_X^{*,*}$ can be naturally extended to functionals on \mathscr{E} . Let $K_L \in \mathrm{Sym}^2(\mathrm{PV}_X^{*,*}), L > 0$, be the heat kernel of the Laplacian $H = [\bar{\partial}, \bar{\partial}^*]$, which is determined by the following equation

$$e^{-LH}\alpha = P_1 \operatorname{Tr}(P_2\alpha)$$

if we formally write $K_L = P_1 \otimes P_2$.

Definition 4.2. The regularized BV operator Δ_L for L > 0 is defined to be the second order operator

$$\Delta_L = \frac{\partial}{\partial (\partial K_L)} : \mathcal{O}(\mathscr{E}) \to \mathcal{O}(\mathscr{E})$$

where $\partial K_L \in \text{Sym}^2(\text{PV}_X^{*,*}) \subset \text{Sym}^2(\mathscr{E})$ is the kernel for the operator ∂e^{-LH} , and we have naturally identified $\text{PV}_X^{*,*}$ as a subspace of \mathscr{E} . The *regularized Batalin-Vilkovisky bracket* is defined via Δ_L by

(4.4)
$$\{S_1, S_2\}_L = \Delta_L (S_1 S_2) - (\Delta_L S_1) S_2 - (-1)^{|S_1|} S_1 \Delta_L S_2$$

for any $S_1, S_2 \in \mathcal{O}(\mathscr{E})$.

Lemma 4.3. If $S_1 \in \mathcal{O}_{loc}(\mathrm{PV}_X^{*,*}), S_2 \in \mathcal{O}(\mathrm{PV}_X^{*,*})$, then the Poisson bracket is identical to the following limit

$$\{S_1, S_2\} = \lim_{L \to 0} \{S_1, S_2\}_L$$

Definition 4.4. The *classical Poisson bracket* $\{,\}$ for functionals on \mathscr{E} is defined to be the pairing

(4.5)
$$\mathcal{O}_{loc}(\mathscr{E}) \times \mathcal{O}(\mathscr{E}) \to \mathcal{O}(\mathscr{E})$$
$$S_1 \times S_2 \to \{S_1, S_2\} = \lim_{L \to 0} \{S_1, S_2\}_L$$

A local functional $S \in \mathcal{O}_{loc}(\mathscr{E})$ satisfies the classical master equation if

(4.6)
$$QS + \frac{1}{2} \{S, S\} = 0$$

Now we are ready to define the classical action for the generalized BCOV theory.

Definition 4.5. The classical BCOV action functional $S^{BCOV} \in \mathcal{O}_{loc}(\mathscr{E})$ is defined by the Taylor coefficients

$$D_n S^{BCOV} \left(t^{k_1} \alpha_1, \cdots, t^{k_n} \alpha_n \right) = \begin{cases} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_0 \operatorname{Tr} \left(\alpha_1 \cdots \alpha_n \right) & \text{if } n \ge 3 \\ 0 & \text{if } n < 3 \end{cases}$$

where

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_0 = \int_{\overline{M}_{0,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} = \binom{n-3}{k_1, \cdots, k_n}$$

Note that the cohomological degree of $D_n S^{BCOV}$ is

$$-2(d-3) - 2n$$

Lemma 4.6. S^{BCOV} satisfies the classical master equation

$$QS^{BCOV} + \frac{1}{2} \left\{ S^{BCOV}, S^{BCOV} \right\} = 0$$

Proof. Note that $\bar{\partial}S^{BCOV} = 0$ since it's a total derivative. We have

$$(QS^{BCOV})[t^{k_1}\alpha_1, \cdots, t^{k_n}\alpha_n]$$

$$= -\sum_i \pm \langle \tau_{k_1} \cdots \tau_{k_i+1} \cdots \tau_{k_n} \rangle_0 \operatorname{Tr} \alpha_1 \cdots \partial \alpha_i \cdots \alpha_n$$

$$= \frac{1}{2}\sum_i \pm \langle \tau_{k_1} \cdots \tau_{k_i+1} \cdots \tau_{k_n} \rangle_0 \operatorname{Tr} \{\alpha_i, \alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_n\}$$

$$= \frac{1}{2}\sum_{i \neq j} \pm \langle \tau_{k_1} \cdots \tau_{k_i+1} \cdots \tau_{k_n} \rangle_0 \operatorname{Tr} \{\alpha_i, \alpha_j\} \alpha_1 \cdots \hat{\alpha}_i \cdots \hat{\alpha}_j \cdots \alpha_n$$

where we have used the formula

$$\operatorname{Tr}(\partial \alpha)\beta = -\frac{1}{2}\operatorname{Tr}\{\alpha,\beta\}$$

which follows from the BV relation $\partial(\alpha\beta) = (\partial\alpha)\beta + (-1)^{|\alpha|}\alpha\partial\beta + \{\alpha,\beta\}$ and the selfadjointness of ∂ with respect to the trace pairing. On the other hand,

$$\{S^{BCOV}, S^{BCOV}\}[t^{k_1}\alpha_1, \cdots, t^{k_n}\alpha_n]$$

$$= \sum_{I \subset \{1, \cdots, n\}} \pm \langle \tau_0 \prod_{i \in I} \tau_{k_i} \rangle_0 \langle \tau_0 \prod_{j \in I^c} \tau_{k_j} \rangle_0 \operatorname{Tr} \left(\partial \prod_{i \in I} \alpha_i\right) \prod_{j \in I^c} \alpha_j$$

$$= -\frac{1}{2} \sum_{I \subset \{1, \cdots, n\}} \pm \langle \tau_0 \prod_{i \in I} \tau_{k_i} \rangle_0 \langle \tau_0 \prod_{j \in I^c} \tau_{k_j} \rangle_0 \operatorname{Tr} \left\{ \prod_{i \in I} \alpha_i, \prod_{j \in I^c} \alpha_j \right\}$$

$$= -\frac{1}{2} \sum_{I \subset \{1, \cdots, n\}} \pm \langle \tau_0 \prod_{i \in I} \tau_{k_i} \rangle_0 \langle \tau_0 \prod_{j \in I^c} \tau_{k_j} \rangle_0 \sum_{i \in I, j \in I^c} \operatorname{Tr} \{\alpha_i, \alpha_j\} \cdots$$

$$= -\frac{1}{2(n-2)} \sum_{I \subset \{1,\cdots,n\}} \pm \langle \tau_0 \prod_{i \in I} \tau_{k_i} \rangle_0 \langle \tau_0 \prod_{j \in I^c} \tau_{k_j} \rangle_0 \sum_{i \in I, j \in I^c} \sum_{k \neq i, j} \operatorname{Tr}\{\alpha_i, \alpha_j\} \cdots$$

$$= -\frac{1}{2(n-2)} \sum_{i,j,k} \sum_{\{i,k\} \subset I, j \in I^c} \pm \langle \tau_0 \prod_{i \in I} \tau_{k_i} \rangle_0 \langle \tau_0 \prod_{j \in I^c} \tau_{k_j} \rangle_0 \operatorname{Tr}\{\alpha_i, \alpha_j\} \cdots$$

$$-\frac{1}{2(n-2)} \sum_{i,j,k} \sum_{i \in I, \{j,k\} \subset I^c} \pm \langle \tau_0 \prod_{i \in I} \tau_{k_i} \rangle_0 \langle \tau_0 \prod_{j \in I^c} \tau_{k_j} \rangle_0 \operatorname{Tr}\{\alpha_i, \alpha_j\} \cdots$$

$$= -\frac{1}{2(n-2)} \sum_{i,j,k} \pm \langle \tau_{k_1} \cdots \tau_{k_j+1} \cdots \tau_{k_n} \rangle_0 \operatorname{Tr}\{\alpha_i, \alpha_j\} \alpha_1 \cdots \hat{\alpha_i} \cdots \hat{\alpha_j} \cdots \alpha_n$$

$$-\frac{1}{2(n-2)} \sum_{i,j,k} \pm \langle \tau_{k_1} \cdots \tau_{k_i+1} \cdots \tau_{k_n} \rangle_0 \operatorname{Tr}\{\alpha_i, \alpha_j\} \alpha_1 \cdots \hat{\alpha_i} \cdots \hat{\alpha_j} \cdots \alpha_n$$

$$= -\sum_{i,j} \pm \langle \tau_{k_1} \cdots \tau_{k_i+1} \cdots \tau_{k_n} \rangle_0 \operatorname{Tr}\{\alpha_i, \alpha_j\} \alpha_1 \cdots \hat{\alpha_i} \cdots \hat{\alpha_j} \cdots \alpha_n$$

where we have used the topological recursive relations

$$<\tau_{k_1+1}\tau_{k_2}\cdots\tau_{k_n}>_0=\sum_{1\in I,\{2,3\}\subset I^c}<\tau_0\prod_{i\in I}\tau_{k_i}>_0<\tau_0\prod_{j\in I^c}\tau_{k_j}>$$

The classical master equation now follows.

4.2. Quantization and higher genus B-model.

4.2.1. Quantization of BCOV theory.

Definition 4.7. The regularized propagator of BCOV theory is defined by the kernel

(4.7)
$$P_{\epsilon}^{L} = -\int_{\epsilon}^{L} du \bar{\partial}^{*} \partial K_{u}$$

Let

$$\partial_{P^L_\epsilon}: \mathcal{O}(\mathscr{E}) \to \mathcal{O}(\mathscr{E})$$

be the operator corresponding to contracting with $P_{\epsilon}^L.$ We have

(4.8)
$$\left[Q,\partial_{P^L_{\epsilon}}\right] = \Delta_{\epsilon} - \Delta_L$$

Definition 4.8. A quantization of the BCOV theory on X is given by a family of functionals

$$\mathbf{F}[L] = \sum_{g \geq 0} \hbar^g \mathbf{F}_g[L] \in \mathscr{O}(\mathscr{E})[[\hbar]]$$

•

for each $L \in \mathbb{R}_{>0}$, with the following properties.

(1) The renormalization group flow equation

$$\mathbf{F}[L] = W\left(P(\epsilon, L), \mathbf{F}[\epsilon]\right)$$

for all $L, \epsilon > 0$. This is equivalent to

$$e^{\mathbf{F}[L]/\hbar} = e^{\hbar \frac{\partial}{\partial P_{\epsilon}^{L}}} e^{\mathbf{F}[\epsilon]/\hbar}$$

(2) The quantum master equation

$$Q\mathbf{F}[L] + \hbar \Delta_L \mathbf{F}[L] + \frac{1}{2} \{\mathbf{F}[L], \mathbf{F}[L]\}_L = 0, \quad \forall L > 0$$

- (3) The locality axiom, as in [Cos11]. This says that $\mathbf{F}[L]$ has a small L asymptotic expansion in terms of local functionals.
- (4) The classical limit condition

$$\lim_{L \to 0} \mathbf{F}_0[L] = S^{BCOV}$$

(5) Degree axiom. The functional $\mathcal{D}_n \mathbf{F}_g$ is of cohomological degree

$$(\dim X - 3)(2g - 2) - 2n$$

(6) We will give \$\mathcal{E}(X) = PV_X^{*,*}[[t]]\$ an additional grading, which we call Hodge weight, by saying that elements in

$$t^m \Omega^{0,*}(\wedge^k TX) = \mathrm{PV}^{k,*}(X)$$

have Hodge weight k + m - 1. We will let $HW(\alpha)$ denote the Hodge weight of an element $\alpha \in \mathscr{E}$.

Then, the functional \mathbf{F}_g must be of Hodge weight

$$(3 - \dim X)(g - 1)$$

(7) The dilaton axiom. Let

$$Eu: \mathcal{O}(\mathscr{E}) \to \mathcal{O}(\mathscr{E})$$

be the Euler vector field, defined by

$$Eu \ \Phi = n\Phi$$

if $\Phi \in \mathcal{O}^{(n)}(\mathscr{E})$. Let $1 \cdot t \in t \operatorname{PV}_X^{*,*}$, which associates a derivation $\frac{\partial}{\partial(1 \cdot t)} \in \operatorname{Der}(\mathscr{E})$. Let's define the dilaton vector field \mathcal{D} by

$$\mathcal{D} = Eu - \frac{\partial}{\partial(1 \cdot t)}$$

Then the dilaton axiom asserts that there exists $\mathbf{G}[L] \in \hbar \mathcal{O}(\mathscr{E})[[\hbar]]$ such that

$$\left(\mathcal{D}+2\left(\hbar\frac{\partial}{\partial\hbar}-1\right)\right)\mathbf{F}[L]=Q\mathbf{G}[L]+\{\mathbf{F}[L],\mathbf{G}[L]\}_{L}+\hbar\Delta_{L}\mathbf{G}[L]$$

This is equivalent to the following equation

$$\left(Q + \hbar\Delta_L + \delta\left(\mathcal{D} + 2\hbar\frac{\partial}{\partial\hbar}\right)\right)e^{\mathbf{F}[L]/\hbar + \delta\mathbf{G}[L]/\hbar} = 0$$

where δ is an odd variable with $\delta^2 = 0$. Moreover, we require the following renormalization group flow equation

$$e^{\mathbf{F}[L]/\hbar + \delta \mathbf{G}[L]/\hbar} = e^{\hbar \frac{\partial}{\partial P_{\epsilon}^{L}}} e^{\mathbf{F}[\epsilon]/\hbar + \delta \mathbf{G}[\epsilon]/\hbar}$$

(8) The string equation axiom. Let

$$T_{(-1)}:\mathscr{E}\to\mathscr{E}$$

be the operator defined by

$$T_{(-1)}(t^{k}\mu) = \begin{cases} t^{k-1}\mu & \text{if } k > 0\\ 0 & \text{if } k = 0 \end{cases}$$

We define the following operator Y[L] on \mathscr{E} depending on the scale L

$$Y[L][\alpha] = \begin{cases} \int_0^L du \bar{\partial}^* \partial e^{-uH} \alpha & \alpha \in \mathrm{PV}_X^{*,*} \\ 0 & \alpha \in t \, \mathrm{PV}_X^{*,*}[[t]] \end{cases}$$

Both $T_{(-1)}$ and Y[L] induce a derivation on $\mathcal{O}(\mathscr{E})$, which we still denote by the same symbols. Let $\mathrm{Tr} \in \mathrm{Sym}^2(\mathscr{E}^{\vee})$ denote the Trace operator. We define the string operator $\mathcal{S}[L]$ by

$$\mathcal{S}[L] = T_{(-1)} - \frac{\partial}{\partial(1)} - Y[L] + \frac{1}{\hbar} \operatorname{Tr}$$

Then the string equation axiom asserts that there exists $\mathbf{K}[L] \in \hbar \mathcal{O}(\mathscr{E})[[\hbar]]$ such that

$$(Q + \hbar \Delta_L + \delta \mathcal{S}[L]) e^{\mathbf{F}[L]/\hbar + \delta \mathbf{K}[L]/\hbar} = 0$$

where δ is an odd variable with $\delta^2 = 0$. Moreover, we require the following renormalization group flow equation

$$e^{\mathbf{F}[L]/\hbar + \delta \mathbf{K}[L]/\hbar} = e^{\hbar \frac{\partial}{\partial P_{\epsilon}^{L}}} e^{\mathbf{F}[\epsilon]/\hbar + \delta \mathbf{K}[\epsilon]/\hbar}$$

Remark 4.9. The reason for the string operator taking the above form is that S[L] is compatible with renormalization group flow equation and quantum master equation in the following sense

$$S[L]e^{\hbar\frac{\partial}{\partial P_{\epsilon}^{L}}} = e^{\hbar\frac{\partial}{\partial P_{\epsilon}^{L}}}S[\epsilon]$$
$$[S[L], (Q + \hbar\Delta_{L})] = 0$$

All the above properties of $\mathbf{F}[L]$ are motivated by mirror symmetry and modeled on the corresponding Gromov-Witten theory on the A-side. This will be discussed in more detail in the next section. The main goal for the quantum BCOV theory is to find $\mathbf{F}[L]$ satisfying the above properties on Calabi-Yau manifolds. We will prove in the next chapter that in the case of elliptic curves, such quantization exists and is also unique up to homotopy.

Let's assume that X is a compact Calabi-Yau manifold and we have already found such a quantization $\mathbf{F}[L]$ of BCOV theory on X. Since X is compact, the following kernel

$$P_L^{\infty} = -\int_L^{\infty} du \bar{\partial}^* \partial K_u$$

is in fact a smooth kernel. This allows us to take the following limit

$$\mathbf{F}[\infty] = \lim_{L \to \infty} \mathbf{F}[L] \in \mathcal{O}(\mathscr{E})[[\hbar]]$$

Observe that $\lim_{L\to\infty} K_L$ is the projection to harmonic parts, hence

$$\lim_{L \to \infty} \partial K_L = 0$$

The quantum master equation at $L = \infty$ then says that

which implies that we have an induced map on Q-cohomology

$$(4.10) D_n \mathbf{F}_g[\infty] : \operatorname{Sym}^n_{\mathbb{C}} \left(\operatorname{H}^*(\mathscr{E}, Q) \right) \to \mathbb{C}$$

where $\mathrm{H}^*(\mathscr{E}, Q)$ is the cohomology of the complex \mathscr{E} with respect to Q.

Lemma 4.10. Given a Kähler metric on X, we have a natural isomorphism

$$\mathrm{H}^*(X, \wedge^* T_X)[[t]] \cong \mathrm{H}^*(X, \wedge^* T_X)[[t]] \cong \mathrm{H}^*(\mathscr{E}, Q)$$

where $H^*(X, \wedge^*T_X)$ is the sheaf cohomology of \wedge^*T_X on X, and $\mathbb{H}^*(X, \wedge^*T_X)[[t]]$ is the space of Harmonic polyvector fields.

Definition 4.11. Given a quantization $\mathbf{F}[L]$ of BCOV theory on X, the associated B-model correlation functions $\mathbf{F}_{g,n,X}^B$ are defined by the commutative diagram

$$\begin{aligned} \mathbf{F}_{g,n,X}^{B} &: \operatorname{Sym}_{\mathbb{C}}^{n} \left(\operatorname{H}^{*} \left(X, \wedge^{*} T_{X} \right) \left[[t] \right] \right) \longrightarrow \mathbb{C} \\ & \cong \bigvee \\ D_{n} \mathbf{F}_{g} [\infty] &: \operatorname{Sym}_{\mathbb{C}}^{n} \left(\operatorname{H}^{*} (\mathscr{E}, Q) \right) \longrightarrow \mathbb{C} \end{aligned}$$

Lemma 4.12. $\mathbf{F}_{g,n,X}^B$ satisfies the following dilaton equation

$$\mathbf{F}_{g,n+1,X}^{B}\left[t,t^{k_{1}}\mu_{1},\cdots,t^{k_{n}}\mu_{n}\right] = (2g-2+n)\mathbf{F}_{g,n,X}^{B}\left[t^{k_{1}}\mu_{1},\cdots,t^{k_{n}}\mu_{n}\right] \quad \forall g,n$$

and the string equation

$$\mathbf{F}_{g,n+1,X}^{B}\left[1, t^{k_{1}}\mu_{1}, \cdots, t^{k_{n}}\mu_{n}\right] = \sum_{i} \mathbf{F}_{g,n,X}^{B}\left[t^{k_{1}}\mu_{1}, \cdots, t^{k_{i}-1}\mu_{i}, \cdots, t^{k_{n}}\mu_{n}\right], \quad \forall 2g+n \ge 3$$

for any $\mu_{i} \in \mathbb{H}^{*}(X, \wedge^{*}T_{X}).$

Proof. The dilaton axiom at $L \to \infty$ says that

$$\left(\mathcal{D}+2\left(\hbar\frac{\partial}{\partial\hbar}-1\right)\right)\mathbf{F}[\infty]=Q\mathbf{G}[\infty]$$

Therefore $\mathcal{D}\mathbf{F}[\infty] = 0$ if we restrict to *Q*-cohomology classes. This proves the dilaton equation. The proof of string equation is similar.

4.2.2. Higher genus mirror symmetry. Let X and X^{\vee} be mirror Calabi-Yau manifolds. The mirror symmetry says that the A-model topological string correlation functions on X are equivalent to B-model topological string correlation functions on X^{\vee} . It's long been known that the A-model correlation functions are given by the Gromov-Witten invariants, and it's proposed in [BCOV94] that B-model correlation functions could be defined via Kodaira-Spencer gauge theory. The formulation of $\mathbf{F}_{g,n,X^{\vee}}^B$ serves for this purpose. Let τ be local coordinates on the moduli space of complex structures of X^{\vee} around the large complex limit, and we use X_{τ}^{\vee} to denote the corresponding Calabi-Yau manifold. Let q be the complexified Kähler moduli on X around the large volume limit. The physics statement of mirror symmetry predicts a mirror map

$$\tau \to q = q(\tau)$$

and an isomorphism of cohomology classes

$$\Phi: \mathrm{H}^*\left(\wedge^* T_X^*\right) \to \mathrm{H}^*\left(\wedge^* T_{X_{\tau}^{\vee}}\right)$$

such that

$$\mathbf{F}_{g,n,X;q}^{A}\left[t^{k_{1}}\alpha_{1},\cdots,t^{k_{n}}\alpha_{n}\right] = \lim_{\bar{\tau}\to\infty}\mathbf{F}_{g,n,X_{\tau}^{\vee}}^{B}\left[t^{k_{1}}\Phi(\alpha_{1}),\cdots,t^{k_{n}}\Phi(\alpha_{n})\right]$$

On the left hand side we have the generating function from the Gromov-Witten theory in the A-model (see Definition 1.2). On the right hand side, there exists certain mysterious $\bar{\tau} \to \infty$ limit that we could be able to take around the large complex limit of X^{\vee} predicted in [BCOV94]. This anti-holomorphic dependence can be understood as a choice of complex conjugate splitting filtration for the Hodge filtration on polyvector fields. We refer to [CL] for the more precise description. This generalizes the well-established genus zero mirror symmetry to higher genus case, with all descendants included.

In section 6, we will prove this mirror symmetry statement for one-dimensional Calabi-Yau manifolds, i.e., elliptic curves. The $\bar{\tau} \to \infty$ limit in this case turns out to be the well-known map from almost holomorphic modular forms to quasi modular forms. It would be extremely interesting to understand the higher dimensional cases in the future. We will construct the quantization $\mathbf{F}[L]$ of BCOV theory on elliptic curves in this section. We will show that there exits a unique quantization $\mathbf{F}[L]$ satisfying dilaton axioms, and $\mathbf{F}[L]$ satisfies a set of Virasoro equations.

5.1. Deformation-obstruction complex.

5.1.1. Translation invariant deformation-obstruction complex. Let E be the elliptic curve

$$E = \mathbb{C}/\left(\mathbb{Z} \oplus \mathbb{Z}\tau\right)$$

where we will fix the complex moduli τ . The space of fields of BCOV theory is

$$\mathscr{E}_E = \mathrm{PV}_E^{*,*}[[t]]$$

Let $J(\mathscr{E}_E)$ be the D_E module of smooth jets of polyvector fields valued in formal power series $\mathbb{C}[[t]]$. By Proposition 3.28, the deformation obstruction complex for the BCOV theory on elliptic curves is given by

$$\Omega^*_E\left(C^*_{red}\left(J(\mathscr{E}_E)[-1]\right)\right)[2]$$

We would like to consider the functionals which are translation invariant. This allows us to consider the following L_{∞} subalgebra of $J(\mathscr{E}_E)[-1]$

$$\mathfrak{g} = J(\mathscr{E}_E)^E[-1] \subset J(\mathscr{E}_E)[-1]$$

where $J(\mathscr{E}_E)^E$ denotes the translation invariant polyvector fields. Let z be the linear coordinate on the universal cover \mathbb{C} of E, then

$$J(\mathscr{E}_E)^E = \mathbb{C}[[z, \bar{z}]][d\bar{z}, \partial_z][[t]]$$

where $d\bar{z} \in PV_E^{0,1}$, $\partial_z \in PV_E^{1,0}$ are the translation invariant polyvector fields on E, and z, \bar{z} represents the jet coordinates. Let

$$D = \mathbb{C}\left[\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right]$$

be the subspace of translation invariant differential operators of D_E . $J(\mathscr{E}_E)^E$ has a naturally induced *D*-module structure. Then the space of translation invariant local functionals on \mathscr{E}_E modulo constants is given by

$$\mathbb{C} \otimes_D \prod_{k>0} \operatorname{Hom}\left(\operatorname{Sym}^k_{\mathbb{C}}\left(J(\mathscr{E}_E)^E\right), \mathbb{C}\right)$$

where \mathbb{C} has the *D*-module structure such that $\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}$ act trivially. Let

$$\Omega^* = \mathbb{C}\left[dz, d\bar{z}\right]$$

be the translation invariant differential forms on E. The Koszul resolution gives the quasiisomorphism of complexes of D-modules

$$\mathbb{C} \cong \Omega^*[2] \otimes_{\mathbb{C}} D$$

Therefore the deformation obstruction complex for the translation invariant theory is quasiisomorphic to the de Rham complex of D-module $C^*_{red}(\mathfrak{g})$

$$\Omega^*\left(C^*_{red}\left(\mathfrak{g}\right)\right)[2]$$

Lemma 5.1. The natural inclusion of translation invariant deformation obstruction complex into the full deformation obstruction complex

$$\Omega^*\left(C^*_{red}\left(J(\mathscr{E}_E)^E[-1]\right)\right)[2] \hookrightarrow \Omega^*_E\left(C^*_{red}\left(J(\mathscr{E}_E)[-1]\right)\right)[2]$$

is quasi-isomorphc.

Proof. The D_E L_{∞} algebra $J(\mathscr{E}_E)[-1]$ is explicitly given by

$$J(\mathscr{E}_E)[-1] = C^{\infty}(E)[[z,\bar{z}]][d\bar{z},\partial_z][[t]][-1]$$

with differential $Q = \bar{\partial} - t\partial$. By considering the $\bar{\partial}$ cohomology, we see that there's a quasi-isomorphism

$$J(\mathscr{E}_E)[-1] \cong C^{\infty}(E)[[z,t]][\partial_z][-1]$$

Similarly, we have

$$J(\mathscr{E}_E)^E[-1] \cong \mathbb{C}[[z,t]][\partial_z][-1]$$

Let $D^{hol} = \mathbb{C}[\frac{\partial}{\partial z}]$ be the translation invariant holomorphic differential operators, and $\Omega^{hol,*}(\mathbb{C}[[z,t]][\partial_z][-1])$ denote the holomorphic de Rham complex of the D^{hol} -module $\mathbb{C}[[z,t]][\partial_z]$. Then it's easy to see that

$$\Omega^* \left(C^*_{red} \left(\mathbb{C}[[z,t]][\partial_z][-1] \right) \right) = \mathbb{C}[d\bar{z}] \otimes_{\mathbb{C}} \Omega^{hol,*} \left(C^*_{red} \left(\mathbb{C}[[z,t]][\partial_z][-1] \right) \right)$$

and

$$\Omega^*_E\left(C^*_{red}\left(C^{\infty}(E)[[z,t]][\partial_z][-1]\right)\right) = C^{\infty}(E)[d\bar{z}] \otimes_{\mathbb{C}} \Omega^{hol,*}\left(C^*_{red}\left(\mathbb{C}[[z,t]][\partial_z][-1]\right)\right)$$

Since $H^*(C^{\infty}(E)[d\bar{z}], \bar{\partial}) = \mathbb{C}[d\bar{z}]$, we find the quasi-isomorphism

$$\Omega^* \left(C^*_{red} \left(\mathbb{C}[[z,t]][\partial_z][-1] \right) \right) \hookrightarrow \Omega^*_E \left(C^*_{red} \left(C^{\infty}(E)[[z,t]][\partial_z][-1] \right) \right)$$

and the lemma follows.

5.1.2. Modified degree assignment. We will modify the degree assignment in \mathscr{E}_E as follows

$$\deg\left(d\bar{z}\right) = 1, \deg\left(\partial_z\right) = -1, \deg(t) = 0$$

and recall that the Hodge weight is defined by

$$\mathrm{HW}(t^k d\bar{z}^m \partial_z^n) = k + n - 1$$

Lemma 5.2. With the modified degree assignment as above, we have

$$\deg Q = 1, \deg \left(V_{S^{BCOV}} \right) = 1$$

and the degree axiom and Hodge weight axiom of $\mathbf{F}_{g}[L]$ on the elliptic curve is equivalent to

$$\deg\left(\mathbf{F}_{g}[L]\right) = 0, \quad \mathrm{HW}\left(\mathbf{F}_{g}[L]\right) = 2 - 2g$$

Lemma 5.3. The tangent space of translation invariant quantization of BCOV theory at genus g is given by

$$\mathrm{H}^{0}_{2-2g}(\Omega^{*}\left(C^{*}_{red}\left(\mathfrak{g}\right)\right)[2])$$

and the obstruction at genus g lies in

$$\mathrm{H}^{1}_{2-2q}(\Omega^{*}\left(C^{*}_{red}\left(\mathfrak{g}\right)\right)[2])$$

Here the subscript 2 - 2g means that we take the homogeneous degree 2 - 2g part of the Hodge weight.

We will use this modified degree assignment throughout this section, which is equivalent to the original cohomology degree and Hodge degree assignments, but more convenient in identifying the tangent space and the obstruction space for the quantization of BCOV theory.

5.1.3. Coupling to dilaton equation. Recall that the dilaton vector field is given by

$$\mathcal{D} = Eu - \frac{\partial}{\partial (1 \cdot t)}$$

The dilaton axiom is equivalent to the following modified quantum master equation

(5.1)
$$\left(Q + \hbar \Delta_L + \delta \left(\mathcal{D} + 2\hbar \frac{\partial}{\partial \hbar}\right)\right) e^{\mathbf{F}[L]/\hbar + \delta \mathbf{G}[L]/\hbar} = 0$$

where δ is an odd variable of cohomological degree one.

Lemma 5.4.

$$\left(Q + \hbar\Delta_L + \delta\left(\mathcal{D} + 2\hbar\frac{\partial}{\partial\hbar}\right)\right)^2 = 0$$

Lemma 5.5. The homotopic dilaton equation is compatible with renormalization group flow, i.e.,

$$\left(Q + \hbar\Delta_L + \delta\left(\mathcal{D} + 2\hbar\frac{\partial}{\partial\hbar}\right)\right)e^{\hbar\partial_{P_{\epsilon}^L}} = e^{\hbar\partial_{P_{\epsilon}^L}}\left(Q + \hbar\Delta_{\epsilon} + \delta\left(\mathcal{D} + 2\hbar\frac{\partial}{\partial\hbar}\right)\right)$$

Proof. This follows from

$$\left[\mathcal{D}+2\hbar\frac{\partial}{\partial\hbar},\hbar\partial_{P_{\epsilon}^{L}}\right] = \left[Eu,\hbar\partial_{P_{\epsilon}^{L}}\right] + 2\hbar\partial_{P_{\epsilon}^{L}} = 0$$

In order to incorporate the dilaton axiom, we will enlarge the space of functionals to

$$\mathcal{O}\left(\mathscr{E}\right)\otimes\mathbb{C}[\delta]$$

by adding the odd variable δ . The dilaton axiom is equivalent to find $\mathbf{F}[L] + \delta \mathbf{G}[L] \in \mathcal{O}(\mathscr{E})[[\hbar]] \otimes \mathbb{C}[\delta]$ which satisfies the renormalization group flow equation, the modified quantum master equation (5.1) and the classical limit condition

$$\lim_{L \to 0} \mathbf{F}[L] + \delta \mathbf{G}[L] = S^{BCOV} \mod \hbar$$

The obstruction theory is also modified correspondingly. Suppose that we have constructed $\mathbf{F}[L] + \delta \mathbf{G}[L] \in \mathcal{O}(\mathscr{E})[[\hbar]] \otimes \mathbb{C}[\delta]$ which satisfies the modified quantum master equation up to genus \hbar^{g-1} , i.e.,

$$\left(Q + \hbar\Delta_L + \delta\left(\mathcal{D} + 2\hbar\frac{\partial}{\partial\hbar}\right)\right)e^{\mathbf{F}[L]/\hbar + \delta\mathbf{G}[L]/\hbar} = \frac{1}{\hbar}O[L]e^{\mathbf{F}[L]/\hbar + \delta\mathbf{G}[L]/\hbar}$$

where

$$O[L] = \hbar^g O_g[L] \mod \hbar^{g+1}$$

The renormalization group flow equation implies that

$$O[L]e^{\mathbf{F}[L]/\hbar + \delta \mathbf{G}[L]/\hbar} = e^{\hbar \partial_{P_{\epsilon}} D} O[\epsilon] e^{\mathbf{F}[\epsilon]/\hbar + \delta \mathbf{G}[\epsilon]/\hbar}$$

which is equivalent to

$$e^{\mathbf{F}[L]/\hbar + \delta \mathbf{G}[L]/\hbar + \eta O[L]/\hbar^{g+1}} = e^{\hbar \partial_{P_{\epsilon}^{L}}} e^{\mathbf{F}[\epsilon]/\hbar + \delta \mathbf{G}[\epsilon]/\hbar + \eta O[\epsilon]/\hbar^{g+1}}$$

where η is an odd variable, $\eta^2 = 0$. This implies as before that $\mathbf{F}_0[L] + \eta O[L]$ satisfies the classical master equation, hence

$$O_g = \lim_{L \to 0} O_g[L] \in \mathcal{O}_{loc}(\mathscr{E}) \otimes \mathbb{C}[\delta]$$

exists as a local functional. On the other hand,

$$\left(Q + \hbar\Delta_L + \delta\left(\mathcal{D} + 2\hbar\frac{\partial}{\partial\hbar}\right)\right) \left(\frac{1}{\hbar}O[L]e^{\mathbf{F}[L]/\hbar + \delta\mathbf{G}[L]/\hbar}\right) = 0$$

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which is equivalent to

$$QO[L] + \hbar\Delta_L O[L] + \delta \left(\mathcal{D} + 2\hbar \frac{\partial}{\partial \hbar} - 2 \right) O[L] + \{\mathbf{F}[L] + \delta \mathbf{G}[L], O[L]\}_L = 0$$

If we pick up the leading power of \hbar , we find

$$QO_g[L] + \delta (\mathcal{D} + 2g - 2) O_g[L] + \{ \mathbf{F}_0[L], O_g[L] \}_L = 0$$

Taking the limit $L \to 0$, we see that the obstruction class O_g satisfies

(5.2)
$$QO_g + \delta \left(\mathcal{D} + 2g - 2 \right) O_g + \{ S^{BCOV}, O_g \} = 0$$

Then the slight modification of the discussion in subsection 3.4.5 leads to

Proposition 5.6. The obstruction space for extending a quantization of BCOV theory at genus g - 1 to genus g which satisfies the dilaton axiom lies in the cohomology class

$$\mathrm{H}^{1}_{2-2g}\left(\mathcal{O}_{loc}(\mathscr{E})\otimes\mathbb{C}[\delta],Q+\delta\left(\mathcal{D}+2g-2\right)+\{S^{BCOV},-\}\right)$$

If the obstruction class is zero, then the space of isomorphic classes of extensions is a torsor under

$$\mathrm{H}^{0}_{2-2g}\left(\mathcal{O}_{loc}(\mathscr{E})\otimes\mathbb{C}[\delta],Q+\delta\left(\mathcal{D}+2g-2\right)+\{S^{BCOV},-\}\right)$$

Here the subscript 2 - 2g denotes the Hodge weight.

5.2. Uniqueness of the quantization. We consider the translation invariant quantization of BCOV theory on E which satisfies the dilaton axiom. The relevant deformation obstruction complex is

(5.3)
$$\Omega^* \left(C^*_{red} \left(\mathfrak{g} \right) \right) [\delta][2]$$

where δ is an odd variable of cohomological degree one which arises from coupling to dilaton equation. Here \mathfrak{g} is the L_{∞} algebra

$$\mathfrak{g} = J(\mathscr{E})^E[-1] = \mathbb{C}[[z, \overline{z}, d\overline{z}, \partial_z]][[t]][-1]$$

where the L_{∞} structure is induced from $Q + \{S^{BCOV}, -\}$.

We consider the quantization at genus g. We focus on the complex $C_{red}^*(\mathfrak{g})[\delta]$ and its cohomology, from which the cohomology of the total deformation obstruction complex can be computed via spectral sequence. The differential for $C_{red}^*(\mathfrak{g})[\delta]$ is given by

$$Q + \delta \left(Eu - \frac{\partial}{\partial (1 \cdot t)} + 2g - 2 \right) + \{ S^{BCOV}, - \}$$

To compute its cohomology, we first observe that $Q = \bar{\partial} - t\partial$, while $\bar{\partial}$ is the only operator which increases the number of $d\bar{z}$ in \mathfrak{g} . By considering the filtration on the number of $d\bar{z}$, it allows us to first take the cohomology with respect to $\bar{\partial}$. By Poincare lemma, this just simplies \mathfrak{g} by

$$\mathrm{H}^*(\mathfrak{g},\bar{\partial}) = \mathbb{C}[[z,\partial_z]][[t]][-1]$$

with the L_{∞} structure given by $-t\partial + \delta \left(Eu - \frac{\partial}{\partial(1\cdot t)} + 2g - 2\right) + \{S^{BCOV}, -\}$. Consider the following filtration

$$F^{k}C^{*}_{red}\left(\mathrm{H}^{*}(\mathfrak{g},\bar{\partial})\right)[\delta] = C^{\geq k}_{red}\left(\mathrm{H}^{*}(\mathfrak{g},\bar{\partial})\right) \oplus \delta C^{\geq k-1}_{red}\left(\mathrm{H}^{*}(\mathfrak{g},\bar{\partial})\right)$$

and the associated spectral sequence. Here

$$C_{red}^{\geq k}\left(\mathbf{H}^{*}(\mathfrak{g},\bar{\partial})\right) = \prod_{n\geq k} C_{red}^{n}\left(\mathbf{H}^{*}(\mathfrak{g},\bar{\partial})\right)$$

On the graded complex $Gr^*\left(C^*_{red}\left(\mathrm{H}^*(\mathfrak{g},\bar{\partial})\right)[\delta]\right)$, the differential is given by

$$d_0 = -t\partial - \delta \frac{\partial}{\partial (1 \cdot t)}$$

Lemma 5.7. The E_1 -term of the spectral sequence is given by

$$E_1 = \mathrm{H}^*\left(C_{red}^*\left(\mathrm{H}^*(\mathfrak{g},\bar{\partial})\right)[\delta], d_0\right) = C_{red}^*\left(\left(\mathbb{C}[[z]] \oplus \mathbb{C}[[t]]\partial_z \oplus \mathbb{C}z\partial_z\right)[-1]\right)$$

Proof. First of all we observe that the map $\delta \frac{\partial}{\partial(1 \cdot t)}$ is surjective, and the kernel is given by

$$C_{red}^* \left(\mathrm{H}^*(\mathfrak{g}, \bar{\partial}) / \mathbb{C}1 \cdot t \right)$$

with differential $-t\partial$. The lemma follows from the simple calculation that

$$\mathrm{H}^*\left(\mathrm{H}^*(\mathfrak{g},\bar{\partial})/\mathbb{C}1\cdot t,t\partial\right)\cong\left(\mathbb{C}[[z]]\oplus\mathbb{C}[[t]]\partial_z\oplus\mathbb{C}z\partial_z\right)\left[-1\right]$$

Now we consider the differential d_1 on E_1 . There are two contributions: the first one comes from $\{S_3^{BCOV}, -\}$ and the second one is induced from $\delta(Eu + 2g - 2)$. The $\{S_3^{BCOV}, -\}$ gives rise to the L_{∞} product

(5.4)
$$l_2\left(z^k, z\partial_z\right) = (k+1)z^k, \quad l_2\left(z^k, \partial_z\right) = kz^{k-1}$$

Claim. The operator δ is transgressed to the following element in $(\mathbb{C}[[z]] \oplus \mathbb{C}[[t]]\partial_z \oplus \mathbb{C}z\partial_z)^{\vee}$ on E_1

$$\delta(z\partial_z) = -1, \quad \delta(z^k) = 0, \quad \delta(t^k\partial_z) = 0$$

Proof. In fact, if we let t^{\vee} be the dual of $1 \cdot t$ such that $\frac{\partial}{\partial (1 \cdot t)} t^{\vee} = 1$, then

$$\delta \Phi = -d_0 \left(t^{\vee} \cdot \Phi \right) - (t\partial) \left(t^{\vee} \cdot \Phi \right) = -(t\partial)(t^{\vee}) \cdot \Phi \mod \operatorname{im} d_0$$

for any $\Phi \in C^*_{red}$ (($\mathbb{C}[[z]] \oplus \mathbb{C}[[t]] \partial_z \oplus \mathbb{C}z \partial_z$) [-1]), and $(t\partial)(t^{\vee})$ is precisely the dual of $z\partial_z$. This proves the claim.

We will assign the following rescaling degree, which we call *scaling weight*, by

(5.5)
$$\operatorname{SW}(z^k) = k, \operatorname{SW}(t^k \partial_z) = -1$$

which naturally induces a grading on E_1 by duality.

Lemma 5.8. (E_1, d_1) is quasi-isomorphic to the complex

$$C_{red}^*\left(\left(\mathbb{C}[[z]]\oplus\mathbb{C}[[t]]\partial_z\right)[-1]\right)_{2-2g}\otimes\mathbb{C}[\delta]$$

where the subscript 2-2g indicates the scalig weight.

Proof. It follows directly from (5.4) and the above claim.

Recall that we have another grading given by the Hodge weight

$$\mathrm{HW}(z^k) = -1, \ \mathrm{HW}(t^k \partial_z) = k$$

It follows that the tangent space and obstruction space for quantization at genus g will have scaling weight 2 - 2g and Hodge weight 2 - 2g. Let

$$\mathfrak{g}^{hol} = (\mathbb{C}[[z]] \oplus \mathbb{C}[[t]]\partial_z) [-1]$$

with the L_{∞} structure induced from $\{S^{BCOV}, -\}$ as follows

$$l_n\left(z^{k_1},\cdots,z^{k_{n-1}},t^{n-2}\partial_z\right) = \left(\sum k_i\right)z^{\sum k_i-1}, \quad n \ge 2$$

Then the relevant deformation obstruction complex is

$$C^*_{red} \left(\mathfrak{g}^{hol} \right)_{2-2g,2-2g}$$

where the subscript refers to the scaling weight and Hodge weight.

Lemma 5.9. Let $H_{2-2g,2-2g}^*(\mathfrak{g}^{hol})$ be the Lie algebra cohomology of \mathfrak{g}^{hol} with scaling weight 2-2g and Hodge weight 2-2g, then

$$\mathbf{H}_{2-2g,2-2g}^{k}\left(\mathfrak{g}^{hol}\right) = 0 \quad if \quad k \le 2, g > 0$$

Proof. We will let e_k, η_k be the dual of $z^k, t^k \partial_z$. Then the Chevalley-Eilenberg complex of \mathfrak{g}^{hol} is

$$C^*\left(\mathfrak{g}^{hol}\right) = \mathbb{C}[e_k,\eta_k]$$

with the differential given by

$$D = \sum_{n \ge 2} D_n$$

where

$$D_{n} = \sum_{k_{1}, \cdots, k_{n}} \left(\sum k_{i} \right) \left(\eta_{n-2} \prod e_{k_{i}} \right) \frac{\partial}{\partial e_{\sum k_{i}-1}}$$

We consider the spectral sequence with respect to the filtration given by the number of e_k 's. The first differential is given by

$$d_0 = D_2 = \sum_{k \ge 1} k \eta_0 e_k \frac{\partial}{\partial e_{k-1}}$$

It's easy to see that the d_0 cohomology gives the basis of E_1 term

$$E_1 = \eta_0 \mathbb{C}[e_0, \eta_1, \eta_2, \cdots] \oplus \bigoplus_{k>0} \eta_0 e_k^2 \mathbb{C}[e_0, \cdots, e_k, \eta_1, \eta_2, \cdots]$$

and the differential d_1 on E_1 is induced by

$$D_{3} = \sum_{k_{1},k_{2}} (k_{1}+k_{2})\eta_{1}e_{k_{1}}e_{k_{2}}\frac{\partial}{\partial e_{k_{1}+k_{2}-1}}$$

$$= 2\sum_{k>0} k\eta_{1}e_{0}e_{k}\frac{\partial}{\partial e_{k-1}} + \sum_{k_{1},k_{2}>0} (k_{1}+k_{2})\eta_{1}e_{k_{1}}e_{k_{2}}\frac{\partial}{\partial e_{k_{1}+k_{2}-1}}$$

$$= 2\sum_{k>0} k\delta_{1}e_{k}\frac{\partial}{\partial e_{k-1}}e_{0} - 2\eta_{1}e_{1} + \sum_{k_{1},k_{2}>0} (k_{1}+k_{2})\eta_{1}e_{k_{1}}e_{k_{2}}\frac{\partial}{\partial e_{k_{1}+k_{2}-1}}$$

The first term acting on E_1 will produce d_0 -exact terms, hence zero. The other terms preserves the basis of E_1 . Therefore

$$d_{1} = -2\eta_{1}e_{1} + \sum_{k_{1},k_{2}>0} (k_{1}+k_{2})\eta_{1}e_{k_{1}}e_{k_{2}}\frac{\partial}{\partial e_{k_{1}+k_{2}-1}}$$
$$= -2\eta_{1}e_{1} + 2\sum_{k>1} (k+1)\eta_{1}e_{1}e_{k}\frac{\partial}{\partial e_{k}} + \sum_{k_{1},k_{2}>1} (k_{1}+k_{2})\eta_{1}e_{k_{1}}e_{k_{2}}\frac{\partial}{\partial e_{k_{1}+k_{2}-1}}$$

By taking the filtration on the number of e_1 's, we find that the d_1 -cohomology has a basis given by

$$E_2 = \eta_0 \mathbb{C}[e_0, \eta_1, \eta_2, \cdots] \oplus \bigoplus_{k>0} \eta_0 \eta_1 e_k^2 \mathbb{C}[e_0, e_2, \cdots, e_k, \eta_2, \eta_3, \cdots]$$

Note that the scaling weight and Hodge weight for η_k and e_k are given by

$$SW(e_k) = -k \quad SW(\eta_k) = 1$$
$$HW(e_k) = 1 \quad HW(\eta_k) = -k$$

and the cohomology degree are

$$\deg e_k = 0, \quad \deg \eta_k = 1$$

Elements in $\eta_0 \mathbb{C}[e_0, \eta_1, \eta_2, \cdots]$ has positive scaling weight, hence doesn't contribute. For elements in cohomology degree 2, given by the form $\eta_0 \eta_1 e_k^2 f(e_0, e_2, \cdots, e_k)$, they have positive Hodge weight, hence also don't contribute. This proves the lemma. Remark 5.10. If we consider $\mathrm{H}_{s,h}^{k}(\mathfrak{g}^{hol})$, where s refers to the scaling weight and h refers to the hodge weight, then the above proof actually gives more vanishing results

(5.6)
$$\mathrm{H}^{k}_{s,h}(\mathfrak{g}^{hol}) = 0 \quad \text{if} \quad k \le 2, s \le 0, h \le 0 \quad \text{or} \quad k \le 2, s = 0$$

This will be used in subsection 5.6 to prove the Virasoro equations.

Theorem 5.11. If there exists a quantization of BCOV theory on the elliptic curve satisfying the dilaton equation, then it's unique up to homotopy.

Proof. This is equivalent to saying that

$$\mathbf{H}^{0}\left(\Omega\left(C_{red}^{*}\left(\mathfrak{g}\right)\right)[\delta][2]\right)_{2-2g}=0$$

where the subscript means the component with Hodge weight 2 - 2g. There's a spectral sequence

$$\mathrm{H}^{i}(E,\mathbb{C})\otimes\mathrm{H}^{j}(C^{*}_{red}\left(\mathfrak{g}\right)[\delta])\to\mathrm{H}^{i+j-2}\left(\Omega\left(C^{*}_{red}\left(\mathfrak{g}\right)\right)[\delta][2]\right)$$

On the other hand, there's a spectral sequence converging to $\mathrm{H}^{k}(C^{*}_{red}(\mathfrak{g})[\delta])$ with E_{2} -term given by

$$\mathbf{H}^{k}\left(C_{red}^{*}\left(\mathfrak{g}^{hol}\right)_{2-2g,2-2g}\right)$$

which is zero for $k \leq 2, g > 0$ by the previous lemma. This proves the theorem.

5.3. Existence of the quantization. In this section, we show the existence of the quantization of BCOV theory on elliptic curves.

5.3.1. Logarithmic BCOV theory on \mathbb{C} . We consider the pair $(\mathbb{C}, 0)$ where 0 is the origin of \mathbb{C} . Let z be the holomorphic coordinate. Consider the sheaf of vector fields

$$T_0\mathbb{C}\subset T\mathbb{C}$$

which is defined to be the subsheaf of vector fields that vanishes at least for order two at the origin. We define the space of relative polyvector fields for the pair $(\mathbb{C}, 0)$ by

$$\mathrm{PV}_{(\mathbb{C},0)}^{*,*} = \bigoplus_{i,j} \mathrm{PV}_{(\mathbb{C},0)}^{i,j} = \bigoplus_{i,j} \Omega^{0,j} \left(\wedge^{i} T_{0} \mathbb{C} \right)$$

Let

$$\Omega_{(\mathbb{C},0)} = \frac{dz}{z}$$

which defines a trace map

$$\operatorname{Tr}_{(\mathbb{C},0)} : \operatorname{PV}_{(\mathbb{C},0)}^{*,*} \to \mathbb{C}$$
$$\mu \to \int_{\mathbb{C}} \left(\mu \vdash \Omega_{(\mathbb{C},0)} \right) \land \Omega_{(\mathbb{C},0)}$$

As in the case of ordinary Calabi-Yau case, the logarithmic volume form $\Omega_{(\mathbb{C},0)}$ induces a well-defined map

$$\partial: \mathrm{PV}_{(\mathbb{C},0)}^{i,j} \to \mathrm{PV}_{(\mathbb{C},0)}^{i-1,j}$$

and a $\bar\partial$ operator

$$\bar{\partial}: \mathrm{PV}_{(\mathbb{C},0)}^{i,j} \to \mathrm{PV}_{(\mathbb{C},0)}^{i,j+1}$$

These operators give $PV_{(\mathbb{C},0)}^{*,*}$ the structure of differential graded Batalin-Vilkovisky algebra as before.

We can extend the BCOV theory to the pair $(\mathbb{C}, 0)$, which we call a logarithmic BCOV theory. The space of fields is

$$\mathscr{E}_{(\mathbb{C},0)} = \mathrm{PV}_{(\mathbb{C},0)}^{*,*}[[t]]$$

and the differential is $Q = \bar{\partial} - t\partial$. The classical action functional

$$S^{BCOV}_{(\mathbb{C},0)} \in \mathcal{O}_{loc}\left(\mathscr{E}_{(\mathbb{C},0)}\right)$$

is defined by the same formula as in the case of BCOV theory, and it satisfies the classical master equation. The renormalization group flow equation and quantum master equation is defined similarly.

Theorem 5.12. There exists a unique quantization of logarithmic BCOV theory for the pair $(\mathbb{C}, 0)$.

Proof. The space of fields can be written as

$$\mathscr{E}_{(\mathbb{C},0)} = C^{\infty}(\mathbb{C}) \otimes \mathbb{C}[d\bar{z},\alpha][[t]]$$

where $\alpha = z^2 \partial_z$. We first give an explicitly description of ∂ -operator. Since

$$\partial \left(f\alpha\right) \vdash \Omega_{(\mathbb{C},0)} = \partial \left(f\alpha \vdash \Omega_{(\mathbb{C},0)}\right) = dz \frac{\partial}{\partial z} \left(fz\right) = z \frac{\partial}{\partial z} \left(zf\right) \vdash \Omega_{(\mathbb{C},0)}$$

we see that

$$\partial = z \frac{\partial}{\partial z} z \frac{\partial}{\partial \alpha}$$

The space of jets of $\mathscr{E}_{(\mathbb{C},0)}$ is

$$J(\mathscr{E}_{(\mathbb{C},0)}) = C^{\infty}(\mathbb{C})[[z',\bar{z}']][d\bar{z},\alpha][[t]]$$

where z', \bar{z}' indicates the jet coordinates. Then $J(\mathscr{E}_{(\mathbb{C},0)})[-1]$ is an L_{∞} algebra, where the L_{∞} structure is induced by $Q + \{S_{(\mathbb{C},0)}^{BCOV}, -\}$. The space of local functionals can be described as

$$\mathcal{O}_{loc}(\mathscr{E}_{(\mathbb{C},0)}) = \omega_{\mathbb{C}} \otimes_{D_{\mathbb{C}}} C^*_{red} \left(J(E_{(\mathbb{C},0)})[-1] \right)$$

where $\omega_{\mathbb{C}}$ denotes the right $D_{\mathbb{C}}$ -module of top differential forms on \mathbb{C} , and $C^*_{red} \left(J(E_{(\mathbb{C},0)})[-1] \right)$ is the reduced Chevalley-Eilenberg complex in the category of $D_{\mathbb{C}}$ -modules

$$C^*_{red}\left(J(E_{(\mathbb{C},0)})[-1]\right) = \prod_{k>0} \operatorname{Sym}_{C^{\infty}(\mathbb{C})}^k \left(J(\mathscr{E}_{(\mathbb{C},0)})^{\vee}\right)$$

where

$$J(\mathscr{E}_{(\mathbb{C},0)})^{\vee} = \operatorname{Hom}_{C^{\infty}(\mathbb{C})} \left(J(\mathscr{E}_{(\mathbb{C},0)}), C^{\infty}(\mathbb{C}) \right)$$

Let

$$D_{\mathbb{C}}^{hol} = \mathcal{O}(\mathbb{C}) \left[\frac{\partial}{\partial z} \right]$$

be the holomorphic differential operators on \mathbb{C} , where $\mathcal{O}(\mathbb{C})$ is the space of holomorphic functions on \mathbb{C} . Let

$$J(\mathscr{E}_{(\mathbb{C},0)})^{hol} = \mathcal{O}(\mathbb{C})[[z',t]][\alpha] \subset J(\mathscr{E}_{(\mathbb{C},0)})$$

Claim. There's a quasi-isomorphism of complexes

$$\mathcal{O}_{loc}(\mathscr{E}_{(\mathbb{C},0)}) \cong \Omega^*_{hol}\left(C^*_{red}\left(J(\mathscr{E}_{(\mathbb{C},0)})^{hol}[-1]\right)\right) [2]$$

where Ω_{hol}^* denotes the holomorphic de Rham complex of the $D_{\mathbb{C}}^{hol}$ module $C_{red}^* \left(J(\mathscr{E}_{(\mathbb{C},0)})^{hol}[-1] \right)$, and the L_{∞} structure is given by $-t\partial + S_{(\mathbb{C},0)}^{BCOV}$.

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Proof of the claim. In fact, we have the quasi-isomorphic embedding of L_{∞} $D_{\mathbb{C}}$ -algebras

$$C^{\infty}(\mathbb{C}) \otimes_{\mathcal{O}(\mathbb{C})} J(\mathscr{E}_{(\mathbb{C},0)})^{hol}[-1] \hookrightarrow J(\mathscr{E}_{(\mathbb{C},0)})[-1]$$

from which we find the quasi-isomorphism of complexes of $D_{\mathbb{C}}$ -modules

$$C^{\infty}(\mathbb{C}) \otimes_{\mathcal{O}(\mathbb{C})} \left(C^*_{red} \left(J(\mathscr{E}_{(\mathbb{C},0)})^{hol}[-1] \right) \right) \cong C^*_{red} \left(J(E_{(\mathbb{C},0)})[-1] \right)$$

Therefore the Koszul resolution gives that

$$\mathcal{O}_{loc}(\mathscr{E}_{(\mathbb{C},0)}) \cong \left(\Omega^*_{\mathbb{C}}[2] \otimes_{C^{\infty}(\mathbb{C})} D_{\mathbb{C}}\right) \otimes_{D_{\mathbb{C}}} C^*_{red} \left(J(\mathscr{E}_{(\mathbb{C},0)})[-1]\right)$$
$$\cong \Omega^*_{\mathbb{C}}[2] \otimes_{\mathcal{O}(\mathbb{C})} \left(C^*_{red} \left(J(\mathscr{E}_{(\mathbb{C},0)})^{hol}[-1]\right)\right)$$
$$\cong \Omega^*_{hol} \left(C^*_{red} \left(J(\mathscr{E}_{(\mathbb{C},0)})^{hol}[-1]\right)\right) [2]$$

This proves the claim.

The L_{∞} algebra structure of $J(\mathscr{E}_{(\mathbb{C},0)})^{hol}[-1]$ is given by $Q + \left\{S^{BCOV}_{(\mathbb{C},0)}, -\right\}$. Explicitly, the differential is

$$Q\left(f(z,z',t)\alpha\right) = -t(z+z')\frac{\partial}{\partial z'}(z+z')f(z,z',t)$$

for $f(z, z', t) \in \mathcal{O}(\mathbb{C})[[z', t]]$, and the non-trivial higher products are

$$l_n\left(t^{k_1}f_1\alpha, t^{k_2}f_2, \cdots, t^{k_n}f_n\right) = \binom{n-2}{k_1, \cdots, k_n}(z+z')\frac{\partial}{\partial z'}(z+z')f_1\cdots f_n$$

for $f_i \in \mathcal{O}(\mathbb{C})[[z']]$.

Since the construction of logarithmic theory is \mathbb{C}^* -equivariant, it follows from a general result on the cohomology of equivariant D-module described in the appendix of [CL] that the de Rham cohomology of the $L_{\infty} D_{\mathbb{C}}^{hol}$ -algebra $J(\mathscr{E}_{(\mathbb{C},0)})^{hol}[-1]$ is determined by its fiber at the origin z = 0. This can be viewed as a homotopy of the theory on \mathbb{C} to the theory near the origin. We refer to [CL] for more detailed proof of this fact.

Let $J(\mathscr{E}_{(\mathbb{C},0)})_0^{hol}[-1]$ be the L_∞ algebra at z=0. Then the differential is given by

$$Q\left(f(z',t)\alpha\right) = -tz'\frac{\partial}{\partial z'}z'f(z',t)$$

It's easy to see by taking a filtration on the homogeneous degree of z' that the Q-cohomology of $J(\mathscr{E}_{(\mathbb{C},0)})_0^{hol}[-1]$ is concentrated on terms without α , hence of pure degree 1. Therefore the standard spectral sequence associated to this filtration shows that

$$\mathrm{H}^{*}\left(C_{red}\left(J(\mathscr{E}_{(\mathbb{C},0)})_{0}^{hol}[-1]\right)\right)$$

is concentrated at degree 0. Hence

$$\mathbf{H}^{k}\left(C_{red}\left(J(\mathscr{E}_{(\mathbb{C},0)})^{hol}[-1]\right)\right) = 0, \text{ if } k \neq 0$$

Since we have the quasi-isomorphism $(\mathcal{O}_{loc}(\mathscr{E}_{(\mathbb{C},0)})) \cong (\Omega^*_{hol}(C^*_{red}(J(\mathscr{E}_{(\mathbb{C},0)})^{hol}[-1]))$ [2]), there's a spectral sequence

$$\mathrm{H}^{i}(\mathbb{C}^{*}) \otimes \mathrm{H}^{j}\left(C_{red}\left(J(\mathscr{E}_{(\mathbb{C},0)})^{hol}[-1]\right)\right) \to \mathrm{H}^{i+j-2}\left(\mathcal{O}_{loc}(\mathscr{E}_{(\mathbb{C},0)})\right)$$

It follows that $\mathrm{H}^k\left(\mathcal{O}_{loc}(\mathscr{E}_{(\mathbb{C},0)})\right) = 0$ if $k \geq 0$. This proves the uniqueness and the existence of logarithmic BCOV theory for $(\mathbb{C},0)$.

5.3.2. Existence of BCOV theory on the elliptic curve. We consider the case for the elliptic curve E. The space of fields is $\mathscr{E} = \mathrm{PV}_{E}^{*,*}[[t]]$, and the deformation obstruction complex is quasi-isomorphic to

$$\Omega^*_E\left(C^*_{red}\left(J(\mathscr{E})[-1]\right)\right)[2]$$

which, by lemma 5.1, is again quasi-isomorphic to the translation invariant deformation obstruction complex

$$\Omega^*\left(C^*_{red}\left(J(\mathscr{E})^E[-1]\right)\right)[2]$$

Theorem 5.13. There exists a quantization of BCOV theory on the elliptic curve \mathscr{E} which satisfies the dilaton axiom.

Proof. The obstruction lies

$$\mathrm{H}^{3}\left(\Omega^{*}\left(C_{red}^{*}\left(J(\mathscr{E})^{E}[-1]\right)\right)\right)$$

and there's a spectral sequence

$$\Omega^{p} \otimes \mathrm{H}^{q}\left(\left(J(\mathscr{E})^{E}[-1]\right)\right) \Longrightarrow \mathrm{H}^{p+q}\left(\Omega^{*}\left(C^{*}_{red}\left(J(\mathscr{E})^{E}[-1]\right)\right)\right)$$

By lemma 5.9, the relevant obstruction for quantization at genus g comes only from $\mathrm{H}^3\left(\left(J(\mathscr{E})^E[-1]\right)\right)$ at E_2 -term. Let U be a small disk on E, and we consider the deformation obstruction complex for the quantization on U

$$\Omega_U^*\left(C_{red}^*\left(J(\mathscr{E})|_U[-1]\right)\right)[2]$$

which, by the same argument in lemma 5.1, is quasi-isomorphic to

$$C^*_{red}\left(J(\mathscr{E})^E[-1]\right)[2]$$

and the obstruction for quantization at genus g lies in $\mathrm{H}^3\left(\left(J(\mathscr{E})^E[-1]\right)\right)$. By the locality property of the obstruction class, we know that the obstruction class for quantization on Erestricts to the obstruction class for quantization on U under the natural restriction map

$$\Omega_E^*\left(C_{red}^*\left(J(\mathscr{E})[-1]\right)\right)[2] \to \Omega_U^*\left(C_{red}^*\left(J(\mathscr{E})|_U[-1]\right)\right)[2]$$

Furthermore, the spectral sequence implies that the map on obstructions is injective. Since the quantization on U coupled to dilaton is unique up to homotopy, which can be proved by similar arguments as in Theorem 5.11, we only need to construct a quantization on U.

Consider the exponential map

$$\exp: \mathbb{C} \to \mathbb{C}^*$$

and assume that we have an isomorphism of small disks

$$\exp: U \to V$$

where V doesn't contain 0. By Theorem 5.12, we can construct a quantization of logarithmic theory on $(\mathbb{C}, 0)$. This quantization restricts to a quantization of logarithmic theory on V, which gives a quantization of the ordinary BCOV theory on U under the pull-back of the exp map. This proves the existence theorem.

5.4. Holomorphicity. In this section we prove that we can quantize the BCOV classical action using local functionals which contain only holomorphic derivatives. Before proving this, we discuss several properties of Feynman graph integrals for such local functionals.

5.4.1. Deformed quantum master equation. We consider the BCOV theory on \mathbb{C} . Let $\mathrm{PV}_{\mathbb{C},c}^{*,*}$ be the space of polyvector fields with compact support. Let $S = \sum_{g\geq 0} \hbar^g S_g$ be local functionals on $\mathrm{PV}_{\mathbb{C},c}^{*,*}[[t]]$ and contains only holomorphic derivatives, where S_0 is the classical BCOV action. Here we mean that each S_g has the following form

$$S_g[\mu_1, \cdots, \mu_n] = \int_{\mathbb{C}} D_1(\mu_1) \cdots D_n(\mu_n)$$

where $D_i \in C^{\infty}_{\mathbb{C}} \left[\frac{\partial}{\partial z} \right]$ is a differential operator containing only holomorphic derivatives in z, and z is the complex coordinate on \mathbb{C} . The propagator is given by

(5.7)
$$P_{\epsilon}^{L}(z_{1}, \bar{z}_{1}, z_{2}, \bar{z}_{2}) = -\int_{\epsilon}^{L} \frac{du}{4\pi u} \left(\frac{\bar{z}_{1} - \bar{z}_{2}}{4u}\right)^{2} e^{-|z_{1} - z_{2}|/4u}$$

and the regularized BV kernel is

(5.8)
$$\partial K_L(z_1, \bar{z}_1, z_2, \bar{z}_2) = \frac{\bar{z}_1 - \bar{z}_2}{4\pi L^2} e^{-|z_1 - z_2|/4L} \left(d\bar{z}_1 \otimes 1 + 1 \otimes d\bar{z}_2 \right)$$

with the regularized BV operator defined as before

$$\Delta_L = \frac{\partial}{\partial(\partial K_L)}$$

 S_0 satisfies the classical master equation

$$QS_0 + \frac{1}{2} \{S_0, S_0\} = 0$$

Lemma 5.14. The limit

(5.9)
$$I[L] = \lim_{\epsilon \to 0} \hbar \log \left(\exp \left(\hbar \partial_{P_{\epsilon}^{L}} \right) \exp \left(S/\hbar \right) \right)$$

exists, and defines a family of effective functionals on $\mathrm{PV}_{\mathbb{C},c}^{*,*}$ satisfying renormalization group flow equation.

Proof. I[L] is given by Feynman graph integrals

(5.10)
$$I[L] = \lim_{\epsilon \to 0} \sum_{\Gamma \text{ connected}} W_{\Gamma}(P_{\epsilon}^{L}, S)$$

where P_{ϵ}^{L} is the propagator, S are the vertices, W_{Γ} is the Feynman integral for the graph Γ , and the summation is over all possible connected Feynman graphs. Since S contains only holomorphic derivatives, the graphs having self-loops don't contribute since the propagator on the self-loop will become zero. Theorem 3.30 implies that the above limit exists, hence I[L] is well-defined. By construction, I[L] satisfies the renormalization group flow equation.

We consider the condition for S such that I[L] satisfies the quantum master equation. Recall that the quantum master equation is equivalent to

$$(Q + \hbar \Delta_L) e^{I[L]/\hbar} = \lim_{\epsilon \to 0} (Q + \hbar \Delta_L) \exp\left(\hbar \partial_{P_{\epsilon}^L}\right) e^{S/\hbar} = 0$$

Using the fact that

$$(Q + \hbar\Delta_L) \exp\left(\hbar\partial_{P_{\epsilon}^L}\right) = \exp\left(\hbar\partial_{P_{\epsilon}^L}\right) (Q + \hbar\Delta_{\epsilon})$$

we get the following equivalent condition for quantum master equation

(5.11)
$$\lim_{\epsilon \to 0} \exp\left(\hbar\partial_{P^L_{\epsilon}}\right) \left(\left(QS + \frac{1}{2} \{S, S\}_{\epsilon} + \hbar\Delta_{\epsilon}S\right) e^{S/\hbar} \right) = 0$$

Lemma 5.15. Let Φ be a smooth function on \mathbb{C}^2 with compact support. Then

$$\lim_{\epsilon \to 0} \int d^2 z_1 \int d^2 z_2 \partial^{n_0} U_{\epsilon}(z_{12}) \left(\prod_{i=1}^k \partial^{n_i} H^L_{\epsilon}(z_{12}) \right) \Phi(z_1, z_2)$$

= $\frac{A(n_0; n_1, \cdots, n_k)}{(4\pi)^k} \int d^2 z_2 \left(\partial^{n_0+1}_{z_1} + \sum_{i=1}^k (n_k+2) \atop \partial z_1 - \Phi(z_1, z_2) \right) \Big|_{z_1=z_2}$

where $z_{12} = z_1 - z_2$ and n_0, n_1, \dots, n_k are some non-negative integers,

$$U_{\epsilon}(z) = \frac{1}{4\pi\epsilon} \left(\frac{\bar{z}}{4\epsilon}\right) e^{-|z|^2/4\epsilon}, \quad H_{\epsilon}^L(z) = \int_{\epsilon}^L \frac{dt}{4\pi t} \left(\frac{\bar{z}}{4t}\right)^2 e^{-|z|^2/4t}$$
The constant $A(n_0, n_1, \cdots, n_k)$ is given by

$$A(n_0; n_1, \cdots, n_k) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^k du_i \frac{\prod_{i=1}^k u_i^{n_i+1}}{\left(1 + \sum_{i=1}^k u_i\right)^{\sum_{j=0}^k (n_j+2)}}$$

Proof. Using integration by parts

$$\begin{split} &\int d^{2}z_{1} \int d^{2}z_{2} \partial^{n_{0}} U_{\epsilon}(z_{12}) \left(\prod_{i=1}^{k} \partial^{n_{i}} H_{\epsilon}^{L}(z_{12}) \right) \Phi(z_{1}, z_{2}) \\ &= \int d^{2}y \int d^{2}z \int_{\epsilon}^{L} \cdots \int_{\epsilon}^{L} \prod_{i=1}^{k} dt_{i} \frac{1}{4\pi\epsilon} \left(\frac{\bar{z}}{4\epsilon} \right)^{n_{0}+1} \left(\prod_{i=1}^{k} \frac{1}{4\pi t_{i}} \left(\frac{\bar{z}}{4t_{i}} \right)^{n_{i}+2} \right) e^{-\frac{|z|^{2}}{4} \left(\frac{1}{\epsilon} + \sum_{i=1}^{k} \frac{1}{t_{i}} \right)} \Phi(y+z, y) \\ &= \int d^{2}y \int d^{2}z \int_{\epsilon}^{L} \cdots \int_{\epsilon}^{L} \prod_{i=1}^{k} dt_{i} \frac{1}{4\pi\epsilon} \left(\frac{1}{\epsilon \left(\frac{1}{\epsilon} + \sum_{i=1}^{k} \frac{1}{t_{i}} \right)} \right)^{n_{0}+1} \left(\prod_{i=1}^{k} \frac{1}{4\pi t_{i}} \left(\frac{1}{t_{i} \left(\frac{1}{\epsilon} + \sum_{i=1}^{k} \frac{1}{t_{i}} \right)} \right)^{n_{i}+2} \right) \\ &= e^{-\frac{|z|^{2}}{4} \left(\frac{1}{\epsilon} + \sum_{j=1}^{k} \frac{1}{t_{j}} \right)} \partial_{z}^{n_{0}+1} + \sum_{i=1}^{k} (n_{i}+2)} \Phi(y+z, y) \end{split}$$

Consider the rescaling $t_i \to t_i \epsilon$, we get

$$= \int d^2 y \int d^2 z \int_1^{L/\epsilon} \cdots \int_1^{L/\epsilon} \prod_{i=1}^k dt_i \frac{1}{\left(1 + \sum_{i=1}^k \frac{1}{t_i}\right)^{n_0+2}} \left(\prod_{i=1}^k \frac{1}{4\pi t_i} \frac{1}{t_i^{n_i+2} \left(1 + \sum_{j=1}^k \frac{1}{t_j}\right)^{n_i+2}} \right) \\ \left(\frac{1 + \sum_{j=1}^k \frac{1}{t_j}}{4\pi \epsilon} e^{-\frac{|z|^2}{4\epsilon} \left(1 + \sum_{j=1}^k \frac{1}{t_j}\right)} \right) \partial_z^{n_0+1 + \sum_{i=1}^k (n_k+2)} \Phi(y+z,y)$$

Taking the limit $\epsilon \to 0$

$$\stackrel{\epsilon \to 0}{\longrightarrow} \int_{1}^{\infty} \dots \int_{1}^{\infty} \prod_{i=1}^{k} \frac{dt_{i}}{4\pi t_{i}^{n_{i}+3}} \frac{1}{\left(1+\sum_{i=1}^{k} \frac{1}{t_{i}}\right)^{\sum_{i=0}^{k}(n_{i}+2)}} \int d^{2}y \left(\partial_{z}^{n_{0}+1+\sum_{i=1}^{k}(n_{k}+2)} \Phi(y+z,y)\right) \bigg|_{z=0}$$

$$= \int_{0}^{1} \cdots \int_{0}^{1} \prod_{i=1}^{k} \frac{du_{i}}{4\pi} \frac{\prod_{i=1}^{k} u_{i}^{n_{i}+1}}{\left(1 + \sum_{i=1}^{k} u_{i}\right)^{\sum_{j=0}^{k} (n_{j}+2)}} \int d^{2}z_{2} \left(\frac{u_{0}^{n_{0}+1} + \sum_{i=1}^{k} (n_{k}+2)}{\partial_{z_{1}}} \Phi(z_{1}, z_{2}) \right) \bigg|_{z_{1}=z_{2}}$$

Corollary 5.16. Given two local functionals S_1, S_2 on $\mathrm{PV}_{\mathbb{C},c}^{*,*}$ having only holomorphic derivatives, the limit

$$\lim_{\epsilon \to 0} \exp\left(\hbar \frac{\partial}{\partial P_{\epsilon}^{L}}\right) \Delta_{\epsilon}(S_{1}, S_{2})$$

exits as local functional on $\mathrm{PV}_{\mathbb{C},c}^{*,*}$ which has only holomorphic derivatives, and it doesn't depend on L.

Definition 5.17. Let S_1, S_2 be two local functionals on $\mathrm{PV}_{\mathbb{C},c}^{*,*}$ having only holomorphic derivatives. We define the deformed BV bracket $\{S_1, S_2\}'$ by

(5.12)
$$\{S_1, S_2\}' = \lim_{\epsilon \to 0} \exp\left(\hbar \frac{\partial}{\partial P_{\epsilon}^L}\right) \Delta_{\epsilon}(S_1, S_2)$$

Note that the classical BV bracket is given by

(5.13)
$$\{S_1, S_2\} = \lim_{\epsilon \to 0} \Delta_{\epsilon}(S_1 S_2)$$

where we have used the fact that $\Delta_{\epsilon}S_1 = 0$. Therefore the deformed BV bracket can be viewed as the quantum corrected version of the classical BV bracket in this setting.

Let Γ be a connected graph without self-loops, $V(\Gamma)$ be the set of vertices, $E(\Gamma)$ be the set of edges, $V = |V(\Gamma)|, E = |E(\Gamma)|$. We index the set of vertices as in section 3.5 by

$$v: \{1, 2, \cdots, V\} \to V(\Gamma)$$

and index the set of edges by

$$e: \{0, 1, 2, \cdots, E-1\} \to E(\Gamma)$$

such that $e(0), e(1), \dots, e(k) \in E(\Gamma)$ are all the edges connecting v(1), v(V). We also fix an orientation of the edges such that given $e \in E(\Gamma)$, the left endpoint $l(e) \in V(\Gamma)$ and the

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right endpoint $r(e) \in V(\Gamma)$ are defined. We consider the following Feynman graph integral by puting U_{ϵ} on e(0), puting H_{ϵ}^{L} to all other edges, and puting a smooth function Φ on $\mathbb{C}^{|V(\Gamma)|}$ with compact support for the vertices. We would like to compute the following limit of the graph integral

$$\lim_{\epsilon \to 0} \prod_{i=1}^{V} \int d^2 z_i \partial^{n_0} U_{\epsilon}(z_{e(0)}) \left(\prod_{i=1}^{E-1} \partial^{n_i} H_{\epsilon}^L(z_{e(i)}) \right) \Phi$$

where we use the notation that

$$z_e = z_i - z_j$$
, if $l(e) = v(i), r(e) = v(j)$

Lemma 5.18. The above limit exists and we have the identity

$$\begin{split} &\lim_{\epsilon \to 0} \prod_{i=1}^{V} \int d^2 z_i \partial^{n_0} U_{\epsilon}(z_{e(0)}) \left(\prod_{i=1}^{E-1} \partial^{n_i} H_{\epsilon}^L(z_{e(i)}) \right) \Phi \\ &= \lim_{\epsilon \to 0} \frac{A(n_0; n_1, \cdots, n_k)}{(4\pi)^k} \prod_{i=2}^{V} \int d^2 z_i \; \partial_{z_1}^{n_0+1+\sum_{i=1}^k (n_i+2)} \left(\left(\prod_{i=k+1}^{E-1} \partial^{n_i} H_{\epsilon}^L(z_{e(i)}) \right) \Phi \right) \bigg|_{z_1 = z_V} \end{split}$$

The constant $A(n_0; n_1, \dots, n_k)$ is defined as in the previous lemma. The limit on the RHS exists due to Theorem 3.30.

Proof.

$$\begin{split} \prod_{i=1}^{V} \int d^{2}z_{i} \partial^{n_{0}} U_{\epsilon}(z_{e(0)}) \left(\prod_{i=1}^{E-1} \partial^{n_{i}} H_{\epsilon}^{L}(z_{e(i)}) \right) \Phi \\ &= \prod_{i=1}^{V} \int d^{2}z_{i} \prod_{i=1}^{E-1} \int_{\epsilon}^{L} dt_{e(i)} \\ &\left(\frac{1}{4\pi\epsilon} \left(\frac{\bar{z}_{e(0)}}{4\epsilon} \right)^{n_{0}+1} \right) \left(\prod_{i=1}^{E-1} \frac{1}{4\pi t_{e(i)}} \left(\frac{\bar{z}_{e(i)}}{4t_{e(i)}} \right)^{n_{i}+2} \right) e^{-\left(\frac{|z_{e(0)}|^{2}}{4\epsilon} + \sum_{j=1}^{E-1} \frac{|z_{e(j)}|^{2}}{4t_{e(i)}} \right)} \Phi \end{split}$$

We will use the same notations as in the proof of Theorem 3.30. The incidence matrix $\{\rho_{v,e}\}_{v\in V(G),e\in E(G)}$ is defined by

$$\rho_{v,e} = \begin{cases} 1 & l(e) = v \\ -1 & r(e) = v \\ 0 & \text{otherwise} \end{cases}$$

We assume that the orientation of e(0) is such that

$$\rho_{v(1),e(0)} = 1, \rho_{v(V),e(0)} = -1$$

The $(V-1) \times (V-1)$ matrix $M_{\Gamma}(t)$ is defined by

$$M_{\Gamma}(t)_{i,j} = \sum_{l=0}^{E-1} \rho_{v(i),e(l)} \frac{1}{t_{e(l)}} \rho_{v(j),e(l)}, \qquad 1 \le i,j \le V-1$$

where we use the convention that $t_{e(0)} = \epsilon$. Under the following linear change of variables

$$\begin{cases} z_i = y_i + y_V & 1 \le i \le V - 1 \\ z_V = y_V \end{cases}$$

and use integration by parts

$$\begin{split} \prod_{i=1}^{V} \int d^{2}z_{i} \prod_{i=1}^{E-1} \int_{\epsilon}^{L} dt_{e(i)} \\ \left(\frac{1}{4\pi\epsilon} \left(\frac{\bar{z}_{e(0)}}{4\epsilon}\right)^{n_{0}+1}\right) \left(\prod_{i=1}^{E-1} \frac{1}{4\pi t_{e(i)}} \left(\frac{\bar{z}_{e(i)}}{4t_{e(i)}}\right)^{n_{i}+2}\right) e^{-\left(\frac{|z_{e(0)}|^{2}}{4\epsilon} + \sum_{j=1}^{E-1} \frac{|z_{e(j)}|^{2}}{4t_{e(i)}}\right)} \Phi \\ &= \int d^{2}y_{V} \prod_{i=1}^{V-1} \int d^{2}y_{i} \prod_{i=1}^{E-1} \int_{\epsilon}^{L} \frac{dt_{e(i)}}{4\pi t_{e(i)}} \exp\left(-\frac{1}{4}\sum_{i,j=1}^{V-1} y_{i}M_{\Gamma}(t)_{i,j}\bar{y}_{j}\right) \\ &= \frac{1}{4\pi\epsilon} \left(\sum_{j=1}^{V-1} \sum_{i=1}^{V-1} \rho_{v(i),e(0)}M_{\Gamma}^{-1}(t)_{i,j}}{\epsilon} \frac{\partial}{\partial y_{j}}\right)^{n_{0}+1} \prod_{\alpha=1}^{E-1} \left(\sum_{j=1}^{V-1} \sum_{i=1}^{V-1} \rho_{v(i),e(\alpha)}M_{\Gamma}^{-1}(t)_{i,j}} \frac{\partial}{\partial y_{j}}\right)^{n_{\alpha}+2} \Phi \end{split}$$

Notice that for $0 \le \alpha \le k$, and $1 \le i \le V - 1$, $\rho_{v(i),e(\alpha)}$ is nonzero only for $\rho_{v(1),e(\alpha)} = 1$. Consider the change of variables

$$t_{e(i)} \to \epsilon t_{e(i)} \qquad 1 \le i \le k$$

$$t_{e(i)} \to t_{e(i)} \qquad k+1 \le i \le E-1$$

we get

$$\int d^{2}y_{V} \prod_{i=1}^{V-1} \int d^{2}y_{i} \prod_{i=1}^{k} \int_{1}^{L/\epsilon} \frac{dt_{e(i)}}{4\pi t_{e(i)}} \prod_{i=k+1}^{E-1} \int_{\epsilon}^{L} \frac{dt_{e(i)}}{4\pi t_{e(i)}} \exp\left(-\frac{1}{4} \sum_{i,j=1}^{V-1} y_{i} \bar{y}_{j} M_{\Gamma}(\tilde{t})_{i,j}\right)$$

$$= \prod_{i=1}^{k} \int_{1}^{L/\epsilon} \frac{dt_{e(i)}}{4\pi} \prod_{i=k+1}^{E-1} \int_{\epsilon}^{L} \frac{dt_{e(i)}}{4\pi} F(t;\epsilon)$$

where \tilde{t} 's are define by

$$\begin{split} \tilde{t}_{e(0)} &= \epsilon \\ \tilde{t}_{e(i)} &= \epsilon t_{e(i)} & \text{if } 1 \leq i \leq k \\ \tilde{t}_{e(i)} &= t_{e(i)} & \text{if } k+1 \leq i \leq E-1 \end{split}$$

$$\begin{split} F(t;\epsilon) \\ &= \prod_{i=1}^{V} \int d^2 y_i \frac{1}{\sum_{i=1}^{E-1} t_{e(i)}} \exp\left(-\frac{1}{4} \sum_{i,j=1}^{V-1} y_i \bar{y}_j M_{\Gamma}(\tilde{t})_{i,j}\right) \\ & \frac{1}{4\pi\epsilon} \left(\sum_{j=1}^{V-1} \frac{\sum_{i=1}^{V-1} \rho_{v(i),e(0)} M_{\Gamma}^{-1}(\tilde{t})_{i,j}}{\epsilon} \frac{\partial}{\partial y_j}\right)^{n_0+1} \prod_{\alpha=1}^{E-1} \left(\sum_{j=1}^{V-1} \frac{\sum_{i=1}^{V-1} \rho_{v(i),e(\alpha)} M_{\Gamma}^{-1}(\tilde{t})_{i,j}}{\tilde{t}_{e(\alpha)}} \frac{\partial}{\partial y_j}\right)^{n_\alpha+2} \Phi \end{split}$$

We first show that $\lim_{\epsilon \to 0} F(t; \epsilon)$ exists. Using integration by parts, we write $F(t; \epsilon)$ as

 $F(t;\epsilon)$

$$= \prod_{i=1}^{V} \int d^{2} y_{i} \frac{1}{\prod_{i=1}^{E-1} t_{e(i)}} \exp\left(-\frac{|y_{1}|^{2}}{4\epsilon} \left(1 + \sum_{\alpha=1}^{k} \frac{1}{t_{e(\alpha)}}\right)\right) \frac{1}{4\pi\epsilon} \left(\frac{\bar{y}_{1}}{4\epsilon}\right)^{n_{0}+1} \prod_{\alpha=1}^{k} \left(\frac{\bar{y}_{1}}{4\epsilon t_{e(\alpha)}}\right)^{n_{\alpha}+2} \\ \exp\left(-\frac{1}{4} \sum_{i,j=1}^{V-1} y_{i} \bar{y}_{j} \sum_{\beta=k+1}^{E-1} \frac{\rho_{v(i),e(\beta)}\rho_{v(j),e(\beta)}}{t_{e(\beta)}}\right) \prod_{\beta=k+1}^{E-1} \left(\frac{\sum_{i=1}^{V-1} \rho_{v(i),e(\beta)}\bar{y}_{i}}{4t_{e(\beta)}}\right)^{n_{\alpha}+2} \Phi \\ = \prod_{i=1}^{V} \int d^{2} y_{i} \frac{1}{\prod_{i=1}^{E-1} t_{e(i)}} \\ \exp\left(-\frac{|y_{1}|^{2}}{4\epsilon} \left(1 + \sum_{\alpha=1}^{k} \frac{1}{t_{e(\alpha)}}\right)\right) \frac{1}{4\pi\epsilon} \left(\prod_{\alpha=1}^{k} \frac{1}{t_{e(\alpha)}}\right)^{n_{\alpha}+2} \frac{1}{\left(1 + \sum_{\alpha=1}^{k} \frac{1}{t_{e(\alpha)}}\right)^{n_{0}+1 + \sum_{\alpha=1}^{k} (n_{\alpha}+2)} \\ \left(\frac{\partial}{\partial y_{1}}\right)^{n_{0}+1 + \sum_{\alpha=1}^{k} (n_{\alpha}+2)} \left(e^{-\frac{1}{4} \sum_{i,j=1}^{V-1} y_{i} \bar{y}_{j}} \sum_{\beta=k+1}^{E-1} \frac{\rho_{v(i),e(\beta)}\rho_{v(j),e(\beta)}}{t_{e(\beta)}} \prod_{\beta=k+1}^{E-1} \left(\sum_{i=1}^{V-1} \rho_{v(i),e(\beta)} \bar{y}_{i}}{4t_{e(\beta)}}\right)^{n_{\alpha}+2} \Phi$$

Therefore under the limit $\epsilon \to 0,$ we get

$$\begin{split} &\lim_{\epsilon \to 0} F(t;\epsilon) \\ &= \prod_{i=2}^{V} \int d^{2} y_{i} \frac{1}{\prod_{i=1}^{E-1} t_{e(i)}} \left(\prod_{\alpha=1}^{k} \frac{1}{t_{e(\alpha)}} \right)^{n_{\alpha}+2} \frac{1}{\left(1 + \sum_{\alpha=1}^{k} \frac{1}{t_{e(\alpha)}} \right)^{\sum_{\alpha=0}^{k} (n_{\alpha}+2)}} \\ & \left(\frac{\partial}{\partial y_{1}} \right)^{n_{0}+1+\sum_{\alpha=1}^{k} (n_{\alpha}+2)} \left(e^{-\frac{1}{4} \sum_{i,j=1}^{V-1} y_{i} \bar{y}_{j}} \sum_{\beta=k+1}^{E-1} \frac{\rho_{v(i),e(\beta)} \rho_{v(j),e(\beta)}}{t_{e(\beta)}}}{\beta_{\beta=k+1}} \prod_{\beta=k+1}^{E-1} \left(\frac{\sum_{i=1}^{V-1} \rho_{v(i),e(\beta)} \bar{y}_{i}}{4t_{e(\beta)}} \right)^{n_{\alpha}+2} \Phi \right) \bigg|_{y_{1}=0} \end{split}$$

Claim.

$$\lim_{\epsilon \to 0} \prod_{i=1}^{k} \int_{1}^{L/\epsilon} \frac{dt_{e(i)}}{4\pi} \prod_{i=k+1}^{E-1} \int_{\epsilon}^{L} \frac{dt_{e(i)}}{4\pi} F(t;\epsilon) = \prod_{i=1}^{k} \int_{1}^{\infty} \frac{dt_{e(i)}}{4\pi} \prod_{i=k+1}^{E-1} \int_{0}^{L} \frac{dt_{e(i)}}{4\pi} \lim_{\epsilon \to 0} F(t;\epsilon)$$

Clearly Eqn (5.14) follows from the claim and Lemma 5.15.

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To prove the claim, first notice that we have the estimate

$$0 \le \frac{M_{\Gamma}^{-1}(t)_{1,j}}{t_{e(\alpha)}} \le \frac{1}{t_{e(\alpha)} \left(\frac{1}{\epsilon} + \sum_{i=1}^{k} \frac{1}{t_{e(i)}}\right)}$$

for $1 \le \alpha \le k, 1 \le j \le V - 1$. In fact, by Lemma 3.34,

$$M_{\Gamma}^{-1}(t)_{1,j} = \frac{\sum_{C \in \operatorname{Cut}(\Gamma; \{v(1), v(j)\}, \{v_V\})} \prod_{e \in C} t_e}{\sum_{T \in \operatorname{Tree}(\Gamma)} \prod_{e \notin T} t_e}$$

$$\leq \frac{\sum_{C \in \operatorname{Cut}(\Gamma; \{v(1), v(j)\}, \{v_V\})} \prod_{e \in C} t_e}{\sum_{\substack{T \in \operatorname{Tree}(\Gamma)\\ e_i \in E(T) \text{ for some } 0 \le i \le k}} \prod_{e \notin T} t_e} \leq \frac{1}{\left(\frac{1}{\epsilon} + \sum_{i=1}^k \frac{1}{t_{e(i)}}\right)}$$

For $0 \le \alpha \le E-1, 1 \le j \le V-1$, $\frac{\sum\limits_{i=1}^{V-1} \rho_{v(i),e(\alpha)} M_{\Gamma}^{-1}(t)_{i,j}}{t_{e(\alpha)}}$ is bounded by a constant by Lemma 3.35. It follows that

$$\begin{split} &|F(t;\epsilon)| \\ &\leq \prod_{i=1}^{V} \int d^{2}y_{i} \frac{1}{\prod_{i=1}^{E-1} t_{e(i)}} \exp\left(-\frac{1}{4} \sum_{i,j=1}^{V-1} y_{i} \bar{y}_{j} M_{\Gamma}(\tilde{t})_{i,j}\right) \\ &\left|\frac{1}{4\pi\epsilon} \left(\sum_{j=1}^{V-1} \frac{\sum_{i=1}^{V-1} \rho_{v(i),e(0)} M_{\Gamma}^{-1}(\tilde{t})_{i,j}}{\epsilon} \frac{\partial}{\partial y_{j}}\right)^{n_{0}+1} \prod_{\alpha=1}^{E-1} \left(\sum_{j=1}^{V-1} \frac{\sum_{i=1}^{V-1} \rho_{v(i),e(\alpha)} M_{\Gamma}^{-1}(\tilde{t})_{i,j}}{\tilde{t}_{e(\alpha)}} \frac{\partial}{\partial y_{j}}\right)^{n_{\alpha}+2} \Phi \\ &\leq \prod_{i=1}^{V} \int d^{2}y_{i} \frac{1}{\prod_{i=1}^{E-1} t_{e(i)}} \exp\left(-\frac{1}{4} \sum_{i,j=1}^{V-1} y_{i} \bar{y}_{j} M_{\Gamma}(\tilde{t})_{i,j}\right) \frac{1}{4\pi\epsilon} \prod_{1\leq\alpha\leq k} \left(\frac{1}{t_{e(\alpha)} \left(1+\sum_{i=1}^{k} \frac{1}{t_{e(i)}}\right)}\right)^{n_{\alpha}+2} \tilde{\Phi} \end{split}$$

where $\tilde{\Phi}$ is some non-negative smooth function on \mathbb{C}^V with compact support. Integrating over y_i 's we get

$$\begin{aligned} |F(t;\epsilon)| &\leq C \frac{1}{\prod\limits_{i=1}^{E-1} t_{e(i)}} \frac{1}{\epsilon \det M_{\Gamma}(\tilde{t})} \prod_{1 \leq \alpha \leq k} \left(\frac{1}{t_{e(\alpha)} \left(1 + \sum\limits_{i=1}^{k} \frac{1}{t_{e(i)}} \right)} \right)^{n_{\alpha}+2} \\ &= C \frac{\epsilon^{k}}{\mathcal{P}_{\Gamma}(\epsilon, \epsilon t_{e(1)}, \cdots, \epsilon t_{e(k)}, t_{e_{k+1}}, \cdots, t_{e(E-1)})} \prod_{1 \leq \alpha \leq k} \left(\frac{1}{t_{e(\alpha)} \left(1 + \sum\limits_{i=1}^{k} \frac{1}{t_{e(i)}} \right)} \right)^{n_{\alpha}+2} \\ &\leq C \frac{1}{\mathcal{P}_{\Gamma}(t_{e_{k+1}}, \cdots, t_{e(E-1)})} \prod_{\alpha=1}^{k} t_{e(\alpha)}} \prod_{\alpha=1}^{k} \frac{1}{t_{e(\alpha)}^{n_{\alpha}+2}} \end{aligned}$$

where C is a constant that only depends on $\tilde{\Phi}$, $\bar{\Gamma}$ is the graph obtained by collapsing the vertices v(1), v(V) and all $e(0), e(1), \dots, e(k)$, and \mathcal{P}_{Γ} is defined in Lemma 3.34. Here we have used the simple combinatorial fact that

$$\mathcal{P}_{\Gamma}(\epsilon, \epsilon t_{e(1)}, \cdots, \epsilon t_{e(k)}, t_{e_{k+1}}, \cdots, t_{e(E-1)}) \ge \epsilon^k \left(\prod_{\alpha=1}^k t_{e(\alpha)}\right) \left(1 + \sum_{\alpha=1}^k \frac{1}{t_{e(\alpha)}}\right) \mathcal{P}_{\bar{\Gamma}}(t_{e_{k+1}}, \cdots, t_{e(E-1)})$$

Since $\overline{\Gamma}$ has no self-loops,

$$\prod_{i=1}^{k} \int_{1}^{\infty} \frac{dt_{e(i)}}{4\pi} \prod_{\alpha=1}^{k} \frac{1}{t_{e(\alpha)}^{n_{\alpha}+3}} \prod_{i=k+1}^{E-1} \int_{0}^{L} \frac{dt_{e(i)}}{4\pi} \frac{1}{\mathcal{P}_{\bar{\Gamma}}(t_{e_{k+1}}, \cdots, t_{e(E-1)})} < \infty$$

Now the claim follows from dominated convergence theorem.

Proposition 5.19. Let S be a local functional on $\mathrm{PV}_{\mathbb{C},c}^{*,*}[[t]]$ with only holomorphic derivatives and $\bar{\partial}S = 0$. Let $\{I[L]\}_{L>0}$ be the effective functional defined by Equation (5.9). Then $\{I[L]\}_{L>0}$ satisfies the quantum master equation

$$QI[L] + \frac{1}{2} \{ I[L], I[L] \}_L + \hbar \Delta_L I[L] = 0$$

if and only if S satisfies the equation

(5.14)
$$QS + \frac{1}{2} \{S, S\}' = 0$$

where $\{-,-\}'$ is the deformed BV bracket defined in Definition 5.17.

Proof. The quantum master equation is equivalent to

$$\lim_{\epsilon \to 0} \exp\left(\hbar \partial_{P_{\epsilon}^{L}}\right) \left(\left(QS + \frac{1}{2} \{S, S\}_{\epsilon} + \hbar \Delta_{\epsilon}S\right) e^{S/\hbar} \right) = 0$$

where $\Delta_{\epsilon}S = 0$ since S is local and contains only holomorphic derivatives. Since $QS = (-t\partial)S$ which also contains only holomorphic derivatives, it follows from Theorem 3.30 and Lemma 5.18 that the equation is equivalent to

$$\lim_{\epsilon \to 0} \exp\left(\hbar \partial_{P_{\epsilon}^{L}}\right) \left(\left(QS + \frac{1}{2} \{S, S\}'\right) e^{S/\hbar} \right) = 0$$

or

$$QS + \frac{1}{2} \{S, S\}' = 0$$

Remark 5.20. Eqn (5.14) can be viewed as quantum corrected equation for the classical master equation. The classical BV bracket contains single contractions between two local functionals, and the quantization deforms the BV bracket to include all multi-contractions.

5.4.2. *Holomorphicity*. Now we prove that we can quantize the BCOV theory on the elliptic curve using local functionals with only holomorphic derivatives.

Definition 5.21. We say that a functional $I \in \mathcal{O}(\mathscr{E})$ has anti-holomorphic degree k if

$$I[\mu_1, \cdots, \mu_n] = 0, \quad \mu_i \in \mathrm{PV}^{a_i, b_i}[[t]]$$

unless $\sum_{i} b_i = k$.

Theorem 5.22. There exists translation invariant local functional $S = \sum_{g \ge 0} \hbar^g S_g \in \mathcal{O}_{loc}(\mathrm{PV}_E^{*,*}[[t]])[[\hbar]]$ such that

- (1) S_0 is the BCOV classical action, S_g contains only holomorphic derivatives for all g > 0, and S has anti-holomorphic degree 1.
- (2) The following limit exists

$$F[L] = \lim_{\epsilon \to 0} \hbar \log \left(\exp \left(\hbar \partial_{P_{\epsilon}^{L}} \right) \exp \left(S/\hbar \right) \right) \in \mathscr{O}(\mathrm{PV}_{E}^{*,*}[[t]])[[\hbar]]$$

which defines a family of effective actions.

(3) F[L] satisfies renormalization flow equation, quantum master equation, and dilaton equation.

Proof. Since the theorem is local, we consider the theory on \mathbb{C} . We prove by induction on g that we can quantize the theory by local functionals S_g which contains only holomorphic derivatives and satisfies

$$\bar{\partial}S_g = 0$$

 S_0 obviously satisfies the property. Suppose we have quantized the theory using $\{S_h\}_{h < g}$ where S_h contains only holomorphic derivatives, has anti-holomorphic degree 1, and $\bar{\partial}S_h =$ 0 for h < g. We denote by

$$S^{(< g)} = \sum_{h=0}^{g-1} \hbar^h S_h$$

By Theorem 3.30,

$$F[L] = \lim_{\epsilon \to 0} \hbar \log \left(\exp \left(\hbar \partial_{P_{\epsilon}^{L}} \right) \exp \left(S^{(\langle g \rangle)} / \hbar \right) \right)$$

exists and satisfies quantum master equation modulo \hbar^g . The obstruction $O_g[L]$ at genus g is given by

$$(Q + \hbar \Delta_L) e^{F[L]/\hbar} = \left(\hbar^{g-1} O_g[L] + O(\hbar^g)\right) e^{F[L]/\hbar}$$

As before, the limit

$$O_g = \lim_{L \to 0} O_g[L]$$

exists as a local functional satisfying

$$QO_g + \{S_0, O_g\} = 0$$

By Proposition 5.19,

$$(-t\partial)S^{($$

Obviously, $\bar{\partial}((t\partial)S^{(\langle g)}) = 0$ by induction. We show that $\frac{1}{2}\{S^{(\langle g)}, S^{(\langle g)}\}'$ also lies in the kernel of $\bar{\partial}$. In fact,

$$\bar{\partial}\{S^{(\langle g)}, S^{(\langle g)}\}' = \lim_{\epsilon \to 0} \bar{\partial} \left(\exp\left(\hbar \partial_{P_{\epsilon}^{L}}\right) \partial_{K_{\epsilon}}(S^{(\langle g)}S^{(\langle g)}) \right) \\
= \lim_{\epsilon \to 0} \left(\exp\left(\hbar \partial_{P_{\epsilon}^{L}}\right) (\bar{\partial} + \hbar \partial_{K_{\epsilon}} - \hbar \partial_{K_{L}}) \partial_{K_{\epsilon}}(S^{(\langle g)}S^{(\langle g)}) \right) \\
= -\hbar \partial_{K_{L}} \lim_{\epsilon \to 0} \left(\exp\left(\hbar \partial_{P_{\epsilon}^{L}}\right) \partial_{K_{\epsilon}}(S^{(\langle g)}S^{(\langle g)}) \right) \\
= -\hbar \partial_{K_{L}}\{S^{(\langle g)}, S^{(\langle g)}\}'$$

Since $\{S^{(\langle g)}, S^{(\langle g)}\}'$ is local and contains only holomorphic derivatives, $\partial_{K_L}\{S^{(\langle g)}, S^{(\langle g)}\}'=0$. It follows that

$$\bar{\partial}O_g = 0$$

and O_g contains only holomorphic derivatives with anti-holomorphic degree 1. By the existence of the theory, we can solve the master equation by

$$O_g = QS_g + \{S_0, S_g\}$$

for some local functional S_g . Now we observe that the $\bar{\partial}$ -cohomology of the deformationobstruction complex is concentrated at terms without anti-holomorphic derivatives. In fact, by Lemma 5.1, the deformation obstruction complex is quasi-isomorphic to

$$\mathbb{C}dzd\bar{z}\otimes_D\left(C_{red}^*\left(J(\mathscr{E}_E)^E[-1]\right)\right)\cong\Omega^*\left(C_{red}^*\left(J(\mathscr{E}_E)^E[-1]\right)\right)[2]$$

The operator $\bar{\partial}: \mathscr{E}_E \to \mathscr{E}_E$ induces a differential, on the above complex. Using

$$\mathrm{H}^*\left(J(\mathscr{E}_E)^E,\bar{\partial}\right) = \mathbb{C}[[z,\partial_z,t]]$$

it's easy to compute the cohomology

$$\mathbf{H}^*\left(\Omega^*\left(C_{red}^*\left(J(\mathscr{E}_E)^E[-1]\right)\right)[2]\right) \cong \left(\mathbb{C}dz[1] \oplus \mathbb{C}dzd\bar{z}\right) \otimes_{D^{hol}} C_{red}^*\left(\mathbb{C}[[z,\partial_z,t]][-1]\right)$$

which corresponds to elements in $\mathbb{C}dzd\bar{z} \otimes_D \left(C_{red}^*\left(J(\mathscr{E}_E)^E[-1]\right)\right)$ which contains only holomorphic derivatives with anti-holomorphic degree at most 1. Since $\bar{\partial}O_g = 0$ and O_g contains only holomorphic derivatives, it follows by considering a filtration on the number of anti-holomorphic derivatives in S_g that we can adjust S_g by adding some $Q + \{S_0, -\}$ -exact term such that it contains holomorphic derivatives only and satisfies $\bar{\partial}S_g = 0$. Moreover, since \mathcal{O}_g has anti-holomorphic degree 1, we can choose S_g to have anti-holomorphic degree 1 as well. This proves the theorem.

5.5. Local renormalization group flow. We consider the BCOV theory on \mathbb{C} . Let z be the linear coordinate on \mathbb{C} . Let R_{λ} be the following rescaling operator on fields

$$R_{\lambda}(t^{k}\alpha(z,\bar{z})d\bar{z}^{n}\partial_{z}^{m}) = \lambda^{n-m}t^{k}\alpha(\lambda z,\lambda\bar{z})d\bar{z}^{n}\partial_{z}^{m}, \ \lambda \in \mathbb{R}^{+}$$

for $t^k \alpha(z, \bar{z}) d\bar{z}^n \partial_z^m \in PV_{\mathbb{C},c}^{*,*}[[t]]$. It induces an action on functionals by

$$R^*_{\lambda}(I)[\mu] = I[R_{\lambda^{-1}}\mu]$$

for $I \in \mathcal{O}(\mathrm{PV}_{\mathbb{C},c}^{*,*}[[t]])$ and $\mu \in \mathrm{PV}_{\mathbb{C},c}^{*,*}[[t]]$.

Lemma 5.23. If $\{F[L]\}_L$ is a family of effective actions satisfying renormalization group flow and quantum master equation, then

$$F_{\lambda}[L] \equiv \lambda^{2\hbar\frac{\partial}{\partial\hbar}-2} R_{\lambda}^* \left(F[\lambda^2 L] \right)$$

also satisfies the renormalization group flow and quantum master equation.

Proof. Since the propagator takes the form

$$P_{\epsilon}^{L}(w_{1}, w_{2}) = -\int_{\epsilon}^{L} \frac{dt}{4\pi t} \left(\frac{\bar{z}_{1} - \bar{z}_{2}}{4t}\right)^{2} e^{-|z_{1} - z_{2}|^{2}/4t} = \int_{\epsilon}^{L} dt \tilde{P}_{t}$$

where

$$\tilde{P}_u = \frac{1}{4\pi u} \left(\frac{\bar{z}_1 - \bar{z}_2}{4u}\right)^2 e^{-|z_1 - z_2|^2/4u}$$

It follows that

(5.15)
$$R_{\lambda}\tilde{P}_{u} = \lambda^{-4}\tilde{P}_{u/\lambda^{2}}, \quad R_{\lambda}P_{\epsilon}^{L} = \lambda^{-2}P_{\epsilon/\lambda^{2}}^{L/\lambda^{2}}$$

The BV kernel is given by

$$\partial K_L(z_1, z_2) = \frac{1}{4\pi L} \left(\frac{\bar{z}_1 - \bar{z}_2}{4L} \right) e^{-|z_1 - z_2|^2/4L} \left(d\bar{z}_1 \otimes 1 + 1 \otimes d\bar{z}_2 \right)$$

hence

(5.16)
$$R_{\lambda}\partial K_L = \lambda^{-2}\partial K_{L/\lambda^2}$$

Moreover,

$$[Q, R_{\lambda}] = 0$$

The renormalization group equation and quantum master equation for F[L] are equivalent to

$$RG : \left(\frac{\partial}{\partial L} + \hbar \frac{\partial}{\partial \tilde{P}_L}\right) e^{F[L]/\hbar} = 0$$
$$QME : \left(Q + \hbar \frac{\partial}{\partial(\partial K_L)}\right) e^{F[L]/\hbar} = 0$$

Rescaling $L \to \lambda^2 L$ and applying the operator R^*_{λ} , we get

$$\begin{split} RG &: \left(\lambda^{-2}\frac{\partial}{\partial L} + \hbar \frac{\partial}{\partial (R_{\lambda}\tilde{P}_{L\lambda^{2}})}\right) e^{R_{\lambda}^{*}F[L\lambda^{2}]/\hbar} = 0\\ QME &: \left(Q + \hbar \frac{\partial}{\partial (R_{\lambda}\partial K_{L\lambda^{2}})}\right) e^{R_{\lambda}^{*}F[L\lambda^{2}]/\hbar} = 0 \end{split}$$

Using Eqn (5.15) and Eqn (5.16)

$$\begin{split} RG &: \left(\frac{\partial}{\partial L} + \lambda^{-2}\hbar\frac{\partial}{\partial\tilde{P}_L}\right)e^{R_{\lambda}^*F[L\lambda^2]/\hbar} = 0\\ QME &: \left(Q + \lambda^{-2}\hbar\frac{\partial}{\partial(\partial K_L)}\right)e^{R_{\lambda}^*F[L\lambda^2]/\hbar} = 0 \end{split}$$

Rescaling $\hbar \to \hbar \lambda^2$, it becomes

$$RG : \left(\frac{\partial}{\partial L} + \hbar \frac{\partial}{\partial \tilde{P}_L}\right) e^{F_{\lambda}[L]/\hbar} = 0$$
$$QME : \left(Q + \hbar \frac{\partial}{\partial(\partial K_L)}\right) e^{F_{\lambda}[L]/\hbar} = 0$$

which says that $\{F_{\lambda}[L]\}\$ satisfies the renormalization group equation and quantum master equation.

Remark 5.24. The above equation defines a flow on the space of the quantization, which is called local renormalization group flow in [Cos11].

Proposition 5.25. Let $S = \sum_{g \ge 0} \hbar^g S_g \in \mathcal{O}_{loc}(\mathrm{PV}_E^{*,*}[[t]])[[\hbar]]$ be the quantized BCOV action, which contains only holomorphic derivatives. Then we can choose S such that S_g contains 2g holomorphic derivatives.

Proof. The problem is local and we can work on the BCOV theory on \mathbb{C} . The effective action is given by

$$F[L] = \lim_{\epsilon \to 0} \hbar \log \left(\exp \left(\hbar \partial_{P_{\epsilon}^{L}} \right) \exp \left(S/\hbar \right) \right)$$

By Theorem 3.30,

$$\lim_{L\to 0} F[L] = S$$

hence

$$\lim_{L \to 0} F_{\lambda}[L] = \lim_{L \to 0} \lambda^{2\hbar \frac{\partial}{\partial \hbar} - 2} R_{\lambda}^* F[\lambda^2 L] = \lambda^{2\hbar \frac{\partial}{\partial \hbar} - 2} R_{\lambda}^* S$$

By Lemma 5.23, $\lambda^{2\hbar\frac{\partial}{\partial\hbar}-2}R^*_{\lambda}S$ also gives a quantization of the classical BCOV action S_0 . It's easy to check that

$$\lambda^{2\hbar\frac{\partial}{\partial\hbar}-2}R_{\lambda}^*S_0 = \lambda^{-2}R_{\lambda}^*S_0 = S_0$$

which allows us to choose S such that

$$\lambda^{2\hbar\frac{\partial}{\partial\hbar}-2}R_{\lambda}^*S = S$$

or equivalently

$$R^*_{\lambda}S_q = \lambda^{2-2g}S_q$$

which says precisely that S_g contains 2g holomorphic derivatives.

Remark 5.26. The rescaling condition $\lambda^{2\hbar\frac{\partial}{\partial\hbar}-2}R^*_{\lambda}S = S$ gives precisely the degree constraint in the deformation-obstruction complex that comes from the dilaton equation. In fact, it's easy to prove that the rescaling $F[L] \to F_{\lambda}[L]$ is also compatible with the dilaton equation. The proposition says that we can construct a quantization of the one-dimensional BCOV theory that is fixed by the local renormalization group flow.

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5.6. Virasoro equations. We will prove in this section that the quantization $\mathbf{F}[L]$ on the elliptic curve E satisfies additional symmetries, i.e., the Virasoro equations. This can be viewed as the mirror equations for the Virasoro constraints of Gromov-Witten invariants on elliptic curves, first discovered by [EHX97], and proved by [OP06b] in general.

We define the following operators E_m, Z_m for $m \ge -1$. If $m \ge 0$, then

$$E_m : \mathrm{PV}_E^{i,j}[[t]] \to \mathrm{PV}_E^{i,j}[[t]]$$

$$t^k \alpha \to t^{m+k} (k+i)_{m+1} \alpha$$

$$Z_m : \mathrm{PV}_E^{i,j}[[t]] \to \mathrm{PV}_E^{i,j+1}[[t]]$$

$$t^k \alpha \to t^{m+k} (k+i)_{m+1} d\bar{z} \wedge \alpha$$

where $(n)_m = n(n+1)\cdots(n+m-1)$ is the Pochhammer symbol. For m = -1, we have

$$\begin{split} E_{-1} : \mathrm{PV}_{E}^{i,j}[[t]] &\to \mathrm{PV}_{E}^{i,j}[[t]] \\ t^{k} \alpha &\to \begin{cases} t^{k-1} \alpha & k > 0 \\ 0 & k = 0 \end{cases} \\ Z_{-1} : \mathrm{PV}_{E}^{i,j}[[t]] &\to \mathrm{PV}_{E}^{i,j+1}[[t]] \\ t^{k} \alpha &\to \begin{cases} t^{k-1} d\bar{z} \wedge \alpha & k > 0 \\ 0 & k = 0 \end{cases} \end{split}$$

Both E_m and Z_m naturally induce the operators acting on $\mathcal{O}(\mathscr{E})$, which we denote by the same symbols.

Definition 5.27. We define the effective Virasoro operators $\{\mathcal{L}_m[L], \mathcal{D}_m[L]\}_{m \geq -1}$

(1) If $m \ge 0$, then

$$\mathcal{L}_m[L] = -(m+1)! \frac{\partial}{\partial(1 \cdot t^{m+1})} + E_m$$

and also

$$\mathcal{D}_m[L] = -(m+1)! \frac{\partial}{\partial (d\bar{z} \cdot t^{m+1})} + Z_m$$

which doesn't depend on the scale L.

(2) If m = -1, the operators $\mathcal{L}_{-1}[L]$ will depend on the scale L. Let Y[L] be the operator

$$Y[L][\alpha] = \begin{cases} \int_0^L du \bar{\partial}^* \partial e^{-uH} \alpha & \alpha \in \mathrm{PV}_E^{*,*} \\ 0 & \alpha \in t \operatorname{PV}_E^{*,*}[[t]] \end{cases}$$

and $\tilde{Y}[L]$ be the operator

$$\tilde{Y}[L][\alpha] = \begin{cases} \int_0^L du \bar{\partial}^* \partial e^{-uH} (d\bar{z} \wedge \alpha) & \alpha \in \mathrm{PV}_E^{*,*} \\ 0 & \alpha \in t \operatorname{PV}_E^{*,*}[[t]] \end{cases}$$

Recall that $S_3^{BCOV} \in \text{Sym}^3(\mathscr{E}^{\vee})$ is the local functional given by the order three component of the classical BCOV action. Then we define $\mathcal{L}_{-1}[L]$ by

$$\mathcal{L}_{-1}[L] = -\frac{\partial}{\partial(1)} + E_{-1} - Y[L] + \frac{1}{\hbar} \frac{\partial}{\partial(1)} S_3^{BCOV}$$

and $\mathcal{D}_{-1}[L]$ by

$$\mathcal{D}_{-1}[L] = -\frac{\partial}{\partial(d\bar{z})} + Z_{-1} - \tilde{Y}[L] + \frac{1}{\hbar} \frac{\partial}{\partial(d\bar{z})} S_3^{BCOV}$$

Note that $\frac{\partial}{\partial(1)}S_3^{BCOV}$ is precisely the Trace map.

Lemma 5.28. The operators $\{\mathcal{L}_m[L], \mathcal{D}_m[L]\}_{m \geq -1}$ satisfy the Virasoro relations

$$[\mathcal{L}_m[L], \mathcal{L}_n[L]] = (m-n)\mathcal{L}_{m+n}[L]$$
$$[\mathcal{L}_m[L], \mathcal{D}_n[L]] = (m-n)\mathcal{D}_{m+n}[L]$$
$$[\mathcal{D}_m[L], \mathcal{D}_n[L]] = 0$$

for all $m, n \geq -1$ and for any L.

Proof. This is a straight-forward check.

Remark 5.29. It should be noted that the above Virasoro relations are only valid for the one-dimensional case, i.e. for elliptic curves.

Recall that the dilaton operator is defined by

$$\mathcal{D} + 2\hbar \frac{\partial}{\partial \hbar} = Eu - \frac{\partial}{\partial (1 \cdot t)} + 2\hbar \frac{\partial}{\partial \hbar}$$

Lemma 5.30. The operators $\{\mathcal{L}_m[L], \mathcal{D}_m[L]\}_{m \geq -1}$ are compatible with renormalization group flow, quantum master equation and the dilaton operator in the following sense

$$\exp\left(\hbar\frac{\partial}{\partial P_{\epsilon}^{L}}\right)\mathcal{L}_{m}[\epsilon] = \mathcal{L}_{m}[L]\exp\left(\hbar\frac{\partial}{\partial P_{\epsilon}^{L}}\right)$$
$$\exp\left(\hbar\frac{\partial}{\partial P_{\epsilon}^{L}}\right)\mathcal{D}_{m}[\epsilon] = \mathcal{D}_{m}[L]\exp\left(\hbar\frac{\partial}{\partial P_{\epsilon}^{L}}\right)$$
$$[\mathcal{L}_{m}[L], Q + \hbar\Delta_{L}] = [\mathcal{D}_{m}[L], Q + \hbar\Delta_{L}] = 0$$
$$[\mathcal{L}_{m}[L], \mathcal{D} + 2\hbar\frac{\partial}{\partial\hbar}] = [\mathcal{D}_{m}[L], \mathcal{D} + 2\hbar\frac{\partial}{\partial\hbar}] = 0$$

Proof. This is a straight-forward check.

Proposition 5.31. Let $\mathbf{F}[L] + \delta \mathbf{G}[L]$ be the quantization of BCOV theory on the elliptic curve E coupled to dilaton. Then for each $m \ge -1$, there exists families of functionals $K_m[L], P_m[L] \in \hbar \mathcal{O}(\mathscr{E})[\delta][[\hbar]]$ satisfying

$$\mathcal{L}_{m}[L]e^{\mathbf{F}[L]/\hbar+\delta\mathbf{G}[L]/\hbar} = \left(Q + \hbar\Delta_{L} + \delta\left(\mathcal{D} + 2\hbar\frac{\partial}{\partial\hbar}\right)\right) \left(\frac{1}{\hbar}K_{m}[L]e^{\mathbf{F}[L]/\hbar+\delta\mathbf{G}[L]/\hbar}\right)$$
$$\mathcal{D}_{m}[L]e^{\mathbf{F}[L]/\hbar+\delta\mathbf{G}[L]/\hbar} = \left(Q + \hbar\Delta_{L} + \delta\left(\mathcal{D} + 2\hbar\frac{\partial}{\partial\hbar}\right)\right) \left(\frac{1}{\hbar}P_{m}[L]e^{\mathbf{F}[L]/\hbar+\delta\mathbf{G}[L]/\hbar}\right)$$

and the renormalization group flow equation

$$e^{\hbar\frac{\partial}{\partial P_{\epsilon}^{L}}}\left(K_{m}[\epsilon]e^{\mathbf{F}[\epsilon]/\hbar+\delta\mathbf{G}[\epsilon]/\hbar}\right) = K_{m}[L]e^{\mathbf{F}[L]/\hbar+\delta\mathbf{G}[L]/\hbar}$$
$$e^{\hbar\frac{\partial}{\partial P_{\epsilon}^{L}}}\left(P_{m}[\epsilon]e^{\mathbf{F}[\epsilon]/\hbar+\delta\mathbf{G}[\epsilon]/\hbar}\right) = P_{m}[L]e^{\mathbf{F}[L]/\hbar+\delta\mathbf{G}[L]/\hbar}$$

Proof. By the Virasoro relations, we only need to prove the case for $\mathcal{L}_{-1}[L], \mathcal{L}_2[L]$, and $\mathcal{D}_{-1}[L]$.

Given m, we will solve $K_m[L]$ by induction on the power of \hbar . The base case for \hbar^0 -order follows from the fact that the classical BCOV action $\mathbf{F}[0]$ satisfies

$$\mathcal{L}_m[0]e^{\frac{1}{\hbar}\mathbf{F}[0]} = \mathcal{D}_m[0]e^{\frac{1}{\hbar}\mathbf{F}[0]} = 0, \quad \forall m \ge -1$$

For g > 0, assume we have found

$$K_m^{< g}[L] = \sum_{i=1}^{g-1} \hbar^i K_{m,i}[L]$$

satisfying the corresponding renormalization group flow equation up to order \hbar^{g-1} , such that

$$\mathcal{L}_{m}[L]e^{\mathbf{F}[L]/\hbar+\delta\mathbf{G}[L]/\hbar} - \left(Q + \hbar\Delta_{L} + \delta\left(\mathcal{D} + 2\hbar\frac{\partial}{\partial\hbar}\right)\right) \left(\frac{1}{\hbar}K_{m}^{< g}[L]e^{\mathbf{F}[L]/\hbar+\delta\mathbf{G}[L]/\hbar}\right)$$
$$= \left(U_{g}[L]\hbar^{g-1} + O(\hbar^{g})\right)e^{\mathbf{F}[L]/\hbar+\delta\mathbf{G}[L]/\hbar}$$

for some $U_g[L]$. By the compatibility of $\mathcal{L}_m[L]$ with renormalization group flow, $\mathbf{F}_0[L] + \epsilon U_g[L]$ satisfies the classical renormalization group flow, where ϵ is an odd variable with $\epsilon^2 = 0$. In particular, the limit

$$\lim_{L \to 0} U_g[L] = U_g$$

exists as a local functional. On the other hand, by the compatibility of $\mathcal{L}_m[L]$ with quantum master equation and dilaton operator, we have

$$\left(Q + \hbar\Delta_L + \delta\left(\mathcal{D} + 2\hbar\frac{\partial}{\partial\hbar}\right)\right) \left(\left(U_g[L]\hbar^{g-1} + O(\hbar^g)\right)e^{\mathbf{F}[L]/\hbar + \delta\mathbf{G}[L]/\hbar}\right) = 0$$

The leading term gives

$$QU_{g}[L] + \delta (\mathcal{D} + 2g - 2) U_{g}[L] + \{\mathbf{F}_{0}[L], U_{g}[L]\}_{L} = 0$$

Taking the limit $L \to 0$, we find

$$QU_g + \delta \left(\mathcal{D} + 2g - 2 \right) U_g + \left\{ S^{BCOV}, U_g \right\} = 0$$

Observe that U_g has the same Hodge weight as $\mathcal{L}_m[L]\mathbf{F}_g[L]$, i.e., 2-2g-m. By Remark 5.10, we see that for m=-1,2, U_g is a trivial element in the cohomology of $Q + \delta (\mathcal{D} + 2g - 2) + \{S^{BCOV}, -\}$. Hence there exists local functional V_g such that

$$U_g = QV_g + \delta \left(\mathcal{D} + 2g - 2 \right) V_g + \left\{ S^{BCOV}, V_g \right\}$$

We can define

$$K_{m,g}[L] = V_g[L], \quad K_m^{\leq g}[L] = K_{m,g}[L] + \hbar^g K_m^{\leq g}[L]$$

where $V_g[L]$ is the effective functional such that $\mathbf{F}_0[L] + \epsilon V_g[L]$ satisfies the classical renormalization group flow for some odd variable ϵ with $\epsilon^2 = 0$. Therefore

$$\mathcal{L}_{m}[L]e^{\mathbf{F}[L]/\hbar+\delta\mathbf{G}[L]/\hbar} - \left(Q + \hbar\Delta_{L} + \delta\left(\mathcal{D} + 2\hbar\frac{\partial}{\partial\hbar}\right)\right) \left(\frac{1}{\hbar}K_{m}^{\leq g}[L]e^{\mathbf{F}[L]/\hbar+\delta\mathbf{G}[L]/\hbar}\right)$$
$$= O(\hbar^{g})e^{\mathbf{F}[L]/\hbar+\delta\mathbf{G}[L]/\hbar}$$

as desired. This proves the Proposition for the case of $\mathcal{L}_m[L]$. The proof for the case of $\mathcal{D}_m[L]$ is similar.

Corollary 5.32. The quantization $\mathbf{F}[L]$ of BCOV theory on the elliptic curve at $L = \infty$ satisfies the following Virasoro equations

(5.17)
$$\mathcal{L}_m[\infty]e^{\mathbf{F}[\infty]/\hbar} = \mathcal{D}_m[\infty]e^{\mathbf{F}[\infty]/\hbar} = 0 \quad on \quad \mathrm{H}^*(\mathscr{E}, Q)$$

for any $m \geq -1$.

Proof. It follows from the previous Proposition that there exists

$$K_m[L] \in \hbar \mathcal{O}(\mathscr{E})[[\hbar]] \quad \forall m \ge -1$$

satisfying certain renormalization group flow equation such that

$$\mathcal{L}_m[L]e^{\mathbf{F}[L]/\hbar} = (Q + \hbar\Delta_L) \left(\frac{1}{\hbar} K_m[L]e^{\mathbf{F}[L]/\hbar}\right)$$

Taking the limit $L \to \infty$, we find

$$\mathcal{L}_m[\infty]e^{\mathbf{F}[\infty]/\hbar} = (QK_m[\infty])e^{\mathbf{F}[\infty]/\hbar}$$

which is zero on Q-closed elements. The proof for $\mathcal{D}_m[\infty]$ is similar.

6. Higher genus mirror symmetry on elliptic curves

Mirror symmetry is a duality between symplectic geometry of Calabi-Yau manifolds (Amodel) and complex geometry of the mirror Calabi-Yau manifolds (B-model). In the case of one-dimensional Calabi-Yau manifolds, i.e. elliptic curves, the mirror map is simple to describe. Let E represent an elliptic curve. In the A-model, we have the moduli of (complexified) Kähler class $[\omega] \in H^2(E, \mathbb{C})$, which can be parametrized by the (complexified) symplectic volume

$$q = \int_E \omega$$

In the B-model, we have the moduli of inequivalent complex structures which is identified with $\mathcal{H}/SL(2,\mathbb{Z})$. Here \mathcal{H} is the upper-half plane, and we represent the elliptic curve E as $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ and identify τ in H under the modular transformation

The mirror map simply identifies the pair (E,q) with the pair (E,τ) via

$$(6.2) q = e^{2\pi i \tau}$$

and mirror symmetry predicts the equivalence between the Gromov-Witten theory of E in the A-model and certain quantum invariants of E in the B-model. We will show in this section that the quantum invariants in the B-model are precisely BCOV invariants constructed from the quantization of the classical BCOV action in the previous section. We prove that the BCOV invariants can be identified with the generating function of descendant Gromov-Witten invariants of the mirror elliptic curve, to all genera. This established the higher genus mirror symmetry on elliptic curves, as originally proposed in [BCOV94]. More precisely, let

$$\tilde{\omega} \in H^2(E, \mathbb{C})$$

be the class of the Poincare dual of a point. Let k_1, \dots, k_n be non-negative integers. We consider the following generating function of descendant Gromov-Witten invariants

(6.3)
$$\sum_{d\geq 0} q^d \left\langle \prod_{i=1}^n \tau_{k_i}(\tilde{\omega}) \right\rangle_{g,d} = \sum_{d\geq 0} \int_{[\overline{M}_{g,n}(E,d)]^{vir}} \prod_{i=1}^n \psi_i^{k_i} ev_i^*(\tilde{\omega})$$

where $\overline{M}_{g,n}(E,d)$ is the moduli space of stable degree d maps from genus g, n-pointed curves to E, and ev_i is the evaluation map at the *i*th marked point. It's proved in [OP06a] that (6.3) is a quasi-modular form in τ of weight $\sum_{i=1}^{n} (k_i + 2)$ under the identification $q = \exp(2\pi i \tau)$.

In the B-model, let $\mathbf{F}^{E_{\tau}}[L] = \sum_{g \ge 0} \hbar^g \mathbf{F}_g^{E_{\tau}}[L]$ be the effective functional on the polyvector fields $\mathrm{PV}_{E_{\tau}}^{*,*}[[t]]$ on the elliptic curve $E_{\tau} = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ constructed in the previous section. Since E_{τ} is compact, we can take the limit $L \to \infty$. Since $\lim_{L\to\infty} K_L=0$, the quantum master equation implies that

which implies that we have well-defined multi-linear maps on the Q-cohomology

$$\mathbf{F}_{g}^{E_{\tau}}[\infty]:\prod_{n}\operatorname{Sym}^{n}(\operatorname{H}^{*}(\operatorname{PV}_{E_{\tau}}^{*,*}[[t]],Q))\to\mathbb{C}$$

Let w be the linear coordinate on \mathbb{C} . We consider the following polyvector fields

(6.5)
$$\omega = \frac{i}{2\operatorname{im} \tau} \partial_w \wedge d\bar{u}$$

which is normalized such that Tr $\omega = \int_{E_{\tau}} (\omega \vee dw) \wedge dw = 1$. We consider

(6.6)
$$\mathbf{F}_{g}^{E_{\tau}}[\infty][t^{k_{1}}\omega,\cdots,t^{k_{n}}\omega]$$

We will prove that it is an almost holomorphic modular form of weight $\sum_{i=1}^{n} (k_i + 2)$. Therefore the following limit makes sense [KZ95]

(6.7)
$$\lim_{\bar{\tau}\to\infty} F_g^{E_{\tau}}[\infty][t^{k_1}\omega,\cdots,t^{k_n}\omega]$$

which gives a quasi-modular form with the same weight. The main theorem in this section is the following **Theorem 6.1.** For any genus $g \ge 2$, n > 0, and non-negative integers k_1, \dots, k_n , we have the identity

(6.8)
$$\sum_{d\geq 0} q^d \left\langle \prod_{i=1}^n \tau_{k_i}(\tilde{\omega}) \right\rangle_{g,d} = \lim_{\bar{\tau}\to\infty} F_g^{E_{\tau}}[\infty][t^{k_1}\omega,\cdots,t^{k_n}\omega]$$

under the identification $q = \exp(2\pi i \tau)$.

It should be noted that the mysterious $\bar{\tau} \to \infty$ limit appears in [BCOV94] to describe the holomorphic anomaly and the large radius limit behavior of the topological string amplitudes. It's argued by physics method in [BCOV94] that the quantum invariants constructed from Kodaira-Spencer gauge theory on Calabi-Yau manifolds can be identified with the Gromov-Witten invariants of its mirror Calabi-Yau under such limit. In our example of elliptic curves, the $\bar{\tau} \to \infty$ limit simply intertwines between the almost holomorphic modular forms and quasi-modular forms. This has also been observed in [ABK08] in the study of local mirror symmetry.

In Theorem 6.1, we only consider the input from $H^2(E, \mathbb{C})$ and its descendants, which is called stationary sector in [OP06a]. In fact, descendant Gromov-Witten invariants with arbitrary inputs on E can be obtained from the stationary sector via Virasoro equations proved in [OP06b]. Since we have proved that the same Virasoro equations hold for BCOV theory (see Corollary 5.32), it follows from Theorem 6.1 that mirror symmetry actually holds for arbitrary inputs.

The rest of this section is devoted to prove Theorem 6.1. We outline the structure as follows. In section 6.1, we analyze the BCOV propagator and give several equivalent descriptions that will be used. In section 6.2, we briefly review the Boson-Fermion correspondence in the theory of lattice vertex algebra. In section 6.3, we use Boson-Fermion correspondence to show that the partition function (6.3), computed by Okounkov-Pandharipande in [OP06a], can be written as Feynman graph integrals with the BCOV propagator. In section 6.4, we prove that (6.6) is an almost holomorphic modular form and analyze the $\bar{\tau} \to \infty$ limit. In section 6.5, we prove Theorem 6.1.

6.1. BCOV propagator on elliptic curves. Let $E_{\tau} = \mathbb{C}/\Lambda$ be the elliptic curve where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, τ lies in the upper half plane. We will use the following convention for

$$(6.9) q = e^{2\pi i \tau}$$

and also use the \mathbb{C}^* coordinate

$$(6.10) z = \exp(2\pi i w)$$

such that $z \sim zq$ on the elliptic curve. We choose the standard flat metric on E_{τ} , and let Δ be the Laplacian. The BCOV propagator is given by the kernel $\mathbf{P}_{\epsilon}^{L} = \int_{\epsilon}^{L} du \bar{\partial}^* \partial e^{-u\Delta}$, which is concentrated on $\mathbf{PV}_{E_{\tau}}^{0,0}$ component. We normalize the integral such that $\mathbf{P}_{\epsilon}^{L}$ is represented by

(6.11)
$$P_{\epsilon}^{L}(w_{1}, w_{2}; \tau, \bar{\tau}) = -\frac{1}{\pi} \int_{\epsilon}^{L} \frac{du}{4\pi u} \sum_{\lambda \in \Gamma} \left(\frac{\bar{w}_{12} - \bar{\lambda}}{4u}\right)^{2} e^{-|w_{12} - \lambda|^{2}/4u}$$

where $w_{12} = w_1 - w_2$. Note that it differs from the standard kernel by a factor $\frac{1}{\pi}$. This factor is purely conventional and this choice will be convenient for the later discussion. Let $E_2(\tau)$ be the second Eisenstein series which is a quasi-modular form of weight 2

$$E_2(\tau) = \frac{3}{\pi^2} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}}' \frac{1}{(m+n\tau)^2} = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}$$

where the sign \sum' indicates that (m, n) run through all $m \in \mathbb{Z}, n \in \mathbb{Z}$ with $(m, n) \neq (0, 0)$. $E_2^*(\tau, \bar{\tau})$ is the almost holomorphic modular form defined by

$$E_2^*(\tau,\bar{\tau}) = E_2(\tau) - \frac{3}{\pi \operatorname{Im} \tau}$$

Note that $E_2(\tau)$ can be recovered from $E_2^*(\tau, \bar{\tau})$ by taking the limit $\bar{\tau} \to \infty$ in the obvious sense.

Lemma 6.2. Under the limit $\epsilon \to 0, L \to \infty$, we have

(6.12)
$$P_0^{\infty}(w_1, w_2; \tau, \bar{\tau}) = -\frac{1}{4\pi^2} \wp(w_1 - w_2; \tau) - \frac{1}{12} E_2^*(\tau, \bar{\tau})$$

if $w_1 - w_2 \notin \Lambda$. Here $\wp(w; \tau)$ is Weierstrass's elliptic function

$$\wp(w;\tau) = \frac{1}{w^2} + \sum_{\lambda \in \Lambda, \lambda \neq 0} \left(\frac{1}{(w-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

Proof. This is a well-known result and we give a proof here. Let's denote by $w_{12} = w_1 - w_2$.

$$\begin{aligned} -\pi \mathbf{P}_{\epsilon}^{L}(w_{1}, w_{2}; \tau, \bar{\tau}) &= \int_{\epsilon}^{L} \frac{dt}{4\pi t} \sum_{m,n \in \mathbb{Z}} \left(\frac{\bar{w}_{12} - (m + n\bar{\tau})}{4t} \right)^{2} \exp\left(-|w_{12} - (m + n\tau)|^{2}/4t\right) \\ &= \int_{\epsilon}^{L} \frac{dt}{4\pi t} \sum_{m \in \mathbb{Z}} \left(\frac{\bar{w}_{12} - m}{4t} \right)^{2} \exp\left(-|w_{12} - m|^{2}/4t\right) \\ &+ \int_{\epsilon}^{L} \frac{dt}{4\pi t} \sum_{n \neq 0} \sum_{m \in \mathbb{Z}} \left(\left(\frac{\bar{w}_{12} - (m + n\bar{\tau})}{4t} \right)^{2} \exp\left(-|w_{12} - (m + n\tau)|^{2}/4t\right) \right) \\ &- \int_{m}^{m+1} dy \left(\frac{\bar{w}_{12} - (y + n\bar{\tau})}{4t} \right)^{2} \exp\left(-|w_{12} - (y + n\tau)|^{2}/4t\right) \right) \\ &+ \int_{\epsilon}^{L} \frac{dt}{4\pi t} \sum_{n \neq 0} \int_{-\infty}^{\infty} dy \left(\frac{\bar{w}_{12} - (y + n\bar{\tau})}{4t} \right)^{2} \exp\left(-|w_{12} - (y + n\tau)|^{2}/4t\right) \\ &= I_{1} + I_{2} + I_{3} \end{aligned}$$

 I_1 is easy to evaluate

$$\lim_{\substack{\epsilon \to 0 \\ L \to \infty}} I_1 = \int_0^\infty \frac{dt}{4\pi t} \sum_{m \in \mathbb{Z}} \left(\frac{\bar{w}_{12} - m}{4t}\right)^2 \exp\left(-|w_{12} - m|^2/4t\right)$$
$$= \sum_{m \in \mathbb{Z}} \frac{1}{(w_{12} - m)^2} \int_0^\infty \frac{dt}{4\pi t} \frac{1}{(4t)^2} \exp\left(-1/4t\right)$$
$$= \frac{1}{4\pi} \sum_{m \in \mathbb{Z}} \frac{1}{(w_{12} - m)^2}$$

Let

$$F(y) = \left(\frac{\bar{w}_{12} - (y + n\bar{\tau})}{4t}\right)^2 \exp\left(-|w_{12} - (y + n\tau)|^2/4t\right)$$
$$= \frac{1}{(w_{12} - y - n\tau)^2} G(u), \qquad u = t/|w_{12} - (y + n\tau)|^2$$

where $G(u) = \frac{1}{(4u)^2} \exp(-1/4u)$ which is a smooth and bounded function on $[0, \infty)$. Then

$$\frac{dF(y)}{dy} = \frac{2}{(w_{12} - y - n\tau)^3} G(u) + \left(\frac{1}{(w_{12} - y - n\tau)^3} + \frac{1}{(w_{12} - y - n\tau)^2(\bar{w}_{12} - y - n\bar{\tau})}\right) uG'(u)$$

It follows that the summation in ${\cal I}_2$ is absolutely convergent. Therefore

$$\lim_{\substack{\epsilon \to 0 \\ L \to \infty}} I_2 = \frac{1}{4\pi} \sum_{n \neq 0} \sum_{m \in \mathbb{Z}} \left(\frac{1}{(w_{12} - m - n\tau)^2} - \int_m^{m+1} dy \frac{1}{(w_{12} - y - n\tau)^2} \right)$$
$$= \frac{1}{4\pi} \sum_{n \neq 0} \sum_{m \in \mathbb{Z}} \frac{1}{(w_{12} - m - n\tau)^2}$$

To evaluate I_3 , notice that

$$\int_{-\infty}^{\infty} dy \left(\frac{\bar{w}_{12} - (y + n\bar{\tau})}{4t}\right)^2 \exp\left(-|w_{12} - (y + n\tau)|^2/4t\right)$$

=
$$\int_{-\infty}^{\infty} dy \frac{y^2 - (\operatorname{Im} w_{12} - n \operatorname{Im} \tau)^2}{(4t)^2} \exp\left(-y^2/4t - (\operatorname{Im} w_{12} - n \operatorname{Im} \tau)^2/4t\right)$$

=
$$-\frac{\sqrt{\pi}((\operatorname{Im} w_{12} - n \operatorname{Im} \tau)^2/t - 2)}{8t^{1/2}} \exp\left(-(\operatorname{Im} w_{12} - n \operatorname{Im} \tau)^2/4t\right)$$

=
$$t \frac{d}{dt} \left(-\frac{\pi}{(4\pi t)^{1/2}} \exp\left(-(\operatorname{Im} w_{12} - n \operatorname{Im} \tau)^2/4t\right)\right)$$

Therefore

$$\lim_{\substack{\epsilon \to 0 \\ L \to \infty}} I_3 = -\lim_{\substack{\epsilon \to 0 \\ L \to \infty}} \frac{1}{4} \sum_{n \neq 0} \left(\frac{1}{(4\pi t)^{1/2}} \exp\left(-(\operatorname{Im} w_{12} - n \operatorname{Im} \tau)^2 / 4t\right) \right) \Big|_{\epsilon}^{L}$$
$$= -\lim_{\substack{\epsilon \to 0 \\ L \to \infty}} \frac{1}{4 \operatorname{Im} \tau} \sum_{n \neq 0} \left(\frac{1}{(4\pi t)^{1/2}} \exp\left(-(a-n)^2 / 4t\right) \right) \Big|_{\epsilon}^{L}, \quad a = \operatorname{Im} w_{12} / \operatorname{Im} \tau, 0 \le a < 1$$

Obviously,

$$\lim_{\epsilon \to 0} \sum_{n \neq 0} \left(\frac{1}{(4\pi\epsilon)^{1/2}} \exp\left(-(a-n)^2/4\epsilon\right) \right) = 0$$

The Poisson summation formula gives

$$\sum_{n \in \mathbb{Z}} \left(\frac{1}{(4\pi L)^{1/2}} \exp\left(-(a-n)^2/4L\right) \right) = \sum_{m \in \mathbb{Z}} \exp\left(-4\pi^2 m^2 L + 2\pi i m a\right)$$

hence

$$\lim_{L \to \infty} \sum_{n \in \mathbb{Z}} \left(\frac{1}{(4\pi L)^{1/2}} \exp\left(-(a-n)^2/4L\right) \right)$$
$$= \lim_{L \to \infty} \sum_{m \in \mathbb{Z}} \exp\left(-4\pi^2 m^2 L + 2\pi i m a\right) = 1$$

Adding the three terms together, we find

$$\lim_{\substack{\epsilon \to 0 \\ L \to \infty}} \left(-\pi P_{\epsilon}^{L}(w_{1}, w_{2}; \tau, \bar{\tau}) \right) = \frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \frac{1}{(w_{12} - m - n\tau)^{2}} - \frac{1}{4 \operatorname{Im} \tau}$$
$$= \frac{1}{4\pi} \wp(w_{12}; \tau) + \frac{1}{12\pi} E_{2}^{*}(\tau; \bar{\tau})$$

We will use the following notation to represent the $\bar{\tau} \to \infty$ limit, which we simply throw away the term involving $\frac{1}{\operatorname{Im} \tau}$

$$\begin{aligned} \mathbf{P}_{0}^{\infty}(w_{1}, w_{2}; \tau, \infty) &\equiv \lim_{\bar{\tau} \to \infty} \mathbf{P}_{0}^{\infty}(w_{1}, w_{2}; \tau, \bar{\tau}) \\ &= -\frac{1}{4\pi^{2}} \wp(w_{1} - w_{2}; \tau) - \frac{1}{12} E_{2}(\tau) \\ &= -\frac{1}{4\pi^{2}} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \frac{1}{(w_{1} - w_{2} - (m + n\tau))^{2}} \end{aligned}$$

or simply $P_0^{\infty}(\tau, \infty)$ if no explicit coordinates are needed. We can also go to the \mathbb{C}^* coordinate z using the formula

$$\sum_{m \in \mathbb{Z}} \frac{1}{(w+m)^2} = -4\pi^2 \frac{z}{(1-z)^2}, \ z = \exp\left(2\pi i w\right)$$

hence

$$P_0^{\infty}(w_1, w_2; \tau, \infty) = \sum_{n \in \mathbb{Z}} \frac{z_1 z_2 q^n}{(z_1 - z_2 q^n)^2}, \quad z_k = \exp\left(2\pi i w_k\right), k = 1, 2$$

If we further assume that w_1, w_2 takes values in $\{a + b\tau | 0 \le a, b < 1\}$, then we have the following relation

$$|qz_2| < |z_1| < |q^{-1}z_2|$$

and we get the power series expression

(6.13)
$$P_0^{\infty}(w_1, w_2; \tau, \infty) = \frac{z_1 z_2}{(z_1 - z_2)^2} + \sum_{m \ge 1} \frac{m z_1^m z_2^{-m} q^m}{1 - q^m} + \sum_{m \ge 1} \frac{m z_1^{-m} z_2^m q^m}{1 - q^m}$$

Later we will use this formula to give the Feynman diagram interpretation of the Gromov-Witten invariants on the elliptic curve. 6.2. Boson-Fermion correspondence. In this section, we discuss some examples of vertex algebra as well as their representations. We collect the basic results on Boson-Fermion correspondence that will be used to prove mirror symmetry. For more details, see [Kac98][MJD00].

6.2.1. Free bosons. The system of free boson is described by the infinite dimensional Lie algebra with basis $\{\alpha_n\}_{n\in\mathbb{Z}}$ and the commutator relations

(6.14)
$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}, \quad n, m \in \mathbb{Z}$$

The irreducible representations $\{\mathbf{H}_p^B\}_{p\in\mathbb{R}}$ are indexed by the real number p called "momentum". For each \mathbf{H}_p^B , there exists an element $|p\rangle \in \mathbf{H}_p^B$, which we call "vacuum", satisfying

(6.15)
$$\alpha_0 |p\rangle = p|p\rangle, \ \alpha_n |p\rangle = 0, \ n > 0$$

and the whole Fock space \mathbf{H}_p^B is given by

$$\mathbf{H}_{p}^{B} = \text{linear span of} \left\{ \alpha_{-i_{1}}^{k_{1}} \alpha_{-i_{2}}^{k_{2}} \cdots \alpha_{-i_{n}}^{k_{n}} |p\rangle \middle| i_{1} > i_{2} > \cdots i_{n} > 0, k_{1}, \cdots, k_{n} \ge 0, n \ge 0 \right\}$$

We will be interested in the Fock space with zero momentum, where the vacuum vector is also annihilated by α_0

$$\alpha_n |0\rangle = 0, \quad \forall n \ge 0$$

 $\{\alpha_{-n}\}_{n>0}$ are called *creation operators*, and $\{\alpha_n\}_{n>0}$ are called *annihilation operators*. We define the normal ordering ::_B by putting all the annihilation operators to the right,

(6.16)
$$: \alpha_n \alpha_m :_B = \begin{cases} \alpha_n \alpha_m & \text{if } n \le 0\\ \alpha_m \alpha_n & \text{if } n > 0 \end{cases}$$

and similarly for the case with more α 's. Here the subscript "B" denotes the bosons in order to distinguish with the fermionic normal ordering that will be discussed later. It's useful to collect α_n 's to form the following *field*

(6.17)
$$\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1}$$

then we have the following relation

$$\alpha(z)\alpha(w) = \sum_{n \ge 1} nz^{-n-1}w^{n-1} + : \alpha(z)\alpha(w) :_B = \frac{1}{(z-w)^2} + : \alpha(z)\alpha(w) :_B, \text{ if } |z| > |w|$$

This provides a convenient way to organize the data of the operators and the normal ordering relations. We can construct the Virasoro operators acting on \mathbf{H}_p^B via normal ordering

(6.18)
$$L_n = \frac{1}{2} \sum_{i \in \mathbb{Z}} : \alpha_i \alpha_{n-i} :_B$$

which satisfies the Virasoro algebra with central charge 1

(6.19)
$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}, \quad \forall n, m \in \mathbb{Z}$$

If we consider the corresponding field

(6.20)
$$L(z) = \sum_{n} L_{n} z^{-n-2}$$

then we can write

(6.21)
$$L(z) = \frac{1}{2} : \alpha(z)^2 :_B$$

 ${\cal L}_0$ is called the "energy operator" and has the following expression

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n \ge 1} \alpha_{-n}\alpha_n$$

which acts on basis of \mathbf{H}_p^B as

$$L_0 \alpha_{-i_1}^{k_1} \alpha_{-i_2}^{k_2} \cdots \alpha_{-i_n}^{k_n} |p\rangle = \left(\frac{1}{2}p^2 + \sum_{a=1}^n k_a i_a\right) \alpha_{-i_1}^{k_1} \alpha_{-i_2}^{k_2} \cdots \alpha_{-i_n}^{k_n} |p\rangle$$

The dual space ${\rm H}_p^{B*}$ can be constructed similarly from the dual vacuum element $\langle p|\in {\rm H}_p^{B*}$ such that

(6.22)
$$\langle p|\alpha_0 = p\langle p|, \ \langle p|\alpha_{-n} = 0, n > 0$$

$$\mathbf{H}_{p}^{B*} = \text{linear span of} \left\{ \left\langle p | \alpha_{i_{n}}^{k_{n}} \cdots \alpha_{i_{2}}^{k_{2}} \alpha_{i_{1}}^{k_{1}} \right| i_{1} > i_{2} > \cdots = i_{n} > 0, \ k_{1}, \cdots, k_{n} \ge 0, n \ge 0 \right\}$$

The natural pairing

$$\mathbf{H}_p^{B*} \otimes \mathbf{H}_p^B \to \mathbb{R}$$

is given by

$$\langle p | \alpha_{j_m}^{l_m} \cdots \alpha_{j_2}^{l_2} \alpha_{j_1}^{l_1} \otimes \alpha_{-i_1}^{k_1} \alpha_{-i_2}^{k_2} \cdots \alpha_{-i_n}^{k_n} | p \rangle \to \langle p | \alpha_{j_m}^{l_m} \cdots \alpha_{j_2}^{l_2} \alpha_{j_1}^{l_1} \alpha_{-i_1}^{k_1} \alpha_{-i_2}^{k_2} \cdots \alpha_{-i_n}^{k_n} | p \rangle$$

and the normalization condition

 $\langle p||p\rangle = 1$

There is a natural identification of the bosonic Fock space of integral momentum with polynomial algebra $\mathbb{C}[z, z^{-1}, x_1, x_2, \cdots]$ as follows. Let

$$H(x) = \exp\left(\sum_{n=1}^{\infty} x_n \alpha_n\right)$$

then

$$\alpha_{-i_1}^{k_1} \alpha_{-i_2}^{k_2} \cdots \alpha_{-i_n}^{k_n} | m \rangle \to \sum_{l \in \mathbb{Z}} z^l \langle l | e^{H(x)} \alpha_{-i_1}^{k_1} \alpha_{-i_2}^{k_2} \cdots \alpha_{-i_n}^{k_n} | m \rangle \in \mathbb{C}[z, z^{-1}, x_1, x_2, \cdots], \quad m \in \mathbb{Z}$$

Under this isomorphism the bosonic operators are represented by

$$\alpha_n \to \frac{\partial}{\partial x_n}, \ \alpha_{-n} \to nx_n, \ n \ge 1$$

and

$$\alpha_0 \to z \frac{\partial}{\partial z}$$

6.2.2. Free Fermions. We consider the free fermionic system that is described by the infinite dimensional Lie superalgebra with odd basis $\{b_n\}, \{c_n\}$, indexed by $n \in \mathbb{Z} + 1/2$, and the anti-commutator relations

(6.23)
$$\{b_n, c_m\} = \delta_{m+n,0}, \ \{b_n, b_m\} = \{c_n, c_m\} = 0, \ \forall n, m \in \mathbb{Z} + 1/2$$

The irreducible representation is given by the Fermionic Fock space \mathbf{H}^{F} , which contains the vacuum $|0\rangle$ satisfying

(6.24)
$$b_n|0\rangle = c_n|0\rangle = 0, \ \forall n \in \mathbb{Z}^{\ge 0} + 1/2$$

and \mathbf{H}^F is constructed by

 $\mathbf{H}^{F} \hspace{.1in} = \hspace{.1in} \text{linear span of} \hspace{.1in}$

$$\{b_{-i_1} \cdots b_{-i_s} c_{-j_1} \cdots c_{-j_t} |0\rangle | 0 < i_1 < i_2 < \cdots < i_s, 0 < j_1 < j_2 < \cdots < j_t, s, t \ge 0\}$$

The normal ordering : $:_F$ is defined similarly with extra care about the signs

(6.25)
$$: b_n c_m :_F = \begin{cases} b_n c_m & \text{if } n < 0 \\ -c_m b_n & \text{if } n > 0 \end{cases}$$

where the subscript "F" refers to the fermions. We can also construct the Virasoro operators acting on \mathbf{H}^F via

(6.26)
$$L_n = \frac{1}{2} \sum_{k+l=n} (l-k) : b_k c_l :_F = \sum_{k \in \mathbb{Z} + 1/2} (n/2 - k) : b_k c_{n-k} :_F, \quad n \in \mathbb{Z}$$

which satisfies the Virasoro algebra with central charge 1

(6.27)
$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0} \quad \forall n, m \in \mathbb{Z}$$

Similar to the bosonic case, we can collect the fermionic operators to form the fermionic fields

(6.28)
$$b(z) = \sum_{n \in \mathbb{Z} + 1/2} b_n z^{-n-1/2}, \ c(z) = \sum_{n \in \mathbb{Z} + 1/2} c_n z^{-n-1/2}$$

such that the normal ordering relations can be written in the simple form

(6.29)
$$b(z)c(w) = \frac{1}{z-w} + :b(z)c(w):_F, \text{ if } |z| > |w|$$

The Virasoro operators can be collected

$$L(z) = \sum_{n \in Z} L_n z^{-n-2}$$

and it's easy to see that

(6.30)
$$L(z) = \frac{1}{2} : \partial b(z)c(z) :_F -\frac{1}{2} : b(z)\partial c(z) :_F$$

The energy operator L_0 has the expression

$$L_0 = \sum_{k \in \mathbb{Z}^{\ge 0} + 1/2} k(b_{-k}c_k + c_{-k}b_k)$$

6.2.3. From fermions to bosons. Consider the above free fermionic system with fields b(z), c(z). We construct the following bosonic field

(6.31)
$$\alpha(z) =: b(z)c(z):_F$$

In mode expansions,

(6.32)
$$\alpha_n = \sum_{k \in \mathbb{Z} + 1/2} : b_k c_{n-k} :_F$$

It's easy to see that the following commutator relations hold as operators on \mathbf{H}^F

(6.33)
$$\begin{aligned} [\alpha_m, \alpha_n] &= m \delta_{m+n,0} \\ [\alpha_m, b_n] &= b_{m+n} \\ [\alpha_m, c_n] &= -c_{m+n} \end{aligned}$$

Therefore $\alpha(z)$ defines a free bosonic field. Moreover, the Virasoro operators coincide for bosons and fermions, i.e.

(6.34)
$$L(z) = \frac{1}{2} : \alpha(z)^2 :_B = \frac{1}{2} : \partial b(z)c(z) :_F -\frac{1}{2} : b(z)\partial c(z) :_F$$

Consider the charge operator α_0 , which corresponds to bosonic momentum operator

$$\alpha_0 = \sum_{k \in \mathbb{Z}^{\ge 0} + 1/2} (b_{-k}c_k - c_{-k}b_k)$$

 α_0 acts on the basis of the Fock space as

$$\alpha_0 b_{-i_1} \cdots b_{-i_s} c_{-j_1} \cdots c_{-j_t} |0\rangle = (s-t) b_{-i_1} \cdots b_{-i_s} c_{-j_1} \cdots c_{-j_t} |0\rangle$$

 \mathbf{H}^F is decomposed into eigenvectors of α_0

(6.35)
$$\mathbf{H}^F = \bigoplus_{m \in \mathbb{Z}} \mathbf{H}_m^F$$

such that each \mathbf{H}_m^F gives a representation of the free bosons. For each \mathbf{H}_m^F , there's a special element given by

(6.36)
$$|m\rangle = \begin{cases} |0\rangle & \text{if } m = 0\\ b_{-m+1/2} \cdots b_{-1/2} |0\rangle & \text{if } m > 0\\ c_{m+1/2} \cdots c_{-1/2} |0\rangle & \text{if } m < 0 \end{cases}$$

.

It's easy to see that

$$\alpha_n | m \rangle = 0, \ \forall n \in \mathbb{Z}^{>0}, \forall m \in \mathbb{Z}$$

Proposition 6.3. The representation H_m^F of free bosons is isomorphic to the Fock space H_m^B with momentum $m \in \mathbb{Z}$ under the identification of vacuums

$$|m\rangle \Leftrightarrow \begin{cases} |0\rangle & \text{if } m = 0\\ b_{-m+1/2} \cdots b_{-1/2} |0\rangle & \text{if } m > 0\\ c_{m+1/2} \cdots c_{-1/2} |0\rangle & \text{if } m < 0 \end{cases}$$

6.2.4. From bosons to fermions. Let P be the creation operator for momentum on bosonic fock space defined by

$$e^P|m\rangle = |m+1\rangle$$

It follows that we have the following commutator relation

$$[\alpha_0, P] = 1$$

We define formally

(6.37)
$$\phi(z) = P + \alpha_0 \log z + \sum_{n \neq 0} \frac{\alpha_{-n}}{n} z^n$$

where $\alpha(z)$ is related to $\phi(z)$ by

$$\alpha(z) = \partial_z \phi(z)$$

Since $\alpha_0|0\rangle = 0$, we view α_0 as annihilation operator and P as creation operator, and extend the bosonic normal ordering by

$$(6.38) \qquad \qquad :\alpha_0 P:_B =: P\alpha_0:_B = P\alpha_0$$

Direct calculation shows

$$\phi(z)\phi(w) = \ln(z-w) + : \phi(z)\phi(w) :_B, \text{ if } |w| < |z|$$

Proposition 6.4. Under the above identification of fermionic Fock space \mathbf{H}^F with bosonic Fock space $\bigoplus_{m \in \mathbb{Z}} \mathbf{H}^B_m$, the fermionic fields can be represented by bosonic fields acting on $\bigoplus_{m \in \mathbb{Z}} \mathbf{H}^B_m$ as

(6.39)
$$b(z) =: e^{\phi(z)} :_B, \quad c(z) =: e^{-\phi(z)} :_B$$

As an example, we can put the product of two fermionic fields into normal ordered form in two ways. Within fermionic fields

$$b(z)c(w) = \frac{1}{z-w} + : b(z)c(w) :_F$$

or using the bosonic representation

$$b(z)c(w) =: e^{\phi(z)} :_B : e^{-\phi(w)} :_B = \frac{1}{z - w} : e^{\phi(z) - \phi(w)} :_B$$

where in the second equality we have used the Wick's theorem (see for example [MJD00]). Therefore

(6.40)
$$: b(z)c(w) :_{F} = \frac{1}{z-w} \left(: e^{\phi(z)-\phi(w)} :_{B} - 1\right)$$

See [MJD00] for a more systematic treatment of the above formula.

6.3. Gromov-Witten invariants on elliptic curves.

6.3.1. Stationary Gromov-Witten invariants. Let E be an elliptic curve. The Gromov-Witten theory on E concerns the moduli space

$$\overline{M}_{g,n}(E,d)$$

parametrizing connected, genus g, n-pointed stable maps to E of degree d. Let

$$ev_i: M_{q,n}(E,d) \to E$$

be the morphism defined by evaluation at the *i*th marked point. Let $\tilde{\omega}$ denote the Poincaré dual of the point class, $\psi_i \in H^2(\overline{M}_{g,n}(E,d),\mathbb{Q})$ the first Chern class of the cotangent line bundle L_i on the moduli space $\overline{M}_{g,n}(E,d)$. By the Virasoro constraints proved in [OP06b], the full descendant Gromov-Witten invariants on E are determined by the *stationary sector*, i.e.,

(6.41)
$$\left\langle \prod_{i=1}^{n} \tau_{k_i} \tilde{\omega} \right\rangle_{g,d} = \int_{\left[\overline{M}_{g,n}(E,d)\right]^{vir}} \prod_{i=1}^{n} \psi_i^{k_i} e v_i^*(\tilde{\omega})$$

where $\left[\overline{M}_{g,n}(E,d)\right]^{vir}$ is the virtual fundamental class of $\overline{M}_{g,n}(E,d)$. The integral vanishes unless the dimension constraint

(6.42)
$$\sum_{i=1}^{n} k_i = 2g - 2$$

is satisfied. Therefore we can omit the subscript g in the bracket $\langle \rangle$. We can also consider the disconnected theory as in [OP06a], where the domain curve of the stable map is allowed to have disconnected components. The bracket $\langle \rangle^{dis}$ will be used for the disconnected Gromov-Witten invariants. It's proved in [OP06a] that the stationary Gromov-Witten invariants can be computed through fermionic vertex algebra, which we now describe. Let H^{F} be the Fock space of free fermionic algebra with fermionic fields $b(z), c(z), \mathrm{H}_{0}^{F}$ is the subspace annihilated by the charge operator α_{0} . Consider the following operator

(6.43)
$$\mathcal{E}(z;\lambda) = \sum_{n \in \mathbb{Z}} \mathcal{E}_n(\lambda) z^{-n-1} =: b(e^{\lambda/2}z)c(e^{-\lambda/2}z):_F + \frac{1}{(e^{\lambda/2} - e^{-\lambda/2})z}$$

In components, we can formally write

(6.44)
$$\mathcal{E}_n(\lambda) = \oint dz z^n \mathcal{E}(z;\lambda) = \begin{cases} \sum_{k \in \mathbb{Z} + \frac{1}{2}} e^{\lambda k} : b_{-k} c_k : + \frac{1}{(e^{\lambda/2} - e^{-\lambda/2})} & \text{if } n = 0\\ \sum_{k \in \mathbb{Z} + \frac{1}{2}} b_{n-k} c_k e^{\lambda (k-n/2)} & \text{if } n \neq 0 \end{cases}$$

Here $\oint = \frac{1}{2\pi i} \int_C$, where C is a circle surrounding the origin. Decomposing in terms of powers of λ , we define

(6.45)
$$\mathcal{E}(z;\lambda) = \sum_{n \ge -1} \lambda^n \mathcal{E}^{(n)}(z)$$

Consider the following n-point partition function for the stationary GW invariants of the elliptic curve E:

(6.46)
$$F_E(\lambda_1, \cdots, \lambda_n; q) = \sum_{d \ge 0} q^d \left\langle \prod_{i=1}^n \left(\sum_{k \ge -2} \lambda_i^k \tau_k(\tilde{\omega}) \right) \right\rangle_d^{dis}$$

The bracket is the disconnected descendant GW invariants.

Proposition 6.5 ([OP06a]). The above partition function can be written as a trace on the fermionic Fock space

(6.47)
$$\sum_{d\geq 0} q^d \left\langle \prod_{i=1}^n \left(\sum_{k\geq -2} \lambda_i^k \tau_k(\tilde{\omega}) \right) \right\rangle_d^{dis} = \operatorname{Tr}_{\mathrm{H}_0^F} q^{L_0} \prod_{i=1}^n \frac{1}{\lambda_i} \oint dz \mathcal{E}(z;\lambda_i)$$

where we use the convention as in [OP06a]

$$\tau_{-2}(\tilde{\omega}) = 1, \quad \tau_{-1}(\tilde{\omega}) = 0$$

In [OP06a], the fermionic Fock space is represented by the infinite wedge space $\Lambda^{\frac{\infty}{2}}V$, where V is a linear space with basis <u>k</u> indexed by the half-integers:

$$V = \bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}\underline{k}$$

For reader's convenience, the notations used in [OP06a] are related to our notations here via

$$\psi_k \to b_{-k}, \ \psi_k^* \to c_k, \ C \to \alpha_0, \ H \to L_0$$

6.3.2. *Bosonization*. Using fermion-boson correspondence, we can have a bosonic description of the Gromov-Witten invariants on the elliptic curve. Following the bosonization rule

$$b(z) =: e^{\phi(z)} :_B, \ c(z) =: e^{-\phi(z)} :_B, \ \text{where } \phi(z) = P + \alpha_0 \log z + \sum_{n \neq 0} \frac{\alpha_{-n}}{n} z^n$$

where P is the creation operator for momentum. Using Eqn (6.40), we can write $\mathcal{E}(z; \lambda)$ in terms of bosonic fields

$$\mathcal{E}(z;\lambda) = \frac{1}{(e^{\lambda/2} - e^{-\lambda/2})z} : e^{\phi(e^{\lambda/2}z) - \phi(e^{-\lambda/2}z)} :_B$$

Let $\mathcal{S}(t)$ be the function

$$S(t) = \frac{e^{t/2} - e^{-t/2}}{t} = \frac{\sinh(t/2)}{t/2}$$

Then

$$\mathcal{E}(z;\lambda) = \frac{1}{\lambda \mathcal{S}(\lambda)z} : \exp\left(\mathcal{S}(\lambda z \partial_z)(\lambda z \alpha(z))\right) :_B, \quad \alpha(z) = \partial_z \phi(z)$$

The following lemma on the interpretation of the factor $\frac{1}{S(\lambda)}$ will be used later in the Feynman diagram representation of the descendant Gromov-Witten invariants.

Lemma 6.6.

$$\frac{1}{\mathcal{S}(\lambda)} = \exp\left(\frac{\lambda^2}{2} \left. \mathcal{S}\left(\frac{1}{2\pi i} \lambda \partial_{w_1}\right) \mathcal{S}\left(\frac{1}{2\pi i} \lambda \partial_{w_2}\right) \left(\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{(2\pi i)^2} \frac{1}{(w_1 - w_2 + n)^2}\right) \right|_{w_1 = w_2}\right)$$
Proof. Since $\mathcal{S}(t)$ is an even function of t, we have

$$\begin{aligned} \frac{\lambda^2}{2} \mathcal{S}\left(\frac{1}{2\pi i}\lambda\partial_{w_1}\right) \mathcal{S}\left(\frac{1}{2\pi i}\lambda\partial_{w_2}\right) \left(\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi i)^2} \frac{1}{(w_1 - w_2 + n)^2}\right) \bigg|_{w_1 = w_2} \\ &= \left. (\lambda/2\pi i)^2 \mathcal{S}((\lambda/2\pi i)\partial_w)^2 \left(\sum_{n\geq 1} \frac{1}{(w+n)^2}\right) \right|_{w=0} \\ &= \left. \sum_{k\geq 1} \frac{2(\lambda/2\pi i)^{2k}}{(2k)!} \left(\frac{\partial}{\partial w}\right)^{2k-2} \left(\sum_{n\geq 1} \frac{1}{(w+n)^2}\right) \right|_{w=0} \\ &= \left. \sum_{k\geq 1} \frac{(\lambda/2\pi i)^{2k}}{k} \sum_{n\geq 1} \frac{1}{n^{2k}} \\ &= \left. \sum_{n\geq 1} \sum_{k\geq 1} \frac{(\lambda/2\pi in)^{2k}}{k} \right|_{k=0} \\ &= -\sum_{n\geq 1} \ln\left(1 + \frac{\lambda^2}{(2\pi)^2n^2}\right) \end{aligned}$$

On the other hand, from the formula $\frac{\sin \lambda}{\lambda} = \prod_{\geq 1} \left(1 - \frac{\lambda^2}{n^2 \pi^2} \right)$, we see that

$$S(\lambda) = \frac{\sinh \lambda/2}{\lambda/2} = \prod_{n \ge 1} \left(1 + \frac{\lambda^2}{n^2 (2\pi)^2} \right)$$

this proves the lemma.

6.3.3. Feynman Diagram Representation. Let w be the \mathbb{C} coordinate where the elliptic curve is defined via the equivalence: $w \sim w + 1 \sim w + \tau$. We identify z with the coordinate on \mathbb{C}^* , such that

$$z = \exp(2\pi i w)$$

Consider the following bosonic lagrangian on $\mathrm{PV}_{E_\tau}^{0,0}$ coming from the above bosonization

(6.48)
$$\sum_{k\geq -1} \lambda^k \mathcal{L}^{(k)}(\mu(w)) \equiv \frac{1}{\lambda} \exp\left(\mathcal{S}\left(\frac{\lambda}{2\pi i}\partial_w\right)(\lambda\mu(w))\right), \quad k\geq -1$$

where on the right hand side, we can expand the lagrangian in terms of powers of λ , which defines $\mathcal{L}^{(k)}$. Let C be a representative of the homology class of the circle [0,1] on the

elliptic curve. Let $I_C^{(k)}$ be the functional on $\mathrm{PV}_{E_\tau}^{0,0}$ given by

$$I_C^{(k)}[\mu] = \int_C dw \mathcal{L}^{(k)}(\mu(w)), \ \ \mu \in \mathrm{PV}_{E_{ au}}^{0,0}$$

Proposition 6.7. The stationary GW invariants can be represented by Feynman integrals

$$\frac{\sum_{d\geq 0} q^d \left\langle \prod_{i=1}^n \tau_{k_i}(\tilde{\omega}) \right\rangle_d^{dis}}{\sum_{d\geq 0} q^d \left\langle 1 \right\rangle_d^{dis}} = \lim_{\bar{\tau} \to \infty} \lim_{\substack{\epsilon \to 0 \\ L \to \infty}} W^{dis} \left(\mathbf{P}_{\epsilon}^L; I_{C_1}^{(k_i+1)}, \cdots, I_{C_n}^{(k_n+1)} \right)$$

where the C_i 's are representatives of the homology class of the cycle [0,1] and are chosen to be disjoint. W^{dis} is given by the weighted summation of all Feynman diagrams (possibly disconnected) with n vertices $I_{C_1}^{(k_1+1)}, \dots, I_{C_n}^{(k_n+1)}$ and the propagator P_{ϵ}^L which is the BCOV propagator. The normalization factor on the LHS is

$$\sum_{d\geq 0} q^d \langle 1 \rangle_d^{dis} = \frac{1}{\prod_{i=1}^{\infty} (1-q^i)}$$

Proof. We will use w's for coordinates on \mathbb{C} and z's for coordinates on \mathbb{C}^* . We use the conventions that are used in section 6.1. The BCOV propagator can be written as

$$\mathbf{P}_{\epsilon}^{L}(w_{1}, w_{2}; \tau, \bar{\tau}) = -\frac{1}{\pi} \int_{\epsilon}^{L} \frac{du}{4\pi u} \sum_{\lambda \in \Gamma} \left(\frac{\bar{w}_{12} - \bar{\lambda}}{4u}\right)^{2} e^{-|w_{12} - \lambda|^{2}/4u}, \text{ where } w_{12} = w_{1} - w_{2}$$

and under the limit $\epsilon \to 0, L \to \infty, \bar{\tau} \to \infty$,

$$P_0^{\infty}(w_1, w_2; \tau, \infty) = \sum_{m \in \mathbb{Z}} \frac{z_1 z_2 q^m}{(z_1 - z_2 q^m)^2} = \frac{z_1 z_2}{(z_1 - z_2)^2} + \sum_{m \ge 1} \frac{m z_1^m z_2^{-m} q^m}{1 - q^m} + \sum_{m \ge 1} \frac{m z_1^{-m} z_2^m q^m}{1 - q^m}$$

where $z_i = \exp(2\pi i w_i)$ and $|q| \le |z_i| < 1$ for i = 1, 2. Let $\{C_i\}_{1 \le i \le n}$ be disjoint cycles lying in the annulus $\{z \in \mathbb{C}^* | |q| < |z| < 1\}$ and representing the generator of the fundamental group of \mathbb{C}^* , such that C_i lies entirely outside C_{i+1} for $1 \le i < n$. By Proposition 6.7 and the boson-fermion correspondence

$$\sum_{d\geq 0} q^d \left\langle \prod_{i=1}^n \left(\sum_{k\geq -2} \lambda_i^k \tau_k(\tilde{\omega}) \right) \right\rangle_d^{dis}$$

= $\operatorname{Tr}_{\mathrm{H}_0^B} q^{L_0} \prod_{i=1}^n \frac{1}{\lambda_i^2} \oint_{C_i} \frac{dz}{z} \frac{1}{\mathcal{S}(\lambda_i)} : \exp\left(\mathcal{S}(\lambda_i z \partial_z)(\lambda_i z \alpha(z))\right) :_B$

$$= \sum_{k_1 \ge 0, k_2 \ge 0, \dots i=1} \prod_{i=1}^{\infty} \frac{q^{ik_i}}{i^{k_i} k_i!}$$

$$(6.49) \quad \left\langle 0 \left| \left(\prod_{i=1}^{\infty} \alpha_i^{k_i} \right) \prod_{i=1}^n \frac{1}{\lambda_i^2} \oint_{C_i} \frac{dz}{z} \frac{1}{\mathcal{S}(\lambda_i)} : \exp\left(\mathcal{S}(\lambda_i z \partial_z)(\lambda_i z \alpha(z))\right) :_B \left(\prod_{i=1}^{\infty} \alpha_{-i}^{k_i} \right) \right| 0 \right\rangle$$

Using Wick's Theorem (see for example [MJD00]), we can put the expression in the bracket into the normal ordered form, and the above summation can be expressed in terms of Feynman diagrams as follows. From the normal ordering relations

$$z_{1}\alpha(z_{1})z_{2}\alpha(z_{2}) = \frac{z_{1}z_{2}}{(z_{1}-z_{2})^{2}} + :z_{1}\alpha(z_{1})z_{2}\alpha(z_{2}):_{B}, |z_{1}| > |z_{2}|$$

$$\alpha_{n}z\alpha(z) = nz^{n} + :\alpha_{n}z\alpha(z):_{B} \quad n > 0$$

$$z\alpha(z)\alpha_{-n} = nz^{-n} + :z\alpha(z)\alpha_{-n}:_{B} \quad n > 0$$

$$\alpha_{n}\alpha_{-n} = n + :\alpha_{n}\alpha_{-n}:_{B} \quad n > 0$$

we see that there're two types of vertices for the Feynman diagrams.

(1) The Type I vertices are given by

$$\exp\left(\mathcal{S}(\lambda_i z \partial_z)(\lambda_i z \alpha(z))\right)$$

for each $\lambda_i, 1 \leq i \leq n$, where $z\alpha(z)$ is viewed as input.

(2) The Type II are vertices of valency two for each m > 0, with two inputs α_m, α_{-m} and weight $\frac{q^m}{m}$, i.e., vertices of the form

$$\frac{q^m}{m}\alpha_m\alpha_{-m}, \quad m > 0$$

The propagators also have three types.

(1) The Type A propagators connect $z_1\alpha(z_1)$ and $z_2\alpha(z_2)$ at two different vertices of the first type and gives the value

$$\frac{z_1 z_2}{(z_1 - z_2)^2}$$

(2) The Type B propagators connects zα(z) from the vertices of the first type and α_m from the vertices of the second type, which gives the value

$$|m|z^m, m \in \mathbb{Z} \setminus \{0\}$$

(3) The Type C propagaotrs connect α_m, α_{-m} from two *different* vertices of the second type, which gives the value

|m|

Since the vertex of Type II has valency two, we can insert any number of vertices of Type II into the propagator of Type A using propagator of Type C. This is equivalent to considering only vertices of Type I from $\exp(S(\lambda_i z \partial_z)(\lambda z \alpha(z)))$ but with propagators

(6.50)
$$\left(\frac{z_1 z_2}{(z_1 - z_2)^2} + \sum_{m \ge 1} \frac{m z_1^m z_2^{-m} q^m}{1 - q^m} + \sum_{m \ge 1} \frac{m z_1^{-m} z_2^m q^m}{1 - q^m}\right)$$

connecting $z_1 \alpha(z_1)$ and $z_2 \alpha(z_2)$ at two different vertices, and

(6.51)
$$\left(\sum_{m\geq 1} \frac{mq^m z_1^m z_2^{-m}}{1-q^m} + \sum_{m\geq 1} \frac{mq^m z_1^{-m} z_2^m}{1-q^m}\right)\Big|_{z_1=z_2}$$

for propagator connecting two $z\alpha(z)$'s at the same vertex.

Now we compare it with the Feynman integral

$$\sum_{k_i \ge -2} \lim_{\bar{\tau} \to \infty} \lim_{\substack{\epsilon \to 0 \\ L \to \infty}} W^{dis} \left(\mathbf{P}_{\epsilon}^L; \lambda_1^{k_i} I_{C_1}^{(k_i+1)}, \cdots, \lambda_n^{k_n} I_{C_n}^{(k_n+1)} \right)$$

The vertices $I_{C_1}^{(k_i+1)}[\mu(w)]$ are precisely the same by construction via the identification of fields

$$\mu(w) = z\alpha(z)$$

The propagator connecting two different vertices $I_{C_1}^{(k_i+1)}$ and $I_{C_1}^{(k_j+1)}$ for $i \neq j$ is

$$\lim_{\bar{\tau} \to \infty} \lim_{\substack{\epsilon \to 0 \\ L \to \infty}} \mathcal{P}_{\epsilon}^{L}(w_1 - w_2; \tau, \bar{\tau}) = \left(\frac{z_1 z_2}{(z_1 - z_2)^2} + \sum_{m \ge 1} \frac{m z_1^m z_2^{-m} q^m}{1 - q^m} + \sum_{m \ge 1} \frac{m z_1^{-m} z_2^m q^m}{1 - q^m} \right)$$

by Eqn (6.13), where $z_i = \exp(2\pi i w_i)$, i = 1, 2. This is precisely (6.50). To consider the self-loop contributions, note that the regularized BCOV propagator is given by the sum

$$\begin{aligned} \mathbf{P}_{\epsilon}^{L}(w_{1},w_{2};\tau,\bar{\tau}) \\ &= -\frac{1}{\pi} \int_{\epsilon}^{L} \frac{du}{4\pi u} \sum_{\lambda \in \Gamma} \left(\frac{\bar{w}_{12}-\bar{\lambda}}{4u}\right)^{2} e^{-|w_{12}-\lambda|^{2}/4u} \\ &= -\frac{1}{\pi} \int_{\epsilon}^{L} \frac{du}{4\pi u} \left(\frac{\bar{w}_{1}-\bar{w}_{2}}{4u}\right)^{2} e^{-|w_{1}-w_{2}/4u} - \frac{1}{\pi} \int_{\epsilon}^{L} \frac{du}{4\pi u} \sum_{\lambda \in \Gamma, \lambda \neq 0} \left(\frac{\bar{w}_{1}-\bar{w}_{2}-\bar{\lambda}}{4u}\right)^{2} e^{-|w_{1}-w_{2}/4u} \end{aligned}$$

Since the vertices $I_{C_1}^{(k_i+1)}$ contains only holomorphic derivatives, the first term doesn't contribute to the self-loops, while the second is smooth around the diagonal $w_1 = w_2$. By Eqn (6.13), under the limit $\lim_{\bar{\tau}\to\infty} \lim_{\substack{\epsilon\to 0\\L\to\infty}}$, the propagator for the self-loop is equivalent to

$$-\frac{1}{4\pi^2} \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{1}{(w_1 - w_2 - m)^2} - \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \sum_{m \in \mathbb{Z}} \frac{1}{(w_1 - w_2 - (m + n\tau))^2}$$
$$= -\frac{1}{4\pi^2} \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{1}{(w_1 - w_2 - m)^2} + \left(\sum_{m \ge 1} \frac{mq^m z_1^m z_2^{-m}}{1 - q^m} + \sum_{m \ge 1} \frac{mq^m z_1^{-m} z_2^m}{1 - q^m}\right)$$

which differs from (6.51) by the first term. By Lemma 6.6, the first term contributes precisely the factor $\frac{1}{S(\lambda)}$ in (6.49). This proves the theorem.

Remark 6.8. If all k_i 's are taken to be 1, then it reduces to Dijkgraaf's theorem in [Dij95], where the RHS are given by cubic Feynman diagrams. Dijkgraaf proves that the corresponding cubic Feynman integrals compute certain Hurwitz numbers on the elliptic curve, which can be identified with the stationary descendant Gromov-Witten invariants with input $\tau_1(\tilde{w})$ under the Hurwitz/Gromov-Witten correspondence [OP06a].

We can further decompose the lagrangian by the number of derivatives

$$\mathcal{L}^{(k)}(\mu) = \sum_{g \ge 0} \mathcal{L}_g^{(k)}(\mu)$$

where $L_g^{(k)}(\mu)$ contains 2g derivatives. Let $I_{C,\hbar}^{(k)}$ be the functional on $\mathrm{PV}_{E_{\tau}}^{0,0}$ taking value in $\mathbb{C}[[\hbar]]$ that is given by

$$I_{C,\hbar}^{(k)}[\mu] = \sum_{g \ge 0} \hbar^g \int_C dw \mathcal{L}_g^{(k)}(\mu(w)), \quad \mu \in \mathrm{PV}_{E_\tau}^{0,0}$$

where C is a cycle representing the class [0, 1] as before.

Corollary 6.9. With the same notations as in Proposition 6.7, we have

$$(6.52) \quad \frac{\frac{1}{\hbar} \sum_{d \ge 0} q^d \hbar^g \left\langle \prod_{i=1}^n \tau_{k_i}(\tilde{\omega}) \right\rangle_{g,d}^{dis}}{\sum_d q^d \langle 1 \rangle_d^{dis}} = \lim_{\bar{\tau} \to \infty} \lim_{\substack{\epsilon \to 0\\ L \to \infty}} \left(\exp\left(\hbar \frac{\partial}{\partial P_{\epsilon}^L}\right) \prod_{i=1}^n \frac{1}{\hbar} I_{C_i,\hbar}^{(k_i)} \right) [0]$$

where on the right hand side, it's understood that the external inputs are zero.

Proof. Proposition 6.7 can be rewritten as

$$(6.53) \qquad \qquad \frac{\sum_{d\geq 0} \sum_{g\geq 0} q^d \left\langle \prod_{i=1}^n \sum_{k_i\geq -2} \lambda_i^{k_i} \tau_{k_i}(\tilde{\omega}) \right\rangle_{g,d}^{dis}}{\sum_d q^d \langle 1 \rangle_d^{dis}} \\ = \lim_{\bar{\tau}\to\infty} \lim_{\substack{\epsilon\to 0\\L\to\infty}} \exp\left(\frac{\partial}{\partial P_{\epsilon}^L}\right) \prod_{i=1}^n \left(\frac{1}{\lambda_i} \sum_{k_i\geq -1} \lambda_i^{k_i} I_{C_i}^{(k)}\right)$$

The theorem follows easily from Eqn (6.48) and the rescaling of Eqn (6.53) under

$$\lambda_i \to \lambda_i \sqrt{\hbar}$$

6.4. $\bar{\tau} \to \infty$ limit. Kodaira-Spencer gauge theory is known to be the closed string field theory of B-twisted topological string. It's argued in [BCOV94] by string theory technique that the B-twisted topological string amplitude would have a meaningful $\bar{t} \to \infty$ limit around the large complex limit of the Calabi-Yau manifold. Here t is certain coordinates on the moduli space of complex structures. We will investigate the meaning of $\bar{\tau} \to \infty$ for the elliptic curve example in this section. 6.4.1. $\bar{\tau} \to \infty$ limit. Let $\omega = \frac{i}{2 \operatorname{Im} \tau} \partial_w \wedge d\bar{w} \in \operatorname{PV}_{E_{\tau}}^{1,1}$, which is normalized such that

 ${\rm Tr}~\omega=1$

Let $F^{E_{\tau}} = \sum_{g \ge 0} \hbar^g F_g^{E_{\tau}}[L]$ be the family of effective action constructed from quantizing the BCOV theory on E_{τ} . Since E_{τ} is compact, we have the well-defined limit

$$F_g^{E_\tau}[\infty] \equiv \lim_{L \to \infty} F_g^{E_\tau}[L]$$

We are interested in the following correlation functions

$$F_g^{E_\tau}[\infty][t^{k_1}\omega,\cdots,t^{k_n}\omega]$$

for some non-negative integers k_1, \dots, k_n satisfying the Hodge weight condition

$$\sum_{i=1}^{n} k_i = 2g - 2$$

Let $\mathcal{H} = \{\tau \in \mathbb{C} | \operatorname{Im} \tau > 0\}$ be the complex upper half-plane. The group $SL(2,\mathbb{Z})$ acts on \mathcal{H} by

$$\tau \to \gamma \tau = \frac{A\tau + B}{C\tau + D}, \text{ for } \gamma \in \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2, \mathbb{Z})$$

Recall that an almost holomorphic modular form [KZ95] of weight k on $SL(2,\mathbb{Z})$ is a function

$$\hat{f}: \mathcal{H} \to \mathbb{C}$$

which grows at most polynomially in $1/\operatorname{Im}(\tau)$ as $\operatorname{Im}(\tau) \to 0$ and satisfies the transformation property

$$\hat{f}(\gamma \tau) = (C\tau + D)^k f(\tau) \text{ for all } \gamma \in \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2, \mathbb{Z})$$

and has the form

$$\hat{f}(\tau,\bar{\tau}) = \sum_{m=0}^{M} f_m(\tau) \operatorname{Im}(\tau)^{-m}$$

for some integer $M \ge 0$, where the functions $f_m(\tau)$'s are holomorphic in τ . The following limit makes sense

$$\lim_{\bar{\tau}\to\infty}\hat{f}(\tau,\bar{\tau})=f_0(\tau)$$

which gives the isomorphism between the rings of almost holomorphic modular forms and quasi-modular forms described in [KZ95].

Lemma 6.10. Let Γ be a connected oriented graph, $V(\Gamma)$ be the set of vertices, $E(\Gamma)$ be the set of edges, and $l, r : E \to V$ be the maps which give each edge the associated left and right vertices. Let $W_{\Gamma,\{n_e\}}(\mathbf{P}^L_{\epsilon})$ be the graph integral

(6.54)
$$W_{\Gamma,\{n_e\}}(\mathbf{P}^L_{\epsilon}) = \prod_{v \in V} \int_{E_{\tau}} \frac{d^2 w_v}{\operatorname{Im} \tau} \prod_{e \in E} \partial^{n_e}_{w_{l(e)}} \mathbf{P}^L_{\epsilon}(w_{l(e)}, w_{r(e)}; \tau, \bar{\tau})$$

where n_e 's are non-negative integers that associates to each $e \in E$, and \mathbf{P}_{ϵ}^L is the regularized BCOV propagator on the elliptic curve E_{τ} . Then $\lim_{\substack{\epsilon \to 0 \\ L \to \infty}} W_{\Gamma,\{n_e\}}(\mathbf{P}_{\epsilon}^L)$ exists as an almost holomorphic modular form of weight $2|E| + \sum_{e \in E} n_e$.

Proof. Given $\gamma \in \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbb{Z})$, the propagator has the transformation property

$$\partial_{w_1}^m P_{\epsilon}^L(w_1, w_2; \gamma\tau, \gamma\bar{\tau}) = (C\tau + D)^{m+2} \left(\partial_{w_1}^m P_{|C\tau + D|^2\epsilon}^{|C\tau + D|^2L} \right) \left((C\tau + D)w_1, (C\tau + D)w_2; \tau, \bar{\tau} \right)$$

Since $\frac{dw \wedge d\bar{w}}{\mathrm{Im}\,\tau}$ is invariant under the transformation

$$\tau \to \frac{A\tau+B}{C\tau+D}, w \to (C\tau+D)u$$

We conclude that $\lim_{\substack{\epsilon \to 0 \\ L \to \infty}} W_{\Gamma,\{n_e\}}(\mathbf{P}^L_{\epsilon})$ has the same transformation property as a modular form of weight

$$2|E| + \sum_{e \in E} n_e$$

Now we show that $\lim_{\substack{\epsilon \to 0 \\ L \to \infty}} W_{\Gamma,\{n_e\}}(\mathbf{P}^L_{\epsilon})$ has polynomial dependence on $\frac{1}{\mathrm{Im} \ \tau}$ by induction on the number of edges. First we observe that each self-loop contributes

$$\lim_{\substack{\epsilon \to 0 \\ L \to \infty}} \left(\partial_{w_1}^n \mathbf{P}_{\epsilon}^L(w_1, w_2; \tau, \bar{\tau}) \Big|_{w_1 = w_2} \right)$$

which is an almost holomorphic modular form of weight n + 2. Therefore we can assume that Γ has no self-loops. Consider the following change of variable

$$w_v = \frac{(i+\tau)u_v + (i-\tau)\bar{u}_v}{2i}$$

and

$$W_{\Gamma,\{n_e\}}(\mathbf{P}^L_{\epsilon}) = \prod_{v \in V(\Gamma)} \int_{[0,1]^2} d^2 u_v \left(\prod_{e \in E(\Gamma)} \partial^{n_e}_{w_{l(e)}} \mathbf{P}^L_{\epsilon}(w_{l(e)}, w_{r(e)}; \tau, \bar{\tau}) \right)$$

We can compute the derivative with respect to $\bar{\tau}$

$$\begin{split} & \frac{\partial}{\partial \bar{\tau}} W_{\Gamma,\{n_e\}}(\mathbf{P}_{\epsilon}^L) \\ = & \prod_{v \in V(\Gamma)} \int_{[0,1]^2} d^2 u_v \sum_{e \in E(\Gamma)} \left(\partial_{\bar{\tau}} \bar{w}_e \frac{\partial}{\partial \bar{w}_{l(e)}} + \frac{\partial}{\partial \bar{\tau}} \right) \partial^{n_e} \mathbf{P}_{\epsilon}^L(w_{l(e)}, w_{r(e)}; \tau, \bar{\tau}) \\ & \left(\prod_{e' \in E(\Gamma) - \{e\}} \partial^{n_{e'}} \mathbf{P}_{\epsilon}^L(w_{l(e')}, w_{r(e')}; \tau, \bar{\tau}) \right) \\ = & \prod_{v \in V(\Gamma)} \int_{E_{\tau}} \frac{d^2 w_v}{\operatorname{Im} \tau} \sum_{e \in E(\Gamma)} \left(\frac{\operatorname{Im} w_e}{\operatorname{Im} \tau} \frac{\partial}{\partial \bar{w}_{l(e)}} + \frac{\partial}{\partial \bar{\tau}} \right) \partial^{n_e} \mathbf{P}_{\epsilon}^L(w_{l(e)}, w_{r(e)}; \tau, \bar{\tau}) \\ & \left(\prod_{e' \in E(\Gamma) - \{e\}} \partial^{n_{e'}} \mathbf{P}_{\epsilon}^L(w_{l(e')}, w_{r(e')}; \tau, \bar{\tau}) \right) \end{split}$$

where $w_e = w_{l(e)} - w_{r(e)}$. It's easy to compute that

(6.55)
$$\left(\frac{\operatorname{Im} w_e}{\operatorname{Im} \tau} \frac{\partial}{\partial \bar{w}_{l(e)}} + \frac{\partial}{\partial \bar{\tau}}\right) \partial^{n_e} P_{\epsilon}^L(w_{l(e)}, w_{r(e)}; \tau, \bar{\tau})$$
$$= -\frac{1}{4\pi} \sum_{\lambda} \frac{\operatorname{Im}(w_e - \lambda)}{\operatorname{Im} \tau} \frac{1}{4\pi t} \partial^{n_e + 1}_{w_e} e^{-|w_e - \lambda|^2/4t} \Big|_{t=\epsilon}^{t=L}$$

 $\frac{\partial}{\partial \bar{\tau}} W_{\Gamma,\{n_e\}}(\mathbf{P}_{\epsilon}^L)$ has two types of contributions corresponding to t = L or $t = \epsilon$ in the above formula.

Let's first consider the term with t = L in (6.55). If $n_e > 0$, then the summation $\sum_{\lambda} \frac{\operatorname{Im}(w_e - \lambda)}{\operatorname{Im} \tau} \frac{1}{4\pi t} \partial_{w_e}^{n_e + 1} e^{-|w_e - \lambda|^2/4t}$ is absolutely convergent, and we have

$$\lim_{L \to \infty} \sum_{\lambda} \frac{\operatorname{Im}(w_e - \lambda)}{\operatorname{Im} \tau} \frac{1}{4\pi L} \partial_{w_e}^{n_e + 1} e^{-|w_e - \lambda|^2/4L} = 0$$

If $n_e = 0$, then

$$\sum_{\lambda} \frac{\operatorname{Im}(w_e - \lambda)}{\operatorname{Im}\tau} \frac{1}{4\pi L} \partial_{w_e} e^{-|w_e - \lambda|^2/4L}$$
$$= \sum_{m \in \mathbb{Z}} \left(\frac{\operatorname{Im} w_e}{\operatorname{Im} \tau} - m \right) \sum_{n \in \mathbb{Z}} \left(\frac{\bar{w}_e - (n + m\bar{\tau})}{16\pi L^2} \right) e^{-|w_e - (n + m\tau)|^2/4L}$$

$$= \sum_{m \in \mathbb{Z}} \left(\frac{\operatorname{Im} w_e}{\operatorname{Im} \tau} - m \right) \sum_{n \in \mathbb{Z}} \left(\left(\frac{\bar{w}_e - (n + m\bar{\tau})}{16\pi L^2} \right) e^{-|w_e - (n + m\tau)|^2/4L} - \int_n^{n+1} dy \left(\frac{\bar{w}_e - (y + m\bar{\tau})}{16\pi L^2} \right) e^{-|w_e - (y + m\tau)|^2/4L} \right) \\ + \sum_{m \in \mathbb{Z}} \left(\frac{\operatorname{Im} w_e}{\operatorname{Im} \tau} - m \right) \int_{-\infty}^{\infty} dy \left(\frac{\bar{w}_e - (y + m\bar{\tau})}{16\pi L^2} \right) e^{-|w_e - (y + m\tau)|^2/4L} \\ = I_1 + I_2$$

Similarly we have $\lim_{L\to\infty} I_1 = 0$. I_2 can be computed using Gaussian integral and we get

$$I_2 = \sum_{m \in \mathbb{Z}} \left(\frac{\operatorname{Im} w_e}{\operatorname{Im} \tau} - m \right) \left(\frac{i(\operatorname{Im} w_e - m \operatorname{Im} \tau)}{8\sqrt{\pi}L^{3/2}} \right) e^{-(\operatorname{Im} w_e - m \operatorname{Im} \tau)^2/4L}$$

It follows that

$$\lim_{L \to \infty} I_2 = \lim_{L \to \infty} \frac{C_1}{(\operatorname{Im} \tau)^2} \sum_{m \in \mathbb{Z}} \frac{\left(\frac{\operatorname{Im} w_e}{\operatorname{Im} \tau} - m\right)^2}{L^{3/2}} e^{-\left(\frac{\operatorname{Im} w_e}{\operatorname{Im} \tau} - m\right)^2/4L}$$
$$= \lim_{L \to \infty} \frac{C_2}{(\operatorname{Im} \tau)^2} \sum_{m \in \mathbb{Z}} (1 - 8m^2 \pi^2 L) e^{-4m^2 \pi^2 L + 2\pi i m \frac{\operatorname{Im} w_e}{\operatorname{Im} \tau}}$$
$$= \frac{C_2}{(\operatorname{Im} \tau)^2}$$

where C_1, C_2 are two constants and in the second step we have used Fourier transformation. Therefore

$$\lim_{L \to \infty} \sum_{\lambda} \frac{\operatorname{Im}(w_e + \lambda)}{\operatorname{Im} \tau} \frac{1}{4\pi L} \partial_{w_e}^{n_e + 1} e^{-|w_e + \lambda|^2/4L} = \begin{cases} 0 & n_e > 0 \\ \\ C \frac{1}{(\operatorname{Im} \tau)^2} & n_e = 0 \end{cases}$$

for some constant C.

Next we consider the term with $t = \epsilon$ in (6.55), which contributes to $\frac{\partial}{\partial \bar{\tau}} W_{\Gamma,\{n_e\}}(\mathbf{P}_{\epsilon}^L)$ as

$$-\frac{1}{4\pi(\operatorname{Im}\tau)^{2}}\sum_{e\in E(\Gamma)}\left(\prod_{v\in V(\Gamma)-l(e)}\int_{E_{\tau}}\frac{d^{2}w_{v}}{\operatorname{Im}\tau}\right)\int_{\mathbb{C}}d^{2}w_{l(e)}$$
$$\operatorname{Im}(w_{e})\frac{1}{4\pi\epsilon}\partial_{w_{e}}^{n_{e}+1}e^{-|w_{e}|^{2}/4\epsilon}\left(\prod_{e'\in E(\Gamma)-\{e\}}\partial^{n_{e'}}\mathcal{P}_{\epsilon}^{L}(w_{l(e')},w_{r(e')};\tau,\bar{\tau})\right)$$

By Proposition 5.18, it reduces to certain graph integral $\lim_{\substack{\epsilon \to 0 \\ L \to \infty}} W_{\Gamma', \{n'_e\}}(\mathbf{P}^L_{\epsilon})$ under the limit $\epsilon \to 0, L \to \infty$ with an extra factor $-\frac{1}{4\pi(\operatorname{Im} \tau)^2}$, where Γ' is obtained from Γ by collapsing the edges connecting l(e) and r(e).

Combining the above two terms, it follows by induction that

$$\bar{\partial}_{\bar{\tau}} \lim_{\substack{\epsilon \to 0\\L \to \infty}} W_{\Gamma,\{n_e\}}(\mathbf{P}_{\epsilon}^L) = \frac{1}{(\operatorname{Im} \tau)^2} \sum_{i=0}^K f_i(\tau) \frac{1}{(\operatorname{Im} \tau)^i}$$

for some functions $f_i(\tau)$ holomorphic in τ and some non-negative integer K. Therefore $W_{\Gamma,\{n_e\}}(\mathbf{P}^L_{\epsilon})$ has polynomial dependence on $\frac{1}{\mathrm{Im}\,\tau}$ as well.

Proposition 6.11. $F_g^{E_{\tau}}[\infty][t^{k_1}\omega, \cdots, t^{k_n}\omega]$, which is viewed as a function on $\tau \in \mathcal{H}$, is an almost holomorphic modular form of weight 2g - 2 + 2n.

Proof. $F_g^{E_{\tau}}[\infty][t^{k_1}\omega, \cdots, t^{k_n}\omega]$ is given by Feynman diagram integrals of the type in the previous Lemma 6.10. We conclude that $F_g^{E_{\tau}}[\infty][t^{k_1}\omega, \cdots, t^{k_n}\omega]$ is an almost holomorphic modular form of weight

$$2|E| + N$$

where E is the number of propagators, N is the total number of holomorphic derivatives appearing in the local functionals for the vertices. By Proposition 5.25 and the Hodge weight condition, this precisely equals

$$\sum_{i=1}^{n} (k_i + 2) = 2g - 2 + 2n$$

It follows that the following limit makes sense

$$\lim_{\bar{\tau}\to\infty} F_g^{E_{\tau}}[\infty][t^{k_1}\omega,\cdots,t^{k_n}\omega]$$

which is a quasi-modular form of weight 2g - 2 + 2n.

6.4.2. Cohomological localization. Let A (resp.B) denote the homology class of the segment [0, 1] (resp. $[0, \tau]$) on the elliptic curve E_{τ} . Let α_A (resp. α_B) be the 1-form representing the

$$[\omega \vdash dw] = \left[\frac{-i}{2\operatorname{Im}\tau}d\bar{w}\right] = \left[\frac{i}{2\operatorname{Im}\tau}(\bar{\tau}\alpha_A - \alpha_B)\right]$$

Consider the isomorphism of complexes

$$\Phi : \left(\mathrm{PV}_{E_{\tau}}^{1,*} \oplus t \, \mathrm{PV}_{E_{\tau}}^{0,*}, Q \right) \to (\mathcal{A}^{*,*}, d)$$
$$\alpha + t\beta \to (\alpha + \beta) \vdash dw$$

where $\mathcal{A}^{*,*}$ is the space of smooth differential forms on E_{τ} . Let

$$\omega_A = \Phi^{-1}(\alpha_A), \quad \omega_B = \Phi^{-1}(\alpha_B)$$

It follows that there exists $\beta \in \mathrm{PV}^{1,*} \oplus t \, \mathrm{PV}^{0,*}$ such that

$$\omega = \frac{i}{2 \operatorname{Im} \tau} (\bar{\tau} \omega_A - \omega_B) + Q\beta$$

Let A_1, \dots, A_n (resp. B_1, \dots, B_n) be disjoint cycles on E_{τ} which lie in the same homology class of A (resp.B). The quantum master equation at $L = \infty$ says

which implies that

$$F_g[\infty][t^{k_1}\omega,\cdots,t^{k_n}\omega] = F_g[\infty][t^{k_1}\frac{i}{2\operatorname{Im}\tau}(\bar{\tau}\omega_{A_1}-\omega_{B_1}),\cdots,t^{k_n}\frac{i}{2\operatorname{Im}\tau}(\bar{\tau}\omega_{A_n}-\omega_{B_n})]$$

Under the limit $\bar{\tau} \to \infty$, we have

(6.57)
$$\lim_{\bar{\tau}\to\infty} F_g[\infty][t^{k_1}\omega,\cdots,t^{k_n}\omega] = \lim_{\bar{\tau}\to\infty} F_g[\infty][t^{k_1}\omega_{A_1},\cdots,t^{k_n}\omega_{A_n}]$$

Since the supports of ω_{A_i} 's are disjoint, and the propagator is concentrated at $PV_{E_{\tau}}^{0,0}$, the RHS can be represented as Feynman graph integrals

$$\lim_{\bar{\tau}\to\infty}\sum_{g\geq 0}\hbar^{g-1}F_g[\infty][t^{k_1}\omega_{A_1},\cdots,t^{k_n}\omega_{A_n}]$$

(6.58) =
$$\sum_{\substack{\Gamma:\text{connected graph}\\|V(\Gamma)|=n}} W_{\Gamma}\left(\hbar \mathbb{P}_{0}^{\infty}(\tau;\infty);\frac{1}{\hbar}\int_{A_{1}}dw\mathcal{J}^{(k_{1})},\cdots,\frac{1}{\hbar}\int_{A_{n}}dw\mathcal{J}^{(k_{n})}\right)$$

where we sum over all connected Feynman graph integrals with n vertices, with propagator $\hbar P_0^{\infty}(\tau; \infty)$, and the *i*th vertex given by $\frac{1}{\hbar} \int_{A_i} dw \mathcal{J}^{(k_i)}$. Here $\mathcal{J}^{(k)} = \sum_{g \ge 0} \hbar^g \mathcal{J}_g^{(k)}$, and $\mathcal{J}_g^{(k)}$ is a lagrangian on $\mathrm{PV}_{E_{\tau}}^{0,0}$ which contains 2g holomorphic derivatives by Proposition 5.25.

We will use α to represent a general element in $PV_{E_{\tau}}^{0,0}$, and write

(6.59)
$$\alpha^{(n)} = \left(\sqrt{\hbar}\frac{\partial}{\partial w}\right)^n \alpha, \quad \alpha^{(0)} \equiv \alpha$$

 $\mathcal{J}^{(k)}(\alpha)$ can be naturally viewed as an element in $\mathbb{C}[\alpha, \alpha^{(1)}, \cdots]/\operatorname{Im} D$, where

$$D = \sum_{i=0}^{\infty} \alpha^{(i+1)} \frac{\partial}{\partial \alpha^{(i)}}$$

represents the operator of total derivative. The initial condition is determined by the classical BCOV action, which says

$$\mathcal{J}_0^{(k)}(\alpha) = \frac{1}{(k+1)!} \alpha^{k+2}$$

If we assign the following degree

$$\deg \alpha^{(n)} = n+1$$

then the Hodge weight condition implies

$$\deg \mathcal{J}^{(k)} = (k+2)$$

In particular, the above degree constraint tells us that

$$\mathcal{J}^{(0)} = \frac{1}{2}\alpha^2$$

and

$$\mathcal{J}^{(1)} = \frac{1}{3!}\alpha^3$$

where the other possible terms don't contribute since they are in the image of D.

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6.4.3. Theory on \mathbb{C}^* and commutativity property. We explore the properties of $\mathcal{J}^{(k)}$ by considering the BCOV theory on \mathbb{C}^* . Let F[L] be the quantization of the BCOV action. We will use z to denote the coordinate on \mathbb{C}^*

$$z = \exp(2\pi i w)$$

where w is the coordinate on \mathbb{C} . The holomorphic volume form is $\frac{dz}{z}$ which defines the trace operator on polyvector fields. We will use $\mathcal{O}_{\mathbb{C}^*}$ to denote the space of holomorphic functions on \mathbb{C}^* . Let C_r be the circle $\{z \in \mathbb{C}^* | |z| = r\}, r > 0$. We associate C_r a non-negative smooth function ρ_{C_r} , which takes constant value outside a small neighborhood of C_r , such that $\rho_{C_r} = 1$ when $|z| \gg r$ and $\rho_{C_r} = 0$ when $|z| \ll r$. Note that $d\rho_{C_r}$ is the generator of $H^1_c(\mathbb{C}^*)$, with

$$\int_{\mathbb{C}^*} d(\rho_C) \wedge \frac{dz}{2\pi i z} = 1$$

and $d\rho_{C_r}$ represents the Poincare dual of C_r . Let ω_{C_r} be the following polyvector field

$$\omega_{C_r} = Q(\rho_{C_r} z \partial_z) \in \mathrm{PV}_{\mathbb{C}^*}^{1,1} \oplus t \, \mathrm{PV}_{\mathbb{C}^*}^{0,0}$$

and consider

$$\mathcal{O}_{C_{r_1},C_{r_2}}^{(k_1,k_2)}[L] = e^{-F[L]/\hbar} \frac{\partial}{\partial (t^{k_1}\omega_{C_{r_1}})} \frac{\partial}{\partial (t^{k_2}\omega_{C_{r_2}})} e^{F[L]/\hbar}$$

where $r_1 \neq r_2$ such that the supports of $d\rho_{C_{r_1}}$ and $d\rho_{C_{r_2}}$ are disjoint.

Lemma 6.12. The effective action $F[L] + \delta \mathcal{O}_{C_{r_1},C_{r_2}}^{(k_1,k_2)}[L]$ satisfies renormalization group flow equation and quantum master equation, where δ is an odd variable with $\delta^2 = 0$.

Proof.

$$\mathcal{O}_{C_{r_1},C_{r_2}}^{(k_1,k_2)}[L]e^{F[L]/\hbar} = \frac{\partial}{\partial(t^{k_1}\omega_{C_{r_1}})} \frac{\partial}{\partial(t^{k_2}\omega_{C_{r_2}})} e^{F[L]/\hbar}$$
$$= e^{\hbar\partial_{P_{\epsilon}^L}} \frac{\partial}{\partial(t^{k_1}\omega_{C_{r_1}})} \frac{\partial}{\partial(t^{k_2}\omega_{C_{r_2}})} e^{F[\epsilon]/\hbar}$$
$$= e^{\hbar\partial_{P_{\epsilon}^L}} \mathcal{O}_{C_{r_1},C_{r_2}}^{(k_1,k_2)}[\epsilon] e^{F[\epsilon]/\hbar}$$

which proves the renormalization group flow equation. Since $Q(t^{k_1}\omega_{C_{r_1}}) = Q(t^{k_2}\omega_{C_{r_2}}) = 0$,

$$(Q + \hbar\Delta_L) \mathcal{O}_{C_{r_1}, C_{r_2}}^{(k_1, k_2)}[L] e^{F[L]/\hbar} = (Q + \hbar\Delta_L) \frac{\partial}{\partial(t^{k_1}\omega_{C_{r_1}})} \frac{\partial}{\partial(t^{k_2}\omega_{C_{r_2}})} e^{F[L]/\hbar}$$
$$= \frac{\partial}{\partial(t^{k_1}\omega_{C_{r_1}})} \frac{\partial}{\partial(t^{k_2}\omega_{C_{r_2}})} (Q + \hbar\Delta_L) e^{F[L]/\hbar}$$
$$= 0$$

which proves the quantum master equation.

We will consider the restriction of $\mathcal{O}_{C_{r_1},C_{r_2}}^{(k_1,k_2)}[L]$ as a functional on $\mathcal{O}_{\mathbb{C}^*} \subset \mathrm{PV}_{\mathbb{C}^*}^{0,0}$ in the following discussion, and we still denoted it by $\mathcal{O}_{C_{r_1},C_{r_2}}^{(k_1,k_2)}[L]$. Since $\omega_{C_{r_1}},\omega_{C_{r_2}}$ have compact support, we can take $L \to \infty$ to obtain $\mathcal{O}_{C_{r_1},C_{r_2}}^{(k_1,k_2)}[\infty]$ as a functional on $\mathcal{O}_{\mathbb{C}^*}$. Note that elements in $\mathcal{O}_{\mathbb{C}^*}$ lie in the kernel of $Q = \bar{\partial} + t\partial$. Quantum master equation implies that $\mathcal{O}_{C_{r_1},C_{r_2}}^{(k_1,k_2)}[\infty]$ only depends on the homology class of C_{r_1}, C_{r_2} and the integers k_1, k_2 if we restrict on $\mathcal{O}_{\mathbb{C}^*}$. Following the convention as in Eqn (6.58), we have the following

Lemma 6.13. Restricting on $\mathcal{O}_{\mathbb{C}^*}$, then

(6.60)
$$\mathcal{O}_{C_{r_1},C_{r_2}}^{(k_1,k_2)}[\infty] = \exp\left(\hbar\partial_{P_0^{\infty}}\right) \left(\int_{C_1} \frac{dz}{2\pi i z} \mathcal{J}^{(k_1)} \int_{C_2} \frac{dz}{2\pi i z} \mathcal{J}^{(k_2)}\right)$$

where $\mathbf{P}^{L}_{\epsilon}$ is the regularized BCOV propagator on \mathbb{C}^{*}

$$\mathbf{P}_{\epsilon}^{L}(z_{1}, z_{2}) = -\frac{1}{\pi} \int_{\epsilon}^{L} \frac{dt}{4\pi t} \sum_{n \in \mathbb{Z}} \left(\frac{\bar{w}_{1} - \bar{w}_{2} + n}{4t}\right)^{2} e^{-|w_{1} - w_{2} + n|^{2}/4t}$$

and

$$\mathbf{P}_0^{\infty}(z_1, z_2) = \lim_{\substack{\epsilon \to 0 \\ L \to \infty}} \mathbf{P}_{\epsilon}^L(z_1, z_2) = \frac{1}{(2\pi i)^2} \sum_{n \in \mathbb{Z}} \frac{1}{(w_1 - w_2 - n)^2} = \frac{z_1 z_2}{(z_1 - z_2)^2}$$

here $z_k = \exp(2\pi i w_k), k = 1, 2.$

Since $\mathcal{O}_{C_{r_1},C_{r_2}}^{(k_1,k_2)}[\infty]$ only depends on the homology class of C_{r_1},C_{r_2} , we have

$$\exp\left(\hbar\partial_{\mathcal{P}_{0}^{\infty}}\right)\left(\int_{C_{r_{1}}}\frac{dz_{1}}{2\pi i z_{1}}\mathcal{J}^{(k_{1})}\int_{C_{r_{2}}-C_{r_{3}}}\frac{dz_{2}}{2\pi i z_{2}}\mathcal{J}^{(k_{2})}\right)=0$$

where $0 < r_3 < r_1 < r_2$. We call this the *commutativity property*.

Lemma 6.14. $\mathcal{J}^{(k)}$'s are uniquely determined (up to total derivative) by the initial conditions $\mathcal{J}^{(k)}(\alpha) = \frac{1}{(k+2)!} \alpha^{k+2} + O(\hbar), \alpha \in \mathcal{O}_{\mathbb{C}^*}$ and the above commutativity property.

Proof. If the propagator connects one $\alpha^{(n)}$ from $\mathcal{J}^{(k_1)}(\alpha)$ and one $\alpha^{(m)}$ from $\mathcal{J}^{(k_2)}(\alpha)$, it replaces the two terms by

$$\hbar \left(\sqrt{\hbar}z_1\partial_{z_1}\right)^n \left(\sqrt{\hbar}z_2\partial_{z_2}\right)^m \frac{z_1z_2}{(z_1-z_2)^2} = (-1)^{m+1} \left(\sqrt{\hbar}z_1\partial_{z_1}\right)^{n+m+1} \left(\frac{\sqrt{\hbar}z_2}{z_1-z_2}\right)^{m+m+1} \left(\frac{\sqrt{\hbar}z_2}{z_1-$$

Using residue we see that

$$\exp\left(\hbar\partial_{P_0^{\infty}}\right)\left(\int_{C_1}\frac{dz_1}{2\pi i z_1}\mathcal{J}^{(k_1)}\int_{C_2-C_2'}\frac{dz_2}{2\pi i z_2}\mathcal{J}^{(k_2)}\right)$$

gives rise to local functional $\int_{C_1} \frac{dz}{2\pi i z} I$ where $I(\alpha) \in \mathbb{C}[\alpha, \alpha^{(1)}, \cdots][\sqrt{\hbar}, \sqrt{\hbar}^{-1}]$. Let

$$u^{(n)}(z_1, z_2) = -\left(\sqrt{\hbar}z_1\partial_{z_1}\right)^n \left(\frac{\sqrt{\hbar}z_2}{z_1 - z_2}\right)$$

We claim that

$$u^{(n)}u^{(m)} = \frac{n!m!}{(n+m+1)!}u^{(n+m+1)} + \sqrt{\hbar}f(u^{(k)},\sqrt{\hbar})$$

where f is a polynomial which is linear in $u^{(k)}$'s. This can be proved by induction on n. For n = 0,

$$u^{(0)}u^{(m)} = \frac{\sqrt{\hbar}z_2}{z_1 - z_2} \left(\sqrt{\hbar}z_1\partial_{z_1}\right)^m \left(\frac{\sqrt{\hbar}z_2}{z_1 - z_2}\right)$$

We use power series to represent

$$u^{(m)} = \sqrt{\hbar}^{m+1} \sum_{k=0}^{\infty} k^m \left(\frac{z_1}{z_2}\right)^k$$

then

$$u^{(0)}u^{(m)} = \sqrt{\hbar}^{m+2} \sum_{k=0}^{\infty} \sum_{j=0}^{k} j^m \left(\frac{z_1}{z_2}\right)^k = \sqrt{\hbar}^{m+2} \sum_{k=0}^{\infty} P(k) \left(\frac{z_1}{z_2}\right)^k$$

where P(k) is a polynomial in k with the highest degree term given by $\frac{1}{m+1}k^{m+1}$. This proves the case for n = 0. The induction follows easily from the formula

$$u^{(n)}u^{(m)} = \left(\sqrt{\hbar}z_1\partial_{z_1}\right)\left(u^{(n-1)}u^{(m)}\right) - u^{(n-1)}u^{(m+1)}$$

This proves the claim.

It follows from the claim that $I \in \sqrt{\hbar}\mathbb{C}[\alpha, \alpha^{(1)}, \cdots][\sqrt{\hbar}]$. We consider the leading $\sqrt{\hbar}$ term in $\exp\left(\hbar\partial_{P_0^{\infty}}\right)\left(\int_{C_1} \frac{dz_1}{2\pi i z_1}\mathcal{J}^{(k_1)}\int_{C_2-C'_2} \frac{dz_2}{2\pi i z_2}\mathcal{J}^{(1)}\right)$, where $\mathcal{J}^{(1)}(\alpha) = \frac{1}{3!}\alpha^3$. If there's only one propagator, it replaces each term $\alpha^{(n)}$ in $\mathcal{J}^{(k_1)}$ by

$$\frac{1}{2} \int_{C_{z_1}} \frac{dz_2}{2\pi i z_2} \left(\sqrt{\hbar} z_1 \partial_{z_1}\right)^{n+1} \left(\frac{\sqrt{\hbar} z_2}{z_1 - z_2} \alpha(z_2)^2\right) = \sqrt{\hbar} \left(\sqrt{\hbar} z_1 \partial_{z_1}\right)^{n+1} \left(\frac{1}{2} \alpha(z_1)^2\right)$$

where C_{z_1} is a small loop around z_1 . If there're two propagators, then it replaces each pair $\alpha^{(n)}, \alpha^{(m)}$ in $\mathcal{J}^{(k_1)}$ by

$$\int_{C_{z_1}} \frac{dz_2}{2\pi i z_2} \left(\sqrt{\hbar} z_1 \partial_{z_1}\right)^{n+1} \left(\frac{\sqrt{\hbar} z_2}{z_1 - z_2}\right) \left(\sqrt{\hbar} z_1 \partial_{z_1}\right)^{m+1} \left(\frac{\sqrt{\hbar} z_2}{z_1 - z_2}\right) \alpha(z_2)$$

$$= \int_{C_{z_1}} \frac{dz_2}{2\pi i z_2} \frac{(n+1)!(m+1)!}{(n+m+3)!} \left(\sqrt{\hbar} z_1 \partial_{z_1}\right)^{n+m+3} \left(\frac{\sqrt{\hbar} z_2}{z_1 - z_2}\right) \alpha(z_2) + \text{higher order in}\sqrt{\hbar}$$

Therefore we find that the leading $\sqrt{\hbar}$ term in $\exp(\hbar P_0^{\infty}) \left(\int_{C_1} \frac{dz_1}{2\pi i z_1} \mathcal{J}^{(k_1)} \int_{C_2 - C'_2} \frac{dz_2}{2\pi i z_2} \mathcal{J}^{(1)} \right)$ is given by

$$\int_{C_1} \frac{dz_1}{2\pi i z_1} E \mathcal{J}^{(k_1)}$$

where E is the operator

$$E = \frac{1}{2} \sum_{k,l \ge 0} \frac{(k+l)!}{k!l!} \alpha^{(k)} \alpha^{(l)} \frac{\partial}{\partial \alpha^{(k+l-1)}} + \sum_{k,l \ge 0} \frac{(k+1)!(l+1)!}{(k+l+3)!} \alpha^{(k+l+3)} \frac{\partial}{\partial \alpha^{(k)}} \frac{\partial}{\partial \alpha^{(l)}} \frac{\partial}{\partial \alpha^{(l$$

By the commutative property, $E\mathcal{J}^{(k_1+1)}(\alpha)$ is a total derivative, i.e., lies in the image of D. The uniqueness now follows from the lemma below.

Lemma 6.15. Consider the graded ring $A = \mathbb{C}[\alpha^{(0)}, \alpha^{(1)}, \cdots] / \text{ im } D$ with grading given by

$$\deg \alpha^{(k)} = k+1$$

where D is the operator of degree 1

$$D = \sum_{n \ge 0} \alpha^{(n+1)} \frac{\partial}{\partial \alpha^{(n)}}$$

Let E be the operator of degree 2 acting on A

$$E = \frac{1}{2} \sum_{k,l \ge 0} \frac{(k+l)!}{k!l!} \alpha^{(k)} \alpha^{(l)} \frac{\partial}{\partial \alpha^{(k+l-1)}} + \sum_{k,l \ge 0} \frac{(k+1)!(l+1)!}{(k+l+3)!} \alpha^{(k+l+3)} \frac{\partial}{\partial \alpha^{(k)}} \frac{\partial}{\partial \alpha^{(l)}} \frac{\partial}{\partial \alpha^{(l$$

There there exists unique $\mathcal{J}^{(k)} \in A$ of degree k + 1 such that

$$E\mathcal{J}^{(k)} = 0$$

Proof. Let $E = E_1 + E_2$ where

$$E_1 = \frac{1}{2} \sum_{k,l \ge 0} \frac{(k+l)!}{k!l!} \alpha^{(k)} \alpha^{(l)} \frac{\partial}{\partial \alpha^{(k+l-1)}},$$

$$E_2 = \sum_{k,l \ge 0} \frac{(k+1)!(l+1)!}{(k+l+3)!} \alpha^{(k+l+3)} \frac{\partial}{\partial \alpha^{(k)}} \frac{\partial}{\partial \alpha^{(l)}}$$

It's easy to check that

$$[D, E_1] = [D, E_2] = 0$$

We write $E_1 = E'_1 + D\alpha^{(0)}$, where

$$E'_{1} = \frac{1}{2} \sum_{k,l>0} \frac{(k+l)!}{k!l!} \alpha^{(k)} \alpha^{(l)} \frac{\partial}{\partial \alpha^{(k+l-1)}} - \alpha^{(1)}$$

We can choose a basis of $A = \mathbb{C}[\alpha^{(0)}, \alpha^{(1)}, \cdots] / \operatorname{im} D$ as

$$\{\alpha^{(i_1)}\alpha^{(i_2)}\cdots(\alpha^{(i_k)})^2\}, \quad 0 \le i_1 \le i_2 \le \cdots \le i_k$$

then E'_1 acts on the above basis in the obvious way, while for the action of E_2 , we need to transform the result of the action to the above basis using the operator D.

Claim ker $E'_1 = Span\{(\alpha^{(0)})^k\}_{k \ge 0}.$

To prove the claim, we consider the filtration by the numbers of $\alpha^{(1)}$

$$F^p A = \left(\alpha^{(1)}\right)^p A,$$

then

$$E_1' \equiv \alpha^{(1)} \left(\sum_{k \ge 2} (k+1)\alpha^{(k)} \frac{\partial}{\partial \alpha^{(k)}} + \alpha^{(1)} \frac{\partial}{\partial \alpha^{(1)}} - 1 \right) : Gr_{F^{\bullet}}^p A \to Gr_{F^{\bullet}}^{p+1} A$$

where $\left(\sum_{k\geq 2}(k+1)\alpha^{(k)}\frac{\partial}{\partial\alpha^{(k)}} + \alpha^{(1)}\frac{\partial}{\partial\alpha^{(1)}} - 1\right)$ is a rescaling operator on A, which is positive on $\alpha^{(i_1)}\alpha^{(i_2)}\cdots(\alpha^{(i_k)})^2$ if $i_k\geq 1$. For $(\alpha^{(0)})^k$, we have

$$E_1'\left(\alpha^{(0)}\right)^k = -\left(\alpha^{(0)}\right)^k \alpha^{(1)} = -\frac{1}{k+1} D\left(\alpha^{(0)}\right)^{k+1}$$

which is zero in A. This proves the claim.

Let $A^{(k)}$ be the degree k part of A which is finite dimensional. We consider the second homogeneous grading on $A^{(k)}$ by giving all $\alpha^{(k)}$ homogeneous degree 1. Then E'_1 is homogeneous of degree 1 and E_2 is homogeneous of degree -1. Let $f \in A^{(k)}$ such that Ef = 0. We decompose

$$f = \sum_{i=0}^{k} f_i$$

where f_k contains is homogeneous of degree *i*. Therefore we have

$$E_1 f_k = 0, E_1 f_{k-1} = 0$$

 $E_2 f_i = -E_1 f_{i-2} , 2 \le i \le k$

It follows from the claim that f_k is a multiple of $(\alpha^{(0)})^k$ and all the other f_i 's are uniquely determined. This proves the uniqueness.

To show the existence, we consider the lagrangian in Eqn (6.48)

$$\int_{C} \frac{dz}{2\pi i z} \sum_{k \ge -1} \lambda^{k+1} \mathcal{L}^{(k)}(\alpha(z))$$

$$= \int_{C} \frac{dz}{2\pi i z} \exp\left(S(\lambda z \partial_{z})(\lambda \alpha(z))\right), \quad S(t) = \frac{\sinh t/2}{t/2}$$

$$= \int_{C} \frac{dz}{2\pi i z} \exp\left(\left(e^{\lambda z \partial_{z}/2} - e^{-\lambda z \partial_{z}/2}\right)\phi(z)\right)\right), \quad \alpha(z) = z \partial_{z}\phi(z)$$

$$= \int_{C} \frac{dz}{2\pi i z} \exp\left(\phi(e^{\lambda/2}z) - \phi(e^{-\lambda/2}z)\right)$$

$$= \int_{C} \frac{dz}{2\pi i z} \exp\left(\phi(e^{\lambda}z) - \phi(z)\right)$$

$$= \int_{C} \frac{dz}{2\pi i z} e^{-\phi(z)} e^{\lambda z \partial_{z}} e^{\phi(z)}$$
$$= \int_{C} \frac{dz}{2\pi i z} \sum_{k \ge 0} \frac{\lambda^{k}}{k!} (z \partial_{z} + \alpha)^{k} \cdot 1$$

where we use the convention that $\mathcal{L}^{(-1)} = 1$. Now we view $\alpha(z)$ as the bosonic field of the free boson system described in section 6.2. Since the normal ordered operator

$$\frac{1}{S(\lambda)} \int_C \frac{dz}{2\pi i z} : \exp\left(S(\lambda z \partial_z)(\lambda \alpha(z))\right) :_B$$

is the bosonization of the fermionic operator

$$\int_C \frac{dz}{2\pi i z} b(e^{\lambda/2} z) c(e^{-\lambda/2} z)$$

which is already simultaneously diagonalized on the standard fermionic basis. It follows that

$$\int_C \frac{dz}{2\pi i z} : \mathcal{L}^{(k)} :_B$$

are commuting operators on the bosonic Fock space, where the normal ordering relation is given by

$$\alpha(z_1)\alpha(z_2) = \frac{z_1 z_2}{(z_1 - z_2)^2} + : \alpha(z_1)\alpha(z_2) :_B \text{ if } |z_1| > |z_2|$$

If we rescale $\lambda \to \sqrt{\hbar}\lambda$, then

$$\int_C \frac{dz}{2\pi i z} : \frac{1}{(k+1)!} (\sqrt{\hbar} z \partial_z + \alpha)^{k+1} 1 :_B$$

are commutating operators on bosonic Fock space if we impose the normal ordering relation

$$\alpha(z_1)\alpha(z_2) = \frac{\hbar z_1 z_2}{(z_1 - z_2)^2} + : \alpha(z_1)\alpha(z_2) :_B \text{ if } |z_1| > |z_2|$$

This is precisely the commutativity property, i.e., we can take

$$\mathcal{J}^{(k)} = \frac{1}{(k+1)!} \left(D + \alpha^{(0)} \right)^{k+1} 1 \in A$$

This proves the existence.

6.5. **Proof of mirror symmetry.** In this section, we prove Theorem 6.1.

Proof of Theorem 6.1. By Corollary 6.9

$$\sum_{d\geq 0} q^d \hbar^g \left\langle \prod_{i=1}^n \tau_{k_i}(\tilde{\omega}) \right\rangle_{g,n,d} = \lim_{\bar{\tau} \to \infty} \lim_{\substack{\epsilon \to 0 \\ L \to \infty}} W\left(\hbar P^L_{\epsilon}(\tau,\bar{\tau}); \frac{1}{\hbar} \int_{C_1} dw \mathcal{L}^{(k_1+1)}, \cdots, \frac{1}{\hbar} \int_{C_n} dw \mathcal{L}^{(k_n+1)}\right)$$

where W is the summation of all connected Feynman diagrams with propagator $\hbar P_{\epsilon}^{L}(\tau, \bar{\tau})$ and n vertices given by

$$\int_{C_i} dw \mathcal{L}^{(k_i+1)}, \quad 1 \le i \le n$$

where C_i 's are cycles on the elliptic curve E_{τ} representing the class [0, 1] and are chosen to be disjoint, and $\mathcal{L}^{(k)}$ is the local functional on $\mathrm{PV}_{E_{\tau}}^{0,0}$ defined in Eqn (6.48).

On the other hand, we have

$$\lim_{\bar{\tau}\to\infty} F_g^{E_{\tau}}[\infty] \left[t^{k_1}\omega, \cdots, t^{k_n}\omega \right] = \lim_{\bar{\tau}\to\infty} \lim_{\substack{\epsilon\to0\\L\to\infty}} W\left(\hbar P_{\epsilon}^L(\tau;\bar{\tau}); \frac{1}{\hbar} \int_{C_1} dw \mathcal{J}^{(k_1)}, \cdots, \frac{1}{\hbar} \int_{C_n} dw \mathcal{J}^{(k_n)}\right)$$

where $\mathcal{J}^{(k)} = \sum_{g \ge 0} \hbar^g \mathcal{J}_g^{(k)}$ are local lagrangians on $\mathrm{PV}_{E_{\tau}}^{0,0}$ which contain only holomorphic derivatives. By lemma 6.14 and the proof of existence in lemma 6.15,

$$\int_{C_i} dw \mathcal{J}^{(k)} = \int_{C_i} dw \mathcal{L}^{(k+1)}$$

This proves the theorem.

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We will fix a ring R which contains \mathbb{Q} . All algebras, modules and tensor products are over R unless otherwise specified.

A.1. L_{∞} structure. Let \mathfrak{g} be a graded module

$$\mathfrak{g} = igoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$$

where \mathfrak{g}_n has degree *n*. Consider the reduced graded symmetric product

(A.1)
$$\bar{S}(\mathfrak{g}[1]) = \bigoplus_{n \ge 1} \operatorname{Sym}^n(\mathfrak{g}[1])$$

where [1] is the shifting operator such that

$$\mathfrak{g}[1]_n = \mathfrak{g}_{n+1}$$

and Sym^{*} is the graded symmetric product. $\overline{S}(\mathfrak{g}[1])$ has a graded commutative co-algebra structure. The co-product Δ is given as follows: for any $v_1, \dots, v_n \in \mathfrak{g}[1], v_1v_2 \dots v_n$ gives an element in Symⁿ ($\mathfrak{g}[1]$), then

(A.2)
$$\Delta : \overline{S} (\mathfrak{g}[1]) \to \overline{S} (\mathfrak{g}[1]) \otimes_R \overline{S} (\mathfrak{g}[1])$$
$$v_1 v_2 \cdots v_n \to \Delta (v_1 v_2 \cdots v_n) = \sum_{I \subset \{1, \cdots, n\}} \epsilon(I, I^c) v_I \otimes v_{I^c}$$

here the summation is over all subset I of the indices $\{1, \dots, n\}$, I^c is the complement of I. For each I, we choose some order for elements in I, I^c . Then if $I = \{i_1, \dots, i_k\}$, we write $v_I = v_{i_1} \cdots v_{i_k}$. $\epsilon(I, I^c)$ is the sign by permuting $v_1 \cdots v_n$ into the order $v_I v_{I^c}$ in $\operatorname{Sym}_R^n(\mathfrak{g}[1])$. It's easy to see that the formula doesn't depend on the choice of the order in I, I^c .

Definition A.1. A structure of L_{∞} algebra on \mathfrak{g} is given by a nilpotent coderivation of degree one Q on $\overline{S}(\mathfrak{g}[1])$ which satisfies

(A.3)
$$Q^2 = 0, \quad \Delta Q = (Q \otimes 1 + 1 \otimes Q) \Delta$$

This is equivalent to saying that the triple $(\bar{S}(\mathfrak{g}[1]), \Delta, Q)$ is a **dg-coalgebra**.

The L_{∞} structure Q is completely determined by

(A.4)
$$Q_k : \operatorname{Sym}_R^k (\mathfrak{g}[1]) \xrightarrow{Q} \bar{S} (\mathfrak{g}[1]) \to \mathfrak{g}[1]$$

where the last map is the projection to $\text{Sym}^1(\mathfrak{g}[1]) = \mathfrak{g}[1]$. In fact, given $v_1, \dots, v_n \in \mathfrak{g}[1]$, we have

(A.5)
$$Q(v_1 \cdots v_n) = \sum_{I \subset \{1, \cdots, n\}} \epsilon(I, I^c) Q_{|I|}(v_I) v_{I^c}$$

where |I| is the size of I.

Definition A.2. A L_{∞} morphism between two L_{∞} algebras $\mathfrak{g}, \mathfrak{g}'$ is a degree zero homomorphism

$$F: \overline{S}(\mathfrak{g}[1]) \to \overline{S}(\mathfrak{g}'[1])$$

which is compatible with coproduct and coderivation, i.e.

$$(F \otimes F) \Delta = \Delta' F, \quad FQ = Q'F$$

where Δ, Δ' and Q, Q' are the coproducts and coderivations on $\bar{S}(\mathfrak{g}[1])$ and $\bar{S}(\mathfrak{g}'[1])$ respectively. In other words, F is a morphism of dg-coalgebras.

Similar to (A.4), F is determined by

(A.6)
$$F_k: S^k(g_1[1]) \xrightarrow{F} \bar{S}(g_2[1]) \to g_2[1]$$

and for any $v_1, \cdots, v_n \in \mathfrak{g}_1[1]$,

(A.7)
$$F(v_1 \cdots v_n) = \sum_r \sum_{\substack{I_1, \cdots, I_r, \\ I_1 \cup \cdots \cup I_r = \{1, \cdots, n\}}} \frac{\epsilon(I_1, \cdots, I_r)}{r!} F_{|I_1|}(v_{I_1}) \cdots F_{|I_r|}(v_{I_r})$$

Here the convention for $\epsilon(I_1, \dots, I_r)$ is similar to (A.2).

Example A.3 (DGLA). A differential graded Lie algebra is a L_{∞} algebra \mathfrak{g} with $Q_k = 0$ for k > 2. If $x \in \mathfrak{g}$, we will use deg(x) to denote the degree of x, and x[1] to denote the

corresponding element of $\mathfrak{g}[1].$ We define a differential on \mathfrak{g} by requiring

$$(dx)[1] = -Q_1(x[1])$$

and a bracket [,] on ${\mathfrak g}$ by requiring

$$([x,y])[1] = (-1)^{\deg(x)}Q_2(x[1]y[1])$$

Now we explore the condition $Q^2 = 0$. For any $x \in \mathfrak{g}$,

-

$$Q^{2}(x[1]) = Q_{1}^{2}x[1] = (d^{2}x)[1] = 0$$

which implies that

$$d^2 = 0$$

For any $x, y \in \mathfrak{g}$,

$$Q(x[1]y[1]) = (-1)^{(\deg(x)+1)(\deg(y)+1)}Q(y[1]x[1])$$

which implies that

$$\begin{split} [x,y] &= -(-1)^{\deg(x)\deg(y)}[y,x] \\ Q^2\left(x[1]y[1]\right) &= Q_1Q_2\left(x[1]y[1]\right) + Q_2\left(\left(Q_1x[1]\right)y[1] + (-1)^{\deg(x)+1}x[1]\left(Q_1y[1]\right)\right) \\ &= -(-1)^{\deg(x)}\left(d[x,y]\right)[1] - (-1)^{\deg(x)+1}\left([dx,y]\right)[1] + ([x,dy])[1] \\ &= 0 \end{split}$$

which implies that

$$d[x, y] = [dx, y] + (-1)^{\deg(x)}[x, dy]$$

For any $x, y, z \in \mathfrak{g}$,

$$\begin{aligned} Q_2^2 \left(x[1]y[1]z[1] \right) &= Q_2 \left(\left(Q_2(x[1]y[1]) \right) z[1] \right) + (-1)^{\deg(x)+1} Q_2 \left(x[1] \left(Q_2(y[1]z[1]) \right) \right) \\ &+ (-1)^{(\deg(y)+1)\deg(x)} Q_2 \left(y[1] \left(Q_2(x[1]z[1]) \right) \right) \\ &= (-1)^{\deg(y)} [[x,y],z][1] + (-1)^{\deg(y)+1} [x,[y,z]][1] \\ &+ (-1)^{\deg(x)\deg(y)+\deg(y)} [y,[x,z]][1] \end{aligned}$$

= 0

which implies

$$[[x, y], z] = [x, [y, z]] - (-1)^{\deg(x) \deg(y)} [y, [x, z]]$$

It's easy to see that all the other relations in $Q^2 = 0$ will follow from the above identities. Therefore L_{∞} algebra structure with $Q_k = 0$ for k > 2 is equivalent to the differential graded Lie algebra structure.

Example A.4 (Differential forms on graded commutative algebra). Let A be a graded commutative k-algebra, where k is the base field. The tangent space is defined to be the space of graded k-derivation

$$T_A = \operatorname{Der}_k(A)$$

Given $D \in T_A$, we will use |D| to denote its degree as a map of graded vector spaces. Then

$$D(ab) = (Da)b + (-1)^{|D||a|}aDb \quad \forall a, b \in A$$

 T_A has a natural structure of graded Lie algebra, with the Lie structure given by

$$[D_1, D_2] = D_1 D_2 - (-1)^{|D_1||D_2|} D_2 D_1$$

and left A-module structure with

$$(aD)(b) = aD(b) \quad \forall a, b \in A$$

The differential forms are defined by

$$\Omega_A^* = \operatorname{Hom}_A \left(S\left(T_A[1] \right), A \right) = \bigoplus_{n \ge 0} \operatorname{Sym}_A^n \left(\Omega_A^1 \right)$$

where $S(T_A[1]) = \prod_{n \ge 0} \operatorname{Sym}_A^n(T_A[1])$ and $\Omega_A^1 = \operatorname{Hom}_A(T_A[1], A)$ is the space of Kähler differentials.

Here we use the following convention for the shifting operator [1]: if M is a graded A-module, M[1] is the degree one shifting of M, then the A-module structure on M[1] is given by

$$a \cdot (m[1]) = (-1)^{|a|} (am)[1], \quad \forall a \in A, m \in M$$

(1) If $f \in \Omega^0_A \equiv A$, then $df \in \Omega^1_A$ is determined by

$$df(X[1]) = (-1)^{|X|} X(f)$$

for $X \in T_A$. By our convention, df is obviously A-linear.

(2) $d^2 = 0$ (3) $d(\alpha\beta) = (d\alpha)\beta + (-1)^{|\alpha|}\alpha d\beta$, for any $\alpha, \beta \in \Omega_A^*$.

A.2. Maurer-Cartan equation.

Definition A.5. Let \mathfrak{g} be a L_{∞} algebra, $b \in \mathfrak{g}[1]_0 = \mathfrak{g}_1$. The Maurer-Cantan equation for b is defined to be

(A.8)
$$\sum_{k\geq 1} \frac{1}{k!} Q_k \left(b^k \right) = 0$$

We will use the notation

$$Q_* = \sum_{k \ge 1} Q_k : \bar{S}(g[1]) \to g[1], \ e^b = \sum_k \frac{b^k}{k!}$$

Then the Maurer-Cartan equation can be written as

(A.9)
$$e^{-b}Q_*(e^b) = 0$$

Let $F: (\mathfrak{g}, Q_1) \to (\mathfrak{g}', Q_2)$ be a L_{∞} morphism of L_{∞} algebras. We will denote by

$$F_* = \sum_k F_k : \bar{S}(\mathfrak{g}[1]) \to \mathfrak{g}'[1]$$

then

$$F(e^{b}) = F(\sum_{k} \frac{b^{k}}{k!})$$

= $\sum_{k} \sum_{r} \frac{1}{r!} \sum_{i_{1}, \cdots, i_{r}, i_{1} + \cdots + i_{r} = k} \frac{F_{i_{1}}(b^{i_{1}}) \cdots F_{i_{r}}(b^{i_{r}})}{i_{1}! \cdots i_{r}!}$

$$= e^{F_*(e^b)}$$

=

Therefore

$$Q'_*(e^{F_*(e^b)}) = Q'_*F(e^b) = F_*Q(e^b) = F_*(Q_*(e^b)e^b)$$

hence the morphism

 $b \to F_*(e^b)$

preserves the solutions of Maurer-Cartan equation.

Definition A.6. The Maurer-Cartan functor of a L_{∞} algebra \mathfrak{g} is the functor

$$\mathcal{MC}_{\mathfrak{g}}: \{ \text{graded commutative Artin algebra} \} \to \text{sets}$$
$$A \to \mathcal{MC}_{\mathfrak{g}}(A) = \left\{ b \in (\mathfrak{g} \otimes m_A) [1]_0 | \sum_{k \ge 1} \frac{1}{k!} Q_k \left(b^k \right) = 0 \right\}$$

from graded commutative Artin algebras to sets, which sends A to the space of solutions of Maurer-Cartan equations of $\mathfrak{g} \otimes m_A$. Here m_A is the nilpotent maximal ideal of A.

 \mathcal{MC}_g is indeed a functor since we have seen that morphism of L_∞ algebras preserves the solutions of Maurer-Cartan equations.

 $\mathcal{MC}_{\mathfrak{g}}$ can also be understood from the viewpoints of functor of points. Let A a graded commutative Artin R-algebra, with m_A the maximal ideal. The dual m_A^{\vee} has a natural coproduct structure which we denote by $\Delta_{m_A^{\vee}}$, and we endow m_A^{\vee} with the zero co-derivation. Let $b \in (\mathfrak{g} \otimes m_A)$ [1]₀, which can be identified with the morphism of graded vector spaces

$$b: m_V^{\vee} \to \mathfrak{g}[1]$$

and it induces a morphism of graded co-algebras

$$\sum_{k\geq 1} \frac{b^k}{k!} \Delta^{k-1}_{m_A^\vee} : m_V^\vee \to \bar{S}\left(\mathfrak{g}[1]\right)$$

here $\Delta_{m_A^\vee}^k$ is defined to be

$$\Delta_{m_A^{\vee}}^k = \left(\Delta_{m_A^{\vee}} \otimes 1^{\otimes k-1}\right) \left(\Delta_{m_A^{\vee}} \otimes 1^{\otimes k-2}\right) \cdots \left(\Delta_{m_A^{\vee}} \otimes 1\right) \Delta_{m_A^{\vee}}$$

and we have $\Delta_{m_A^{\vee}}^k = 0$ for k sufficiently large since m_A is nilpotent. Then the condition for b satisfying Maurer-Cartan equation says precisely that the map $\sum_{k\geq 1} \frac{b^k}{k!} \Delta_{m_A^{\vee}}^{k-1}$ is a morphism of dg-coalgebras. Therefore

$$\mathcal{MC}_{\mathfrak{g}}(A) = \operatorname{Hom}_{DGC}\left(m_{A}^{\vee}, \bar{S}\left(\mathfrak{g}[1]\right)\right)$$

where DGC refers to dg-coalgebra.

Example A.7. If \mathfrak{g} is a DGLA. Let $x \in \mathfrak{g}_1$. The Maurer-Cartan equation is

$$Q_1(x[1]) + \frac{1}{2}Q_2(x[1]x[1]) = 0$$

which is equivalent to

$$dx + \frac{1}{2}[x, x] = 0$$

This can be viewed as the deformation of the differential d to

$$d \to d_x = d + [x, -]$$

and Maurer-Cartan equation says that $d_x^2 = 0$.

A.3. Homotopy. Let \mathfrak{g} be a L_{∞} *R*-algebra. We consider the graded vector spaces

 $\mathfrak{g}[t, dt]$

Here t be a variable of degree 0, and dt is of degree 1. We have an induced dg-coalgebra structure on

 $\bar{S}\left(\mathfrak{g}[1][t,dt]\right)$

The co-derivation is given by

 $Q + d_t$

where Q is the co-derivation induced from that on $\overline{S}(\mathfrak{g}[1])$ and $d_t = dt \wedge \frac{\partial}{\partial t}$. This gives a L_{∞} structure on $\mathfrak{g}[t, dt]$. For $t_0 \in \mathbb{R}$, we have the evaluation map of L_{∞} algebras

$$\operatorname{Eval}_{t=t_0} : \mathfrak{g}[t, dt] \to \overline{S}(\mathfrak{g})$$
$$P(t) + Q(t)dt \to P(t_0)$$

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Definition A.8. Two L_{∞} homomorphisms $F_1, F_2 : \mathfrak{g} \to \mathfrak{g}'$ are said to be **homotopic** to each other, if there exists a L_{∞} morphism $F : \mathfrak{g} \to \mathfrak{g}'[t, dt]$ such that $F_1 = \operatorname{Eval}_{t=0} \circ F$ and $F_2 = \operatorname{Eval}_{t=1} \circ F$. A L_{∞} homomorphism $F : \mathfrak{g} \to \mathfrak{g}'$ is said to be **homotopy equivalence** if there exists a L_{∞} morphism $G : \mathfrak{g}' \to \mathfrak{g}$ such that the compositions $F \circ G$ and $G \circ F$ are homotopic to identities.

Proposition A.9. If $F : \mathfrak{g} \to \mathfrak{g}'$ is a L_{∞} homomorphism which induces a quasi-isomorphism of complexes

$$F:\left(\bar{S}\left(\mathfrak{g}[1]\right),Q_{1}\right)\to\left(\bar{S}\left(\mathfrak{g}'[1]\right),Q'_{1}\right)$$

then F is a homotopy equivalence.

Proof. See for example [Kon03, Fuk].

A.4. Deformation functor.

Definition A.10. Let \mathfrak{g} be a L_{∞} algebra, and b_1, b_2 be two elements of $\mathfrak{g}[1]_0$ which satisfy the Maurer-Cartan equation. We say that b_1 is **gauge equivalent** to b_2 if there exists $\tilde{b} \in$ $(\mathfrak{g}[1][t, dt])_0$ such that \tilde{b} satisfies Maurer-Cartan equation and $\operatorname{Eval}_{t=0}(\tilde{b}) = b_1, \operatorname{Eval}_{t=1}(\tilde{b}) = b_2$.

Example A.11. Let's consider the case that \mathfrak{g} is a DGLA. Let x(t) + y(t)dt be an element of $\mathfrak{g}[t, dt]$ which solves the Maurer-Cartan equation. Then

$$(d+d_t)(x(t)+y(t)dt) + \frac{1}{2}[x(t)+y(t)dt, x(t)+y(t)dt] = 0$$

or equivalently

$$dx(t) + \frac{1}{2}[x(t), x(t)] = 0$$
$$\frac{\partial}{\partial t}x(t) = dy(t) + [x(t), y(t)]$$

If we write $d_{x(t)} = d + [x(t), -]$, then the above equations can be rewritten as

$$d_{x(t)}^2 = 0, \quad \frac{\partial}{\partial t} d_{x(t)} = [d_{x(t)}, [y(t), -]]$$

Given a L_{∞} algebra, we define its deformation functor by

$$\mathcal{D}ef_{\mathfrak{g}}: \{ \text{graded commutative Artin } R\text{-algebra} \} \to \text{sets}$$

 $A \to \mathcal{D}ef_{\mathfrak{g}}(A) = \mathcal{MC}_{\mathfrak{g}}(A) / \sim$

where the equivalence relation \sim is the gauge equivalence.

In explicit deformation problems, the deformation functor is usually realized as a deformation functor of certain L_{∞} algebra. The good thing about L_{∞} structure is that homotopy equivalent L_{∞} algebras characterize essentially the same deformation space. More precisely

Proposition A.12. If \mathfrak{g} is homotopy equivalent to \mathfrak{g}' as L_{∞} algebras, then the deformation functor $\mathcal{D}ef_{\mathfrak{g}}$ is equivalent to the deformation functor $\mathcal{D}ef_{\mathfrak{g}'}$.

Proof. See for example [Kon03, Fuk].

Appendix B. *D*-modules and Jets

Throughout this section, X will be a smooth oriented manifold of dimension n, and E will be a graded vector bundle on X.

B.1. *D*-module. Let D_X denote the sheaf of algebra of differential operators. If we choose local coordinates x_1, \dots, x_n on X, then a local section of D_X can be written as

$$\sum_{I} f_{I}(x) \frac{\partial}{\partial x^{I}}$$

where $I = \{i_1, \dots, i_n\}$ is a multi-index, $\frac{\partial}{\partial x^I} = \left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n}$, and $f_I(x)$ is zero for all but finitely many I's.

Definition B.1. A D_X -module is a sheaf of D_X modules on X.

Example B.2. D_X can be viewed as a free D_X -module of rank 1, where the D_X -module structure is given by composition of differential operators.

Example B.3. The sheaf of smooth functions on X, which we denote by $C^{\infty}(X)$, is naturally a left D_X -module.

Example B.4. If E is a vector bundle on X with a flat connection ∇ , then E can be viewed as a left D_X -module. Locally, the D_X -module structure is generated by

$$D_X \rightarrow \operatorname{End}(E)$$

 $\frac{\partial}{\partial x_i} \rightarrow \nabla_{x_i}$

Example B.5. Let X be an oriented smooth manifold, ω_X be the sheaf of smooth topforms on X. Then ω_X has the structure of right D_X -modules. Given a section α of ω_X , α is uniquely determined by the map

$$\begin{array}{rcl} \alpha: C^\infty_c(X) & \to & \mathbb{C} \\ & f & \to & \int_X \alpha f \end{array}$$

where $C_c^{\infty}(X)$ is the space of smooth functions with compact supports. Let P be a differential operator, then the right action of P on α is defined by requiring

$$\begin{array}{rcl} \alpha \cdot P : C^\infty_c(X) & \to & \mathbb{C} \\ & f & \to & \int_X \alpha P(f) \end{array}$$

Example B.6 (Tensor product). Let M, N be two left D_X -modules, we can define the tensor product as a left D_X -module by

$$M \otimes_{C^{\infty}(X)} N$$

The D_X -module structure is generated locally by

$$\partial_{x_i} (m \otimes n) = (\partial_{x_i} m) \otimes n + m \otimes \partial_{x_i} n$$

where m, n are sections of M, N respectively.

Definition B.7. Let M be a left D_X -module. The **de Rham complex** of M is the defined to be the complex

$$\Omega^*_X(M) \equiv \Omega^*_X \otimes_{C^\infty(X)} M$$

where the differential is defined by

(B.1)
$$d(\omega \otimes m) = (d\omega) \otimes m + \sum_{i} dx_i \wedge \omega \otimes \partial_{x_i} m$$

Example B.8. There's a natural complex of right D_X -modules

 α

$$0 \to \Omega^0_X \otimes_{C^{\infty}(X)} D_X \to \Omega^1_X \otimes_{C^{\infty}(X)} D_X \to \dots \to \Omega^n_X \otimes_{C^{\infty}(X)} D_X \to \omega_X \to 0$$

where the last morphism is determined via

$$\otimes P : C_c^{\infty}(X) \to \mathbb{C}$$

 $f \to \int_X \alpha P(f)$

which defines an element of ω_X . It turns out that the above complex is exact, therefore we have obtained the quasi-isomorphism of complexes of right D_X -modules

(B.2)
$$\omega_X \simeq \Omega_X^*(D_X)[n]$$

where n is the dimension of X.

B.2. Jet bundles. Let E be a vector bundle on X. We will not distinguish between E as a vector bundle or E as a locally free sheaf. The sheaf of jets Jet(E) is locally given by

$$\{f_I^{\alpha}e_{\alpha}\}_I$$

where $\{e_{\alpha}\}$ is a local basis of E, $I = \{i_1, \cdots, i_n\}_{I_i \in \mathbb{Z}^{\geq 0}}$ runs over the set of multi-indices, and f_I^{α} is a local smooth function. There's a natural map

$$\Gamma(E) \to \operatorname{Jet}(E)$$

which in local coordinates x_1, \cdots, x_n is

$$f^{\alpha}e_{\alpha} \to \{\partial_I f^{\alpha}e_{\alpha}\}_I$$

here $\partial_I f = \frac{\partial}{\partial x^I} f$. There's a natural D_X -module structure on Jet(E) generated by

$$\partial_i \{\partial_I f^\alpha e_\alpha\}_I = \{\partial_i f_I^\alpha e_\alpha - f_{iI}^\alpha e_\alpha\}_I$$

which gives the exact sequence of sheaves

$$0 \to E \to \operatorname{Jet}(E) \to \Omega^1_X(\operatorname{Jet}(E)) \to \Omega^2_X(\operatorname{Jet}(E)) \to \cdots$$

or equivalently we have the quasi-isomorphism of complex of $C^{\infty}(X)$ sheaves

$$E \simeq \Omega^*_X(\operatorname{Jet}(E))$$

There's an intrinsic way to describe Jet(E) as follows. We consider the product $X \times X$ with two projections p_1, p_2



Let $\Delta \hookrightarrow X \times X$ be the diagonal, and I_{Δ} be the ideal sheaf of Δ . Then

$$\operatorname{Jet}(E) = \varprojlim_{k} p_{1*} \left(C^{\infty}(X \times X) / I_{\Delta}^{k} \otimes p_{2}^{*} E \right)$$

and the D_X -module structure comes from the natural action of differential operators on the left copy X of $X \times X$. If we choose local coordinates x_1, \dots, x_n on an open subset U of X, a local basis $\{e_\alpha\}$ of E, and let x'_1, \dots, x'_n be another copy of x_1, \dots, x_n , then $\{x_i, x'_i\}$ is a local coordinate system of $U \times U$. Then

$$\operatorname{Jet}(E)|_U = C^{\infty}(U)[[\delta x_1, \cdots, \delta x_n]] \otimes \operatorname{Span}_{\mathbb{C}} \{e_{\alpha}\}$$

where $\delta x_i = x'_i - x_i$. An element of $\text{Jet}(E)|_U$ can be described by $\{f_I^{\alpha} e_{\alpha}\}_I$ via

$$\{f_I^{\alpha}(x)e_{\alpha}\}_I \to \sum_I f_I(x) \frac{\delta x^I}{I!} e_{\alpha}$$

where $\delta x^{I} = \delta x_{1}^{i_{1}} \cdots \delta x_{n}^{i_{n}}, I! = i_{1}! \cdots i_{n}!$ if $I = \{i_{1}, \cdots, i_{n}\}$. The map $E \to \text{Jet}(E)$ is simply given by the Taylor series

$$\{f^{\alpha}(x)e_{\alpha}\} \rightarrow \{f^{\alpha}(x')e_{\alpha}\} = \sum_{I} f_{I}(x)\frac{\delta x^{I}}{I!}e_{\alpha}$$

We can also consider the projective dual

$$\operatorname{Jet}(E)^{\vee} = \operatorname{Hom}_{C^{\infty}(X)}(\operatorname{Jet}(E), C^{\infty}(X))$$

In local coordinates, if we represent an element of Jet(E) by

$$f = \sum_{I} f_{I}^{\alpha} \frac{\delta x^{I}}{I!} e_{\alpha} \in C_{M}^{\infty}(E)[[x_{1}, \cdots, x_{n}]]$$

and represent an element of $\operatorname{Jet}(E)^{\vee}$ by

$$g = g^I_\alpha \frac{\partial}{\partial x^I} e^\alpha$$

where $\{e^{\alpha}\}$ is the dual basis of $\{e_{\alpha}\}$. Then we have the natural pairing

$$< g, f > = \sum_{I, \alpha} g^I_{\alpha} f^{\alpha}_I$$

Since both $\operatorname{Jet}(E)$ and $C^{\infty}(X)$ have the structure of D_X -modules, we have an induced D_X -module on $\operatorname{Jet}(E)^{\vee}$. More precisely, in local coordinates, the action of ∂_{x_i} on $\operatorname{Jet}(E)^{\vee}$ is given by requiring

$$\partial_{x_i} < g, f > = < \partial_{x_i}g, f > + < g, \partial_{x_i}f >$$

from which we find

$$\partial_x \left(g^I_\alpha \frac{\partial}{\partial x^I} e^\alpha \right) = \partial_i g^I_\alpha \frac{\partial}{\partial x^I} e^\alpha + g^I_\alpha \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^I} e^\alpha$$

This is exactly the natural D_X -module structure that we would expect. In particular, it shows that $\text{Jet}(E)^{\vee}$ is a locally free D_X -module whose ranks equals the rank of the vector bundle E.

We can also describe differential operators in terms of jet bundles. Let E, F be two vector bundles on X, then the differential operators from E to F, denoted by Diff(E, F), is just a D_X morphism from Jet(E) to Jet(F) [Cos11]

(B.3)
$$\operatorname{Diff}(E, F) \cong \operatorname{Hom}_{D_X}(\operatorname{Jet}(E), \operatorname{Jet}(F))$$

$$0 \longrightarrow E \longrightarrow \operatorname{Jet}(E) \longrightarrow \Omega^{1}_{X}(\operatorname{Jet}(E)) \longrightarrow \cdots$$
$$\downarrow^{P_{\varphi}} \qquad \qquad \downarrow^{\varphi} \qquad \qquad \downarrow^{1 \otimes \varphi}$$
$$0 \longrightarrow F \longrightarrow \operatorname{Jet}(F) \longrightarrow \Omega^{1}_{X}(\operatorname{Jet}(F)) \longrightarrow \cdots$$

B.3. Local functionals. Let

$$\mathscr{E} = \Gamma(X, E)$$

be the space of smooth sections in E. Recall that a local functional on E of order k is given by a map

$$S: \operatorname{Sym}^k \mathscr{E} \to \mathbb{C}$$

which takes the form

$$S[\alpha_1, \cdots, \alpha_n] = \int_X D_1(\alpha_1) \cdots D_k(\alpha_k) d\operatorname{Vol}$$

where $\alpha_i \in \mathscr{E}$, $D_i : \mathscr{E} \to C^{\infty}(X)$ are some smooth differential operators, and d Vol is a volume form. It's easy to see that such local functional can be described in terms of jet bundle by

$$\omega_M \otimes_{D_M} \operatorname{Sym}_{C^{\infty}(X)}^k(\operatorname{Jet}(E)^{\vee})$$

where the tensor product over D_M takes care of the fact that total derivatives are zero. It follows that the space of local functionals can be described by

$$\mathscr{O}_{loc}(\mathscr{E}) = \omega_X \otimes_{D_X} \prod_{k \ge 1} \operatorname{Sym}_{C^{\infty}(X)}^k (\operatorname{Jet}(E)^{\vee})$$

where we have neglected the constant functional when k = 0. Since $\text{Jet}(E)^{\vee}$ is free D_M -module, the Koszul resolution (B.2) gives the quasi-isomorphism

$$\mathscr{O}_{loc}(\mathscr{E}) \cong \Omega^*_X \left(\widehat{\operatorname{Sym}}_{C^{\infty}(X)}^{* \ge 1} (\operatorname{Jet}(E)^{\vee}) \right) [n]$$

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